A PROOF OF JAKOBSON'S THEOREM

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ABSTRACT. We give a proof of Jakobson's theorem: with positive probability on the parameter, a real quadratic map leaves invariant an absolutely continuous ergodic invariant probability measure with positive Lyapunov exponent.

CONTENTS

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2000 *Mathematics Subject Classification.* Primary: 37C15 (Topological and differentiable equivalence, conjugacy, invariants, moduli, classification); Secondary: 37E05 (maps of the interval), 37J40 (Perturbations, normal forms, small divisors, KAM theory, Arnold diffusion), 11J70 (Continued fractions and generalizations).

1. INTRODUCTION

1.1. **Statement of the theorem.** In the 1960's, Sinai, Ruelle and Bowen developed the ergodic theory of uniformly hyperbolic dynamical systems. In the simplest setting of a uniformly expanding map of a torus, one obtains a unique ergodic invariant probability measure absolutely continuous w.r.t. the Lebesgue measure. In the 1970's, a systematic study of unimodal maps of the interval was initiated. The quadratic family $P_c(x) = x^2 + c$ appeared as a central object, from the point of view of real as well as complex 1-dimensional dynamics. When the critical point escapes to infinity, the same is true for almost all orbits. When P_c has an attractive periodic orbit, it attracts almost all non escaping orbits. Does there exist, for a typical parameter c, another kind of dynamical behavior?

Jakobson [J] provided a positive answer:

Theorem 1.1. *There exists a set Λ of positive Lebesgue measure such that, for $c \in \Lambda$, the quadratic polynomial P_c has an ergodic invariant absolutely continuous probability measure with positive Lyapunov exponent, supported on the interval $[P_c(0), P_c^2(0)]$. One has actually*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\mathrm{Leb}(\Lambda \cap [-2, -2+\varepsilon]) = 1.$$

After Jakobson's original paper, a number of different proofs appeared [BC1] [Ry]. Jakobson's theorem was the subject of my lectures at Collège de France in 1997-98. A first handwritten version of the following notes was produced at the time, and was made available over the years to those who asked me. It is perhaps not too late for a most systematic diffusion effort.

1.2. **Some facts about quadratic polynomials.** We refer to [DH] and [M] as general references for the results in this subsection, with the exception of the last paragraph.

For a complex parameter c, we denote by P_c the complex quadratic polynomial $P_c(z) = z^2+c$. Recall that the *filled-in Julia set* $K(c)$ is the set of points in $\mathbb{C}$ which have a bounded orbit under iteration of P_c. It is a non-empty full compact subset of the complex plane invariant under P_c. Its boundary is the *Julia set* $J(c)$. When c is real, we define $K_{\mathbb{R}}(c)$ to be the intersection $K(c) \cap \mathbb{R}$. Similarly, we define $J_{\mathbb{R}}(c) := J(c) \cap \mathbb{R}$.

The *Mandelbrot set* $\mathcal{M}$ is the set of parameters c such that the critical point 0 of P_c belongs to $K(c)$. By a theorem of Douady-Hubbard, this happens iff $K(c)$ is connected. The Mandelbrot set is a non-empty full compact subset of the complex plane. When the parameter c does not belong to $\mathcal{M}$, $K(c) = J(c)$ is a Cantor set and the restriction of P_c to K_c is an expanding map conjugated to the full unilateral shift on two symbols.

In the rest of this subsection, we only consider real parameters. The intersection of $\mathcal{M}$ with the real line is equal to the interval $[-2, 1/4]$. For $c > 1/4$, the Julia set is disjoint from the real line. When $c < -2$, the Julia set is contained in the real line.

For $c = 1/4$, P_c has a single fixed point at $z = 1/2$, which is *parabolic* in the sense that $DP_c(1/2) = 1$. For $c < 1/4$, the two fixed points of P_c are real. It is customary to denote the larger one by $\beta := \frac{1}{2}(1+\sqrt{1-4c})$ and the smaller one by $\alpha := \frac{1}{2}(1-\sqrt{1-4c})$. The fixed point β is repulsive for all $c < 1/4$. The fixed point α is attractive for $1/4 > c > -3/4$, repulsive for $c < -3/4$, with a flip bifurcation occurring at $c = -3/4$. The real filled-in Julia set $K_{\mathbb{R}}(c)$ is equal to the interval $[-\beta, \beta]$ for $c \in [-2, 1/4]$.

The basin of any attractive periodic orbit must contain the critical point. Therefore there is at most one attractive periodic orbit. Let $\mathcal{A}$ be the set of real parameters c such that P_c has an attractive periodic orbit. It is an open subset of $(-2, 1/4)$. When $c \in \mathcal{A}$, the real Julia set $J_{\mathbb{R}}(c)$ is an expanding invariant Cantor set equal to the complement in $K_{\mathbb{R}}(c)$ of the basin of the attractive periodic orbit. Conversely, a parameter $c \in [-2, 1/4]$ such that the real Julia set is expanding belongs to $\mathcal{A}$.

A deep theorem conjectured by Fatou and proved independently by Graczyk-Swiatek ([GS1], [GS2]) and Lyubich ([L1]), asserts that the open set $\mathcal{A}$ is dense in $[-2, 1/4]$. Their result is posterior to Jakobson's theorem. Observe that Jakobson's theorem implies that $\mathcal{A}$ does not have full Lebesgue measure in $[-2, 1/4]$. More recently, Lyubich has shown ([L2]) that almost all parameters in $[-2, 1/4]$ either belong to $\mathcal{A}$ or satisfy the conclusions of Jakobson's theorem.

1.3. **Plan of the proof.** We describe now the content of the rest of this paper.

In Section 2, we introduce some of the main concepts for the proof of Jakobson's theorem. Denote by A the *central interval* whose endpoints are the negative fixed point α and its inverse image $-\alpha$. An interval J is *regular of order* $n > 0$ if there is a branch g_J of P_c^{-n} which is a diffeomorphism on some fixed combinatorially defined neighborhood $\widehat{A}$ of A and sends A onto J. A parameter c is *regular* if the central interval is covered by regular intervals of order $\leqslant n$, except for a set of exponentially small measure.

Regular parameters satisfy the conclusions of Jakobson's theorem. One uses the maximal regular intervals contained in the central interval to define on A a Bernoulli map T which is a return map for P (but not the first return map). It is very classical that such a map has a unique absolutely continuous invariant probability measure with analytic density. As the return time relating T to P_c is integrable, one is able to spread the T-invariant measure on A into a P-invariant measure supported on $[P_c(0), P_c^2(0)]$. This measure is still absolutely continuous. Its density w.r.t. the Lebesgue measure is integrable but not square-integrable. The Lyapunov exponent of this measure is positive.

In the last three sections of the paper, we assume that the parameter c is very close to -2 (and > -2). This amounts to say that the return time M of the critical point in the central interval A is large. In the first part of Section 3, the first iterates of P_c for such a parameter are considered. It is shown in particular that , for $2 \leqslant n \leqslant M-2$, there are a couple of maximal regular intervals $C_n^{\pm}$ of order n contained in A. These intervals are called the *simple* regular intervals. Their union covers A except for a small symmetric interval around 0 of approximate size 2^{-M}.

To go further, we introduce the main definition of the paper: a parameter c is said to be *strongly regular* if the postcritical orbit can be decomposed into regular returns into the central interval A, and if most of these returns occur in the simple regular intervals $C_n^{\pm}$. More specifically we ask that the fraction of total time spent in non simple returns is at most $2^{-\sqrt{M}}$ (to compare with the approximate size 2^{-M} of the gap left out by the simple regular intervals). For a strongly regular parameter, the derivatives of the iterates along the postcritical orbit grow exponentially fast in a very controlled way.

In Section 4, we prove that strongly regular parameters are regular, and thus satisfy the conclusions of Jakobson's theorem. For $n > 0$, an interval J is said to be n-singular if J is contained in A, its endpoints are consecutive elements of $P_c^{-n-1}(\alpha)$, and J is not contained in a regular interval of order $\leqslant n$. One has to show that, if c is a strongly regular parameter, the union of all n-singular intervals has exponentially small Lebesgue measure. This is done by induction on n, the starting point being provided by the estimates on simple regular intervals of Section 3. We divide the n-singular intervals into several classes: *peripheral , lateral, and central*. The central ones are so close to 0 that the crudest estimate of the Lebesgue measure of their union is sufficient. On the other hand, each peripheral or lateral n-singular interval J is dynamically related to a m-singular interval J^* with $m < n$. The control on the postcritical orbit (Section 3) allows to conclude the induction step.

In the last Section, we prove that, in the parameter interval $(c^{(M)}, c^{(M-1)})$ where the return time of 0 in A is exactly equal to M, most parameters are strongly regular. More precisely, for any $\theta < 1/2$, the set of non strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ has relative Lebesgue measure $O(2^{-\theta M})$.

We first transfer to parameter space the "puzzle" structure of phase space. In order to do this we estimate the variation w.r.t. the parameter of the relevant inverse branches of the iterates of P_c. The next step is to transfer to parameter space the measure estimates of Section 4 on the measure of n-singular intervals. There is a rather subtle point here: while it is easy to transfer estimates for single intervals, for sets which are union of many disjoint components, we need to control the sum of the maximal measure (w.r.t. the parameter) of the components rather than the maximal measure of the set itself. Fortunately, the combinatorial nature of the arguments of Section 4 allows this control, except for the central n-singular intervals where a rough but sufficient control of the number of components is used.

The last part of the proof is an easy and classical large deviation argument: once we know that the order of a given regular return of the postcritical orbit in A is $> n$ with exponentially small probability, it is easy to control the measure of non strongly regular parameters.

2. Regular parameters and Bernoulli maps

2.1. **Regular points and regular parameters.** Consider a parameter $c \in [-2, 0)$ for the real quadratic family. The polynomial P_c has two fixed points α, β which verify $-\beta < \alpha < 0$. The critical value $c = P_c(0)$ satisfies $-\beta \leqslant P_c(0) < \alpha$.

Therefore, there exists $\alpha^{(1)} \in (-\beta, \alpha)$ such that $P{-1}_c(-\alpha) = \{\alpha^{(1)}, -\alpha^{(1)}\}$. We define

$$A := [\alpha, -\alpha] , \qquad \widehat{A} := (\alpha^{(1)}, -\alpha^{(1)}) .$$

Definition 2.1. Let c be a parameter in $[-2, 0)$, and let n be a positive integer. A point $x \in [-\beta, \beta]$ is *n-regular* if there exists an integer m, with $0 < m \leqslant n$, and an open interval $\widehat{J}$ with $x \in \widehat{J}$, such that the restriction of P_c^m to $\widehat{J}$ is a diffeomorphism onto $\widehat{A}$ and $P_c^m(x) \in A$.

Definition 2.2. A parameter $c \in [-2, 0)$ is *regular* if there exist $\theta, C > 0$ such that, for every $n > 0$:

$$\mathrm{Leb}\{x \in A, x \text{ is not } n\text{−regular}\} \leqslant C\, e^{-\theta n}.$$

The set of regular parameters is denoted by $\mathcal{R}$.

Theorem 2.3. *The set of regular parameters has positive Lebesgue measure. More precisely, we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathrm{Leb}(R \cap [-2, -2 + \varepsilon]) = 1 .$$

Before proving Theorem 2.3, we will in the next subsections describe the dynamics of P_c for $c \in \mathcal{R}$. The classical formulation of Jakobson's theorem (Theorem 1.1) will then be an immediate consequence of Theorem 2.3.

2.2. **The special parameter** $c = -2$**.** The polynomial $P := P_{-2}$ is a Tchebycheff polynomial :

$$P(2\cos\theta) = 2\cos 2\theta \ .$$

Define $h(x) = (4 - x^2)^{-1/2}$, for $x \in (-2, 2)$.

We have $\alpha = -1, \ \beta = 2, \ DP(\beta) = 4, \ P(0) = -\beta$ and :

$$|DP(x)| = 2\,\frac{h(x)}{h(P(x))}$$

for $0 < |x| < 2$. This implies

$$|DP^n(x)| = 2^n \frac{h(x)}{h(P^n(x))}$$

for all $n \geqslant 0$, $x \in (-2, 2)$ such that $P^n(x) \in (-2, 2)$.

For later reference, we describe the first return map R to the interval $A = [-1, 1]$.

For $n \geqslant 0$, put

$$\alpha^{(n)} = -2\,\cos\frac{\pi}{3.2^n}, \quad \widetilde{\alpha}^{(n+1)} = -2\,\sin\frac{\pi}{3.2^{n+1}}.$$

We then have $\alpha = \alpha^{(0)} = \widetilde{\alpha}^{(1)}$ and

$$P(\pm\alpha^{(n)}) = -\alpha^{(n-1)}, \quad P(\pm\widetilde{\alpha}^{(n)}) = \alpha^{(n-1)}, \quad \forall\, n > 0.$$

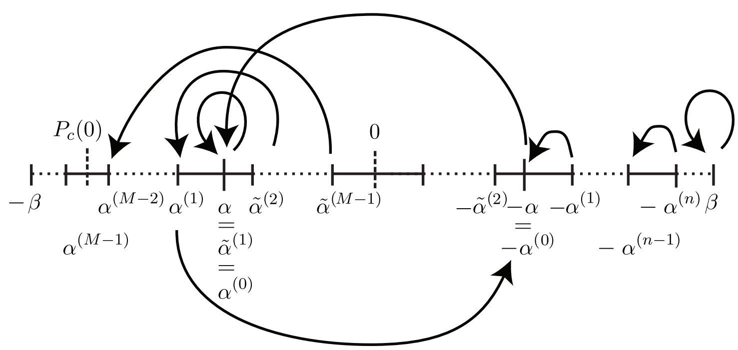

FIGURE 1

The sequence $(\alpha^{n)})_{n\geqslant 0}$ is decreasing and converges to -2. The sequence $(\widetilde{\alpha}^{n)})_{n\geqslant 1}$ is increasing and converges to 0.

The return map R to A is given by (see Fig.)

- $R(\pm\alpha) = P(\pm\alpha) = \alpha$;
- for $n > 1$, the restriction of R to $(\widetilde{\alpha}^{(n-1)}, \widetilde{\alpha}^{(n)}]$ and to $[-\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n-1)})$ is P^n, with $P^n(\pm\widetilde{\alpha}^{(n)}) = -\alpha$, $P^n(\pm\widetilde{\alpha}^{(n-1)}) = \alpha$. The restriction of P^n to $[\widetilde{\alpha}^{(n-1)}, \widetilde{\alpha}^{(n)}]$(resp.$[-\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n-1)}]$) is an orientation preserving (resp. orientation-reserving) diffeomorphism onto A.
- the critical point 0 does not come back to A, as $P(0) = -\beta$.

We also observe that the restriction of P^n to $[\widetilde{\alpha}^{(n-1)}, \widetilde{\alpha}^{(n)}]$ (or $[-\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n-1)}]$) extends to an open neighbourhood of this interval to give a diffeomorphism onto $(-\beta, \beta)$. Therefore, for $n > 1$, the set of n-regular points in A is exactly $[\alpha, \widetilde{\alpha}^{(n)}] \cup [-\widetilde{\alpha}^{(n)}, -\alpha]$. The complement in A is $(\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n)})$. The Lebesgue measure of this interval is $2|\widetilde{\alpha}^{(n)}| = 4\sin\frac{\pi}{3.2^n}$. Hence $c = -2$ is a regular parameter.

2.3. **Regular intervals.** Let c be a parameter in $[-2, 0)$ and let n be a nonnegative integer.

Definition 2.4. A compact interval $J \subset [-\beta, \beta]$ is *regular* of order n if there exists an open interval $\widehat{J} \supset J$ such that the restriction of P_c^n to $\widehat{J}$ is a diffeomorphism onto $\widehat{A}$, with $P_c^n(J) = A$. The integer n and the interval $\widehat{J}$ are uniquely determined by this property. We write $n = \mathrm{ord}(J)$ for the order of J, and $g_J : \widehat{A} \to \widehat{J}$ for the diffeomorphism inverse to $P_c^n/\widehat{J}$.

For $n \geqslant 0$, let

$$\Delta_n = P_c^{-n}(\{-\alpha, +\alpha\}) = P_c^{-(1+n)}(\{\alpha\}) .$$

We have

$$\Delta_0 = \{-\alpha, +\alpha\}, \qquad \Delta_1 = \{\pm\alpha, \pm\alpha^{(1)}\}.$$

Proposition 2.5. *Let $J = [\gamma^-, \gamma^+]$ be a regular interval of order n. Let $\widehat{J} = (\widehat{\gamma}^-, \widehat{\gamma}^+)$ the associated neighborhood.*

(1) *$\gamma^- < \gamma^+$ are consecutive points of Δ_n.*
(2) *$\widehat{\gamma}^- < \gamma^- < \gamma^+ < \widehat{\gamma}^+$ are consecutive points of Δ_{n+1}.*
(3) *If $n > 0$, $P_c(J)$ is regular of order $(n-1)$.*
(4) *If $J \subset A$, then $\widehat{J} \subset \widehat{A}$; if $J \subset$ int A, then $\widehat{J} \subset$ int A.*
(5) *Let J' is another interval, regular of order n'. If $J \subset A$, the interval $J'' = g_{J'}(J)$ is regular of order $n'' = n + n'$, with $\widehat{J}'' = g_{J'}(\widehat{J})$, $g_{J''} = g_{J'} \circ g_J$.*
(6) *If J' is another regular interval, either $J \subset J'$ holds , or $J' \subset J$, or int$(J) \cap$ int$(J') = \emptyset$.*

Proof. (1) No point of int J is sent by P_c^n to Δ_0.
(2) No point of $\widehat{J}$, except from $\gamma^{\pm}$, is sent by P_c^n to Δ_1.
(3) Clear.
(4) If $\alpha \leqslant \gamma^- < \gamma^+ \leqslant -\alpha$ (resp. $\alpha < \gamma^- < \gamma^+ < -\alpha$), then the inequalities $\alpha^{(1)} \leqslant \widehat{\gamma}^+ < \widehat{\gamma}^- \leqslant -\alpha^{(1)}$ (resp. $\alpha \leqslant \widehat{\gamma}^+ < \widehat{\gamma}^- \leqslant -\alpha$) hold by part (2).
(5) From part (4), the inverse branch $g_{J'} \circ g_J$ is defined on $\widehat{A}$.
(6) Assume for instance that the order n' of J' is $\geqslant n$. The endpoints of $P_c^n(J')$ are consecutive points of $\Delta_{n'-n}$, hence this interval is either contained in A or disjoint from int A. In the first case, J' is contained in J. In the second case, J' is disjoint from int J.

□

The set of n-regular points is exactly the union of regular intervals with order in $(0, n]$.

2.4. **The Bernoulli map associated to a regular parameter.** Let $c \in [-2, 0)$ be a regular parameter. Let $\mathcal{J}$ be the family of intervals J which are contained in A, regular of positive order, and maximal with these two properties.These intervals have disjoint interiors.
Let

$$W = \bigcup_{\mathcal{J}} \mathrm{int}\, J.$$

As c is regular, we have $\mathrm{Leb}(A - W) = 0$. Define $N : W \to \mathbb{N}$ and $T : W \to A$ by:

$$N(x) = ord(J), \quad T(x) = P_c^{ord(J)}(x), \qquad \text{for } x \in \operatorname{int} J,\ J \in \mathfrak{J}.$$

For every $J \in \mathfrak{J}$, the restriction of T to $\operatorname{int} J$ is a diffeomorphism onto $\operatorname{int} A$, with inverse g_J.

We can identify the disjoint union

$$\mathfrak{J}^{(\infty)} = \bigsqcup_{m \geqslant 0} \mathfrak{J}^m$$

with the family of **all** regular intervals contained in A. Indeed, let

$$\underline{J} = (J_1, \ldots, J_m) \in \mathfrak{J}^m.$$

The composition $g_{J_1} \circ \cdots \circ g_{J_m}$ is the inverse branch associated to a regular interval that we will also denote by $\underline{J}$. For $m = 0$, we have $\underline{J} = A$ and $ord(\underline{J}) = 0$; otherwise, we have $\underline{J} \subset J_1$.

2.5. **Schwarzian derivative.**

Definition 2.6. For a C^3 diffeomorphism f from an interval I onto its image, the *Schwarzian derivative Sf* is defined by

$$\begin{aligned} Sf &= D^2 \log|Df| - \frac{1}{2}(D\log|Df|)^2 \\ &= \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2 \\ &= -2|Df|^{1/2} D^2(|Df|^{-1/2}). \end{aligned}$$

The composition rule for the Schwarzian derivative is:

$$S(f \circ g) = Sf \circ g.(Dg)^2 + Sg.$$

In particular, if Sf and Sg have the same constant sign, the same holds for $S(f \circ g)$. We have also

$$S(f^{-1}) = -Sf \circ f^{-1}(Df^{-1})^2.$$

The Schwarzian derivative Sf vanishes on I iff f is the restriction to I of a Möbius transformation $x \mapsto \frac{ax+b}{cx+d}$, $ad - bc \neq 0$.

If $0 \notin I$, and f has the form $f(x) = a|x|^\alpha + b$ for some $\alpha \neq 0$, $a \neq 0$, then

$$Sf(x) = \frac{1 - \alpha^2}{2}\, x^{-2}.$$

Therefore, for any $c \in \mathbb{R}$, $n > 0$, the Schwarzian derivative $S(P_c^n)$ is negative on any interval on which P_c^n is a diffeomorphism. In particular,

Lemma 2.7. *For any regular interval J, the inverse branch g_J satisfies $Sg_J > 0$ on $\widehat{A}$.*

2.6. **Bounded distortion.**

Let $f : I \to \mathbb{R}$ be a C^3-diffeomorphism onto its image. Assume that $Sf \geqslant 0$ on I. Then we have, for all $x \in I$

$$|D \log |Df|(x)| \geqslant 2\, d(x, \mathbb{R} - I)^{-1}.$$

Taking for f the inverse branch g_J associated to some regular interval J, we obtain

Lemma 2.8. *For any regular interval J, the inverse branch g_J satisfies*

$$|D \log |Dg_J|(x)| \leqslant C_0,$$

with $C_0 = 2|\alpha - \alpha^{(1)}|^{-1}$.

Let $g : J \to J'$ be a C^2 diffeomorphism between bounded intervals. Assume that g satisfies, for some $C \geqslant 0$, any $x \in J$

$$|D \log |Dg|(x)\Big| \leqslant C. \tag{2.1}$$

Then the derivative has *bounded distortion*:

$$\max_J |Dg| \leqslant e^{C|J|} \min_I |Dg|. \tag{2.2}$$

It follows that, for any measurable subset E of J,

$$\frac{\mathrm{Leb}(E)}{|J|} e^{-C|J|} \leqslant \frac{\mathrm{Leb}(g(E))}{|J'|} \leqslant \frac{\mathrm{Leb}(E)}{|J|} e^{C|J|}. \tag{2.3}$$

In our setting, let $\underline{J} = (J_1, \ldots, J_m) \in \mathcal{J}^m$ be a regular interval of positive order contained in A. Let $\underline{J'} = (J_1, \ldots, J_{m-1})$.

Proposition 2.9. *One has*

$$\begin{aligned} 1 - \frac{|J|}{|J'|} &\geqslant e^{-C_0|A|}(1 - \frac{|J_m|}{|A|}), \\ |J| &\leqslant (1-c_1)^m |A|, \\ |DT^m(x)| &\geqslant e^{-C_0|A|}(1-c_1)^{-m}, \end{aligned}$$

for all $x \in J$. In these inequalities, the constant C_0 is the same than in Lemma 2.8 *and*

$$c_1 := e^{-C_0|A|}(1 - \frac{\max_{\mathcal{J}} |J|}{|A|}).$$

Proof. The first inequality is (2.3), with $g = g_{\underline{J'}}$ and $E = A - J_m$. taking into account the definition of c_1, the second inequality follows. As the mean value of DT^m on J is $|A|/|J|$, the last inequality is a consequence of the second one and (2.2). □

2.7. **The transfer operator.** Let μ be a finite measure on A, which is absolutely continuous with respect to the Lebesgue measure on A. We write $d\mu = h(x)dx$, with density $h \in L^1(A)$. The image $T_*\mu$ is still absolutely continuous with respect to the Lebesgue measure; its density $\mathcal{L}h$ is given by the image of h under the *transfer operator*:

$$\mathcal{L}h = \sum_{\mathcal{J}} h \circ g_J |Dg_J|.$$

More generally, for $m \geqslant 0$, the density $\mathcal{L}^m h$ of $T_*^m \mu$ is given by

$$\mathcal{L}^m h = \sum_{\mathcal{J}^m} h \circ g_{\underline{J}} |Dg_{\underline{J}}|.$$

We will actually consider the operator $\mathcal{L}$ on a much smaller subspace.

Let $U = (\mathbb{C} \setminus \mathbb{R}) \cup \widehat{A}$. For any $n > 0$, the critical values of P_c^n are real. Therefore, if J is a regular interval, the associated inverse branch g_J extends to a univalent map, still denoted by g_J, from U to $\mathbb{C}$, which satisfies

$$g_J(\mathbb{C} \setminus \mathbb{R}) \subset \mathbb{C} \setminus \mathbb{R}, \qquad g_J(\widehat{A}) = \widehat{J}.$$

For a regular interval J, we denote by ε_J the sign of Dg_J on $\widehat{A}$ and define $\bar{g}_J := \varepsilon_J \frac{|A|}{|J|}(g_J - g_J(0))$. The action of $\mathcal{L}$ on holomorphic functions on U is defined by

$$\mathcal{L}\varphi := \sum_{\mathcal{J}} \varphi \circ g_J . \varepsilon_J \, Dg_J.$$

The family $(\bar{g}_J)_{J\in\mathcal{J}^{(\infty)}}$ is a normal family on U: it is formed by univalent functions vanishing and with uniformly bounded derivative at 0 (from the bounded distortion property).

Let m be a nonnegative integer. As $\sum_{\mathcal{J}^m} |\underline{J}| = |A|$, the series

$$h_m := \mathcal{L}^m\mathbf{1} = \sum_{\mathcal{J}^m} \varepsilon_{\underline{J}} Dg_{\underline{J}}$$

defines a holomorphic function h_m on U. From (2.2), it satisfies

$$e^{-C_0|A|} \leqslant h_m(x) \leqslant e^{C_0|A|}\ ,\ \forall x \in A.$$

The family $(h_m)_{m\geqslant 0}$, lying in the closed convex hull of a normal family, is still normal. Therefore, we can extract from the sequence

$$h^{(m)} = \frac{1}{m}\sum_{0}^{m-1} h_k, \qquad m > 0$$

a subsequence converging uniformly on compact subsets of U to some limit h_T. The function h_T is holomorphic on U and still satisfies

$$e^{-C_0|A|} \leqslant h_T(x) \leqslant e^{C_0|A|}\ , \qquad \forall x \in A. \tag{2.4}$$

From the relation

$$\mathcal{L}h^{(m)} = h^{(m)} + \frac{1}{m}(h_m - 1),$$

one obtains at the limit

$$h_T(z) = \sum_{\mathcal{J}} h_T \circ g_J(z).\, \varepsilon_J\, Dg_J(z), \qquad \forall z \in U. \tag{2.5}$$

2.8. **Absolutely continuous invariant measure for T.** Considering only the restriction of h_T to A, define on A the measure $d\mu_T := h_T(x)dx$.

Proposition 2.10. *The positive measure μ_T is invariant under T, ergodic, equivalent to the Lebesgue measure on A, and has total mass $|A|$.*

Proof. The transfer operator preserves positivity and the L^1-norm of positive functions, hence μ_T has total mass $|A|$. The other statements are immediate except for the ergodicity of μ_T which is proved by the following standard argument.

Let E be a T-invariant measurable subset of positive measure of A. Let $x_0 \in E = \cap_{m\geqslant 0} T^{-m}(E)$ be a point of density of E. Let $\varepsilon > 0$; as m increases to ∞, the length of the interval $J^{(m)} \in \mathcal{J}^m$ which contains x_0 goes to 0 (Proposition 2.9). As x_0 is a point of density of E, we have, for m large enough

$$\mathrm{Leb}(J^{(m)} \cap E) \geqslant (1-\varepsilon)\ |J^{(m)}|.$$

Applying (2.3) to $g_{J^{(m)}}$ gives

$$\mathrm{Leb}(E) \geqslant (1 - \varepsilon\ e^{C_0|A|})\ |A|.$$

As ε is arbitrary, we have proved that $\mu_T(A - E) = 0$.

□

Recall that T is related to P_c by:

$$T(x) = P_c^{N(x)}(x), \qquad \text{for } x \in W.$$

Lemma 2.11. *The functions N and $\log|DT|$ belong to $L^p(\mu_T)$ for all $1 \leqslant p < \infty$.*

Proof. Both functions are defined on W. As the parameter c is regular, W has full Lebesgue measure in A and there exist $\theta, C > 0$ such that, for all $n > 0$

$$\mathrm{Leb}\{x \in W,\ N(x) > n) \leqslant Ce^{-\theta n}.$$

Therefore the function N belongs to $L^p(\mu_T)$ for all $1 \leqslant p < \infty$.

The function $\log|DT|$ is bounded from below by Proposition 2.9. For $c \in [-2, 0)$, we have $\beta \leqslant 2$ and $|DP_c| \leqslant 4$ on $[-\beta, \beta]$. This implies $\log|DT(x)| \leqslant N(x)\log 4$ for $x \in W$. Therefore $\log|DT|$ belongs to $L^p(\mu_T)$ for all $1 \leqslant p < \infty$.

□

Define

$$N_T = \frac{1}{|A|}\int N d\mu_T, \quad \lambda_T = \frac{1}{|A|}\int \log|DT| d\mu_T, \quad \lambda_P = \frac{\lambda_T}{N_T}.$$

The Birkhoff sum

$$N_k(x) = \sum_{j=0}^{k-1} N(T^j(x)), \qquad \text{for } x \in \bigcap_{j<k} T^{-j}(W)\ .$$

is related to the iteration of T through

$$T^k(x) = P_c^{N_k(x)}(x)\ .$$

Proposition 2.12. *One has $N_T \geqslant 2$, $\lambda_T \geqslant 2\lambda_P > 0$ and, for Lebesgue almost every $x \in A$*

$$\begin{aligned}
\lim_{k\to+\infty} \frac{1}{k} N_k(x) &= N_T, \\
\lim_{k\to+\infty} \frac{1}{k} \log|DT^k(x)| &= \lambda_T, \\
\lim_{k\to+\infty} \frac{1}{n} \log|DP_c^n(x)| &= \lambda_P.
\end{aligned}$$

Proof. The return time to A is everywhere $\geqslant 2$ hence $N_T \geqslant 2$. The inequality $\lambda_T \geqslant \log(1-c_1)^{-1} > 0$ is a consequence of Proposition 2.9. As μ_T is ergodic and equivalent to the Lebesgue measure on A, the assertions about N_k and $\log|DT^k|$ follow from Birkhoff's ergodic theorem. They imply, for almost all $x \in A$

$$\lim_{k\to+\infty} \frac{1}{N_k(x)} \log|DP_c^{N_k(x)}(x)| = \lambda_P.$$

As $|DP_c| \leqslant 4$ on $[-\beta, \beta]$, the intermediate derivatives are controlled by

$$\log|DP_c^{N_{k+1}}| - (N_{k+1} - n)\log 4 \leqslant \log|DP_c^n| \leqslant \log|DP_c^{N_k}| + (n - N_k)\log 4$$

for $N_k \leqslant n \leqslant N_{k+1}$. As

$$\lim_{k\to+\infty} \frac{N_{k+1}(x)}{N_k(x)} = 1$$

holds for almost all $x \in A$, the last assertion of the proposition is proven.

□

2.9. **Absolutely continuous invariant measure for** P**.** From the T-invariant measure μ_T on A, we construct a P_c-invariant measure μ_P on $[-\beta,+\beta]$.

Let φ be a continuous function on $[-\beta,+\beta]$. For $x \in W$, define

$$S\varphi(x) = \sum_{0 \leqslant j < N(x)} \varphi(P_c^j(x)) \,.$$

As $|S\varphi(x)| \leqslant \|\varphi\|_\infty \; N(x)$, the function $S\varphi$ belongs $L^p(\mu_T)$ for $1 \leqslant p < \infty$. The formula

$$\int \varphi \; d\mu_P = \int S\varphi \; d\mu_T$$

defines a positive linear form on $C^0([-\beta,+\beta])$, hence a finite positive measure μ_P on $[-\beta,+\beta]$.

Proposition 2.13. *The positive measure μ_P has total mass $N_T|A|$. Its support is the interval $[P_c(0), P_c^2(0)]$. It is invariant under P_c and ergodic. It is equivalent to the Lebesgue measure on $[P_c(0), P_c^2(0)]$. The density h_P is given by*

$$h_P(x) = \sum_{J \in \mathfrak{J}} \sum_{0 \leqslant n < ord(J)} \mathbf{1}_{P_c^n(J)} \; h_T \circ (P_c^n|_J)^{-1} \; |D((P_c^n|_J)^{-1})|.$$

The density is bounded from below on $[P_c(0), P_c^2(0)]$, integrable but not square-integrable.

Proof. The assertion on the total mass follows immediately from the definition.

As the parameter c is regular, we have $P_c(0) < \alpha^{(1)}$ (otherwise there is no regular interval of positive order contained in A) and $P_c^2(0) > -\alpha$, hence

$$\bigcup_{n \geqslant 0} P_c^n(A) = [P_c(0), P_c^2(0)] = P_c(A) \cup P_c^2(A) \,.$$

It follows that the support of μ_P is the interval $[P_c(0), P_c^2(0)]$.

From

$$S(\varphi \circ P_c) = S\varphi + \varphi \circ T - \varphi \,;$$

and the invariance of μ_T under T, one deduces the invariance of μ_P under P_c.

The measure μ_P is ergodic: if φ is a measurable function invariant under P_c, its restriction to A is T-invariant, hence almost everywhere constant on A, and then φ is almost everywhere constant on $[P_c(0), P_c^2(0)]$.

The formula defining μ_P shows that it is absolutely continuous with respect to the Lebesgue measure, the density h_P being given by the formula of the proposition:

$$h_P(x) = \sum_{J \in \mathfrak{J}} \sum_{0 \leqslant n < \mathrm{ord}(J)} \mathbf{1}_{P_c^n(J)} \; h_T \circ (P_c^n|_J)^{-1} \; |D((P_c^n|_J)^{-1})|.$$

The density is integrable as μ_P has finite mass.

Considering successively the terms with $n = 0,1,2$ in the sum defining h_P, we obtain from (2.4) in subsection 2.7

$$\begin{aligned}
h_P(x) &\geqslant \quad e^{-C_0|A|} && \text{a.e. on } A \,, \\
h_P(x) &\geqslant \tfrac{1}{2} e^{-C_0|A|}(x - P_c(0))^{-1/2} && \text{a.e. on } [P_c(0), \alpha] \,, \\
h_P(x) &\geqslant \tfrac{1}{8} e^{-C_0|A|}(P_c^2(0) - x)^{-1/2} && \text{a.e. on } [-\alpha, P_c^2(0)].
\end{aligned}$$

Thus the density h_P is bounded from below and not square-integrable. $\square$

Remark 2.14. For $c=-2$, we have

$$h_P(x)=\frac{6}{\pi}(4-x^2)^{-1/2}, \qquad h_T=h_P|_{[-1,1]}.$$

3. Strongly regular parameters

3.1. **The sequence $c^{(m)}$.** For $c\in[-2,0)$, we define (cf. subsection 2.2) a sequence $\alpha^{(m)}=\alpha^{(m)}(c)$ of preimages of the fixed point α of P_c by $\alpha^{(0)}:=\alpha$ and

$$P_c(\pm\alpha^{(m)})=-\alpha^{(m-1)}, \qquad \alpha^{(m)}<0,$$

for $m>0$.

The sequence $(\alpha^{(m)})_{m\geqslant 0}$ is decreasing and converges to $-\beta$.

Proposition 3.1.
(1) *For $c\in[-2,-\frac{3}{2}]$, the preimage $\alpha^{(m)}(c)$ belongs to $[-2,-\frac{3}{2}]$ for $m>0$ and satisfies $1/3\leqslant\partial\alpha^{(m)}/\partial c\leqslant 1/2$ for $m\geqslant 0$.*
(2) *For $m>1$, the equation $P_c(0)=c=\alpha^{(m-1)}(c)$ has a unique root $c^{(m)}$ in $[-2,-\frac{3}{2}]$. This root is simple. The sequence $(c^{(m)})_{m>1}$ is decreasing. For $c\in(c^{(m+1)},c^{(m)})$, the critical value $P_c(0)$ belongs to the interval $(\alpha^{(m)}(c),\ \alpha^{(m-1)}(c))$.*
(3) *The sequence $(c^{(m)})_{m>1}$ converges to -2. More precisely, one has, for some constant $C>0$ and all $m>1$*

$$C^{-1}4^{-m}\leqslant c^{(m)}+2\leqslant C\,4^{-m}.$$

Proof. (1) One has $\alpha^{(m)}(c)\geqslant-\beta\geqslant-2$ for all $m\geqslant 0$, $c\in[-2,0)$.

The fixed point $\alpha=\alpha^{(0)}$ satisfies $\partial\alpha/\partial c=(1-4c)^{-1/2}\in[\frac{1}{3},\frac{1}{2}]$ for $c\in[-2,-\frac{3}{2}]$.

From the inductive definition of $\alpha^{(m)}(c)$, one obtains the recurrence relation

$$\partial\alpha^{(m)}/\partial c=-\frac{1}{2\alpha^{(m)}}\ (1+\partial\alpha^{(m-1)}/\partial c). \tag{3.1}$$

As $\alpha^{(m)}\in[-2,0)$, we obtain inductively that $\partial\alpha^{(m)}/\partial c\geqslant 1/3$ for all $m\geqslant 0$.

Therefore, for all $m>0$, $c\in[-2,-\frac{3}{2}]$, the inequalities $\alpha^{(m)}(c)\leqslant\alpha^{(1)}(c)\leqslant\alpha^{(1)}(-\frac{3}{2})$ hold. For $c=-3/2$, the preimage $\alpha^{(1)}$ satisfies

$$(\alpha^{(1)})^2=\frac{3}{2}-\alpha=1+\frac{1}{2}\sqrt{7}>\frac{9}{4}.$$

This proves that $\alpha^{(m)}(c)\leqslant-3/2$ for all $m>0$, $c\in[-2,-\frac{3}{2}]$. This inequality in turn allows to see inductively from (3.1) that $\partial\alpha^{(m)}/\partial c\leqslant 1/2$ for all $m\geqslant 0$.

(2) The function $c\mapsto P_c(0)-\alpha^{(1)}(c)$ takes a negative value at $c=-2$, a positive value at $c=-3/2$ and its derivative on $[-2,-\frac{3}{2}]$ belongs to $[\frac{1}{2},\frac{2}{3}]$. Therefore it has a unique zero $c^{(2)}$ in $[-2,-\frac{3}{2}]$. This zero is simple. The critical value $P_c(0)$ belongs to $[-2,\alpha^{(1)}(c)]$ for $c\in[-2,c^{(2)}]$.

For $m>2$, the induction step is similar. Consider the function $c\mapsto P_c(0)-\alpha^{(m-1)}(c)$ on the interval $[-2,-\frac{3}{2}]$. Its derivative belongs to $[\frac{1}{2},\frac{2}{3}]$. Its value at -2 is negative. Its value at $c^{(m-1)}$ is $\alpha^{(m-2)}(c^{(m-1)})-\alpha^{(m-1)}(c^{(m-1)})>0$. Therefore it has a unique zero $c^{(m)}$ in $[-2,-\frac{3}{2}]$, which belongs to $(-2,c^{(m-1)})$ and is simple. The critical value $P_c(0)$ belongs to $[-2,\alpha^{(m-1)}(c)]$ for $c\in[-2,c^{(m)}]$.

(3) The function $c \mapsto P_c(0) - \alpha^{(m-1)}(c)$ takes the value $-2 - \alpha^{(m-1)}(-2) = -4\sin^2 \frac{\pi}{3.2^m}$ at $c = -2$ (cf. subsection 2.2). It vanishes at $c^{(m)}$ and its derivative in between belongs to $[\frac{1}{2}, \frac{2}{3}]$. This implies the estimate of the proposition.

□

3.2. **Simple regular intervals.** Let M be a **large** integer. In the rest of the paper, we let the parameter c vary in the interval $(c^{(M)}, c^{(M-1)})$. The first points of the postcritical orbit satisfy

$$\begin{aligned} \alpha^{(M-1)} < \quad & P_c(0) \quad < \alpha^{(M-2)} \\ -\alpha^{(M-n-1)} < \quad & P_c^n(0) \quad < -\alpha^{(M-n)}, \qquad \text{for } 1 < n < M, \\ \alpha < \quad & P_c^M(0) \quad < -\alpha. \end{aligned}$$

Thus M is the first return time of 0 in A.

We define for $1 \leqslant n < M$ preimages $\widetilde{\alpha}^{(n)} \in \Delta_n$ by the conditions

$$P_c(\pm\,\widetilde{\alpha}^{(n)}) = \alpha^{(n-1)}, \qquad \widetilde{\alpha}^{(n)} < 0.$$

The finite sequence defined by these conditions verifies $\widetilde{\alpha}^{(1)} = \alpha$ and $\widetilde{\alpha}^{(n)} < \widetilde{\alpha}^{(n+1)}$ for $1 \leqslant n < M - 1)$.

We also define, for $1 < n < M$:

$$\begin{aligned} C_n^+ &= [\widetilde{\alpha}^{(n-1)}, \widetilde{\alpha}^{(n)}], \\ C_n^- &= [-\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n-1)}]. \end{aligned}$$

Proposition 3.2. *For any $c \in (c^{(M)}, c^{(M-1)})$, the intervals $[\alpha^{(n)}, \alpha^{(n-1)}]$, $[-\alpha^{(n-1)}, -\alpha^{(n)}]$ are regular of order n for any $n > 0$; the intervals C_n^+, C_n^- are regular of order n for $1 < n < M - 1$.*

Proof. The restriction of P_c^n to $[\alpha^{(n)}, \alpha^{(n-1)}]$ is a diffeomorphism onto A. Let θ_n be the preimage of 0 by this diffeomorphism, and define also $\theta_0 = 0$. The restriction of P_c^n to the neighborhood $(-\beta, \theta_{n-1})$ of $[\alpha^{(n)}, \alpha^{(n-1)}]$ is a diffeomorphism onto $(P_c(0), \beta)$, which contains $\widehat{A}$ for $M > 2$. This shows that $[\alpha^{(n)}, \alpha^{(n-1)}]$ is regular of order n. By symmetry, $[-\alpha^{(n-1)}, -\alpha^{(n)}]$ is also regular of order n.

Consider now $C_n^{\pm}$ for $1 < n < M - 1$. The restriction of P_c^{n-1} to the neighborhood $(\alpha^{(n)}, \theta_{n-2})$ of $[\alpha^{(n-1)}, \alpha^{(n-2)}] = P_c(C_n^{\pm})$ is a diffeomorphism onto $(P_c(0), -\alpha^{(1)})$. As $n < M - 1$, the critical value is $\leqslant \alpha^{(n)}$. The two components of $P_c^{-1}(\alpha^{(n)}, \theta_{n-2})$ are neighborhoods of $C_n^{\pm}$ and are sent diffeomorphically by P_c^n onto $(P_c(0), -\alpha^{(1)}) \supset \widehat{A}$. Therefore C_n^+ and C_n^- are regular of order n.

□

Definition 3.3. The intervals $C_n^{\pm}$, for $1 < n < M - 1$ are called the *simple* regular intervals.

In the rest of the paper, we use the letter C to denote various positive constants **independent** of M. The dependance on M in the estimates will always be explicit.

Proposition 3.4. *For $c \in (c^{(M)}, c^{(M-1)})$, the following estimates hold:*

$$\begin{aligned} C^{-1}4^{-M} \leqslant \quad & \beta(c) + P_c(0) \quad \leqslant C4^{-M}, \\ C^{-1}4^{-n} \leqslant \quad & \alpha^{(n)}(c) - P_c(0) \quad \leqslant C4^{-n}, \qquad \textit{for } 0 \leqslant n < M - 2, \\ C^{-1}2^{-n} \leqslant \quad & |\widetilde{\alpha}^{(n)}(c)| \quad \leqslant C2^{-n}, \qquad \textit{for } 0 < n < M - 1. \end{aligned}$$

Proof. One has $\partial\beta/\partial c \in [-\frac{1}{2}, -\frac{1}{3}]$ for $c \in [-2, -\frac{3}{4}]$, hence the derivative of the function $c \mapsto \beta(c) + P_c(0)$ belongs to $[\frac{1}{2}, \frac{2}{3}]$. It vanishes at $c = -2$. Thus the first estimate is a consequence of Proposition 3.1, part (3).

For the bound from below in the second estimate, we have

$$\alpha^{(n)}(c) - P_c(0) \geqslant \alpha^{(n)}(c) - \alpha^{(n+1)}(c) \geqslant 4^{-n-1}|A|$$

as $P_c^{n+1}([\alpha^{(n+1)}(c), \alpha^{(n)}(c)]) = A$ and $|DP_c| \leqslant 4$ on $[-\beta, \beta]$. For the bound from above, the function $c \mapsto \alpha^{(n)}(c) - P_c(0)$ has a negative derivative and its value at -2 is $4\sin^2 \frac{\pi}{3.2^{n+1}}$ (cf. subsection 2.2).

Finally, the third estimate is an immediate consequence of the second.

□

For $c \in (c^{(M)}, c^{(M-1)})$, $x \in [-\beta(c), \beta(c)]$ we define (cf. subsection2.2)

$$h_c(x) := (\beta^2 - x^2)^{-1/2}.$$

Proposition 3.5. (1) *The derivative of P_c satisfies*

$$|DP_c(x)| = 2\frac{h_c(x)}{h_c(P_c(x))}\left(1 + \frac{\beta + c}{x^2}\right)^{-1/2}.$$

(2) *For $n > 0$, $x \in [\alpha^{(n)}, \alpha^{(n-1)}]$*

$$\left|\log|DP_c^n(x)| - \log 2^n \frac{h_c(x)}{h_c(P_c^n(x))}\right| \leqslant C\, n\, 4^{-M}.$$

(3) *For $1 < n < M-1$, $x \in C_n^{\pm}$*

$$\left|\log|DP_c^n(x)| - \log 2^n \frac{h_c(x)}{h_c(P_c^n(x))}\right| \leqslant C 4^{n-M}.$$

Proof. (1) The formula for $|DP_c(x)|$ is a simple calculation using that β is a fixed point. Observe that $\beta + c \asymp 4^{-M}$ from the previous proposition.

(2) In view of the chain rule for derivatives , we have to control $\log(1+(\beta+c)(P_c^m(x))^{-2})$ for $0 \leqslant m < n$. As $P_c^m(x)$ is bounded away from 0 under our hypotheses, we obtain the estimate of the proposition.

(3) Here also we have to control $\log(1 + (\beta + c)(P_c^m(x))^{-2})$ for $0 \leqslant m < n$. For $1 < n < M-1$, $x \in C_n^{\pm}$, $P_c^m(x)$ is again bounded away from 0 except for $m = 0$. In this case, the last estimate in Proposition 3.4 gives the required result.

□

3.3. **Strongly regular parameters.** We define a new condition on the parameter $c \in (c^{(M)}, c^{(M-1)})$, called *strong regularity*. It will later be shown that strongly regular parameters are regular, and that the relative measure of strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ goes to 1 as M goes to $+\infty$.

For any parameter $c \in (c^{(M)}, c^{(M-1)})$, regular or not, we may still denote, as in subsection 2.4, by:

- $\mathfrak{J}$ the family of regular intervals, of positive order, contained in A, and maximal with these properties;
- W the union $\bigsqcup_{\mathfrak{J}} \operatorname{int} J$;
- $N : W \to \mathbb{N}_{>1}$ the function equal to the order of J in $\operatorname{int} J$, for all $J \in \mathfrak{J}$;
- $T : W \to A$ the map whose restriction to $J \in \mathfrak{J}$ is equal to $P_c^{\operatorname{ord}(J)}$.

Lemma 3.6. *For every $n \in [2, M-2]$, $C_n^{\pm}$ are the two elements of $\mathfrak{J}$ of order n. Any other element of $\mathfrak{J}$ has order $> M$.*

Proof. Let J be an element of $\mathfrak{J}$ distinct from the $C_n^{\pm}$, $2 \leqslant n \leqslant M-2$. Its interior is disjoint from the $C_n^{\pm}$, hence it is contained in $(\widetilde{\alpha}^{(M-2)}, -\widetilde{\alpha}^{(M-2)})$. For points in $(\widetilde{\alpha}^{(M-2)}, -\widetilde{\alpha}^{(M-2)})$, the return time to A is $(M-1)$ (for $C_{M-1}^{\pm}$) or M (for $(\widetilde{\alpha}^{(M)}, -\widetilde{\alpha}^{(M)})$). The order of J is therefore at least $(M-1)$.

If the order of J was $M-1$, J should be equal to C_{M-1}^{+} or C_{M-1}^{-}. But $P_c(C_{M-1}^{\pm})$ is equal to the regular interval $J = [\alpha^{(M-2)}, \alpha^{(M-3)}]$. The corresponding inverse branch satisfies $g_J(-\alpha^{(1)}) = \alpha^{(M-1)} < P_c(0)$. Therefore C_{M-1}^{+} and C_{M-1}^{-} are not regular.

If the order of J was M, J should be contained in $[\widetilde{\alpha}^{(M)}, -\widetilde{\alpha}^{(M)}]$, because the minimal return time to $\operatorname{int} A$ is 2. But the image $P_c^M([\widetilde{\alpha}^{(M)}, -\widetilde{\alpha}^{(M)}])$ is equal to $[\alpha, T(0)]$ which is strictly contained in A.

We conclude that the order of J is $> M$. $\square$

The first return time of the critical point in A is equal to M. Although 0 is obviously not contained in any regular interval of positive order, we still define

$$N(0) := M, \qquad T(0) := P_c^M(0).$$

Assume that $T(0) \in \bigcap_{0 \leqslant k < K} T^{-k}(W)$ for some integer $K > 0$. This allows to define $T^{k+1}(0) := T^k(T(0))$ for $0 \leqslant k \leqslant K$. For $0 < k \leqslant K$, we denote by $J(k)$ the element of $\mathfrak{J}$ such that $T^k(0) \in \operatorname{int} J(k)$. For $0 < k \leqslant K+1$, we denote by N_k the Birkhoff sum

$$N_k = \sum_{0}^{k-1} N(T^\ell(0)).$$

We thus have $T^k(0) = P_c^{N_k}(0)$ for $0 < k \leqslant K+1$.

We can now formulate the main definition of this paper.

Definition 3.7. A parameter $c \in (c^{(M)}, c^{(M-1)})$ is *strongly regular* if $T(0) \in \bigcap_{k \geqslant 0} T^{-k}(W)$ and the sequence $(N(T^k(0)))_{k>0}$ satisfies, for all $k \geqslant 1$

$$\sum_{\substack{0<\ell\leqslant k \\ N(T^\ell(0))>M}} N(T^\ell(0)) \leqslant 2^{-\sqrt{M}} \sum_{0<\ell\leqslant k} N(T^\ell(0)). \tag{3.2}$$

In vague words, the postcritical orbit is a concatenation of regular returns to A, and the non simple regular returns fill only a very small proportion of total time.

We also need this definition for a finite initial piece of the postcritical orbit.

Definition 3.8. A parameter $c \in (c^{(M)}, c^{(M-1)})$ is *strongly regular up to level K* if $T(0) \in \bigcap_{0 \leqslant k < K} T^{-k}(W)$ and inequality (3.2) holds for $1 < k \leqslant K$.

Remark 3.9. Let c be a strongly regular parameter. Then $N(T^k(0)) > M$ holds iff $T^k(0)$ belongs to the central interval $(\widetilde{\alpha}^{(M-2)}, -\widetilde{\alpha}^{(M-2)})$ of size $\asymp 2^{-M}$, and we have in this case from (3.2) that

$$N(T^k(0)) \leqslant \frac{2^{-\sqrt{M}}}{1 - 2^{-\sqrt{M}}}\,(N_k - M).$$

In particular, one must have $N_k \geqslant 2^{\sqrt{M}}\,M$. All returns to A of the postcritical orbit are simple up to time $2^{\sqrt{M}}\,M$.

When $T(0) \in \bigcap_{0 \leqslant k < K} T^{-k}(W)$, we construct for $0 < k \leqslant K+1$ a decreasing sequence $B(k)$ of regular intervals containing the critical value. We define $B(1) := [\alpha^{(M-1)}, \alpha^{(M-2)}]$, which is regular of order $(M-1)$ by Proposition 3.2. For $1 \leqslant k \leqslant K$, $B(k+1)$ is defined inductively by its inverse branch:

$$g_{B(k+1)} = g_{B(k)} \circ g_{J(k)}.$$

By Proposition 2.5, part (5), $B(k+1)$ is regular of order $N_{k+1} - 1$.

3.4. **Derivatives along the postcritical orbit.** Strong regularity implies a strong form of the Collet-Eckmann condition.

Proposition 3.10. *Assume that $c \in (c^{(M)}, c^{(M-1)})$ is strongly regular up to level K. Then we have, for $1 \leqslant k \leqslant K+1$ and all $x \in A$*

$$|\log(|Dg_{B(k)}(x)| \, \frac{h_c(g_{B(k)}(x))}{h_c(x)}) + (N_k - 1)\log 2| \leqslant C\, M^{-1}\, N_k.$$

Proof. Proposition 3.5, part (2), provides the initial step of the induction: for $x \in A$

$$|\log(|Dg_{B(1)}(x)| \, \frac{h_c(g_{B(1)}(x))}{h_c(x)}) + (M-1)\log 2| \leqslant C\, M\, 4^{-M}. \tag{3.3}$$

The same proposition, part (3), provides a control of the simple regular returns: when $J(k)$ is simple, for $x \in A$

$$|\log(|Dg_{J(k)}(x)| \, \frac{h_c(g_{J(k)}(x))}{h_c(x)}) + \mathrm{ord}(J(k))\log 2| \leqslant C\, 4^{\mathrm{ord}(J(k)) - M}. \tag{3.4}$$

We need a similar control (with a worst error term!) for non simple regular returns.

Lemma 3.11. *For all $J \in \mathcal{J}$, $x \in A$, one has*

$$|\log(|Dg_J(x)| \, \frac{h_c(g_J(x))}{h_c(x)}) + \mathrm{ord}(J)\log 2| \leqslant \mathrm{ord}(J)\log 2 + C. \tag{3.5}$$

Proof. First observe that, as $x,\ g_J(x)$ belong to A, the term $\log \frac{h_c(g_J(x))}{h_c(x)}$ is bounded. The derivative Dg_J satisfies $|Dg_J| \geqslant 4^{-\mathrm{ord}(J)}$ because $|DP_c| \leqslant 4$ on $[-\beta, \beta]$, and also $|Dg_J| \leqslant C$ from general univalent function theory and $g_J(A) \subset A$. This gives the estimate of the lemma. □

Applying the chain rule to $g_{B(k)}$ now gives, for $1 \leqslant k \leqslant K+1$, $x \in A$

$$|\log(|Dg_{B(k)}(x)| \, \frac{h_c(g_{B(k)}(x))}{h_c(x)}) + (N_k - 1)\log 2| \leqslant R_0 + R_s + R_{ns}.$$

Here, the error term R_0 is associated to $B(1)$ and equal to $C\, M\, 4^{-M}$ from (3.3). The error term R_s is associated to the simple $J(\ell)$, $0 \leqslant \ell < k$. From (3.4), we obtain

$$\begin{aligned} R_s &\leqslant C 4^{-M} \sum_{n=2}^{M-2} 4^n \#\{\ell, \mathrm{ord}(J(\ell)) = n\} \\ &\leqslant C 4^{-M} \sum_{n=2}^{M-2} 4^n \frac{N_k}{n} \\ &\leqslant C\, M^{-1}\, N_k. \end{aligned}$$

Finally, the error term R_s is associated to the non simple $J(\ell)$, $0 \leqslant \ell < k$. From (3.5) and the strong regularity condition, we have

$$\begin{aligned} R_{ns} &\leqslant \sum_{\substack{0<\ell<k \\ N(T^\ell(0))>M}} (C + N(T^\ell(0))) \\ &\leqslant (1 + C\,M^{-1})\,2^{-\sqrt{M}}\,(N_k - M). \end{aligned}$$

We have indeed $R_0 + R_s + R_{ns} \leqslant C\,M^{-1} N_k$. □

4. STRONGLY REGULAR PARAMETERS ARE REGULAR

This paragraph is the central part of the proof. It is devoted to the key measure estimate needed to prove that strongly regular parameters are regular.

4.1. **Intermediate neighborhoods for regular intervals.** To every regular interval J of order n is associated a neighborhood $\widehat{J}$ which is sent onto $\widehat{A} = (\alpha^{(1)}, -\alpha^{(1)})$ by P_c^n. For technical reasons, we will need some combinatorially defined intermediate neighborhood $\breve{J}$ with $J \Subset \breve{J} \Subset \widehat{J}$.

For n odd (resp. even), let $\widehat{\alpha}^{(n)}$ be the element of Δ_n which is immediately to the left (resp. to the right) of α. One has

$$\widehat{\alpha}^{(0)} = -\alpha\ ,\ \widehat{\alpha}^{(1)} = \alpha^{(1)}\ ,\ \widehat{\alpha}^{(2)} = \widetilde{\alpha}^{(2)}\ ,\ P_c(\widehat{\alpha}^{(n+1)}) = \widehat{\alpha}^{(n)} \qquad \text{for } n \geqslant 0\ .$$

We define $\breve{A} := (\widehat{\alpha}^{(3)}, -\widehat{\alpha}^{(3)})$. It satisfies $A \Subset \breve{A} \Subset \widehat{A}$. For a regular interval J of order n, we denote by $\breve{J}$ the image $g_J(\breve{A})$. The left endpoint of $\breve{J}$ is the point in Δ_{n+3} immediately to the left of the left endpoint of J. Similarly for the right endpoints.

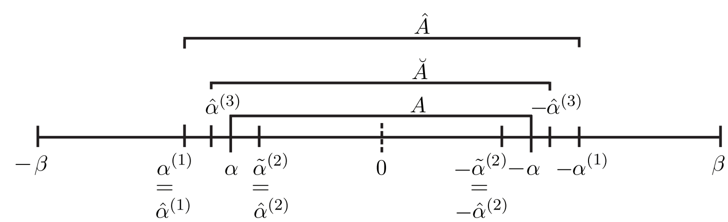

FIGURE 2

4.2. **Singular intervals.** Let $c \in (c^{(M)}, c^{(M-1)})$, and let $n > 1$. Consider an interval $J \subset A$ which have for endpoints two consecutive points of Δ_n. By Proposition 2.5, if $\mathrm{int} J$ intersects some regular interval of positive order $m \leqslant n$, J is contained in such an interval.

Definition 4.1. Let $n > 1$. An interval $J \subset A$ bounded by two consecutive points of Δ_n is n-*singular* if it is not contained in a regular interval of positive order $\leqslant n$. The set of n-singular intervals is denoted by $\mathcal{E}(n)$.

The complement in A of the set of n-regular points, whose measure we need to control in order to prove that c is regular, is equal to the (disjoint) union of the n-singular intervals.

For $2 \leqslant n \leqslant M-2$, $[\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n)}]$ is the only n-singular interval. For $n = M-1$ or $n = M$, there are three n-singular intervals which are C^+_{M-1}, C^-_{M-1} and $[\widetilde{\alpha}^{(M-1)}, -\widetilde{\alpha}^{(M-1)}]$.

To analyze inductively the set $\mathcal{E}(n)$, we introduce two sequences of combinatorially defined neighborhoods of the critical point 0. For $k \geqslant 1$, define

$$A(k) := P_c^{-1}(B(k)), \qquad \breve{A}(k) := P_c^{-1}(\breve{B}(k)).$$

The interval $A(k)$ belongs to $\mathcal{E}(N_k)$: it is equal to the connected component of $A \setminus \Delta_{N_k}$ which contains 0. The left (resp. right) endpoint of $\breve{A}(k)$ is the point immediately to the left (resp. to the right) of the left (resp. right) endpoint of $A(k)$ in Δ_{N_k+3}.

Let $n \geqslant M+3$. We assume that the parameter c is strongly regular up to a level K such that $N_K + 3 \leqslant n < N_{K+1} + 3$. We divide the set $\mathcal{E}(n)$ into several subclasses.

Definition 4.2. An interval $J \in \mathcal{E}(n)$ is *central* if J is contained in $\breve{A}(K)$, *peripheral* if J and $\breve{A}(1)$ have disjoint interiors, *lateral* otherwise.

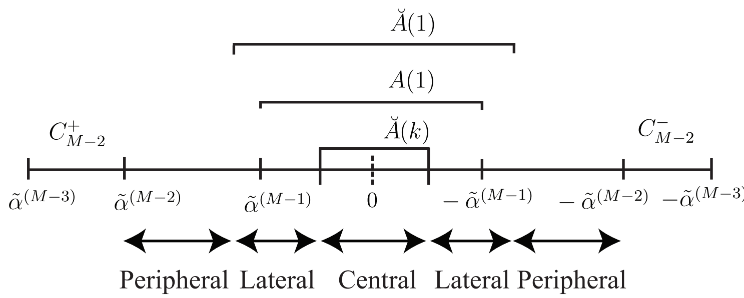

FIGURE 3

We will relate peripheral and lateral elements of $\mathcal{E}(n)$ to singular intervals of lower order, providing the basis of an inductive argument in the end of this section. In the next proposition, we denote by B_0 the regular interval (of order $(M-2)$) $[\alpha^{(M-2)}, \alpha^{(M-3)}]$.

Proposition 4.3. *Let $J \in \mathcal{E}(n)$ be a peripheral interval. Then $J^+ := P_c(J)$ is either of the form $J^+ = g_{B_0}(J^*)$, for some $J^* \in \mathcal{E}(n-M+1)$, or of the form $J^+ = g_{B_0} \circ g_{C_2^-}(J^*)$, for some $J^* \in \mathcal{E}(n-M-1)$.*

Proof. The interval J is contained in $[\widetilde{\alpha}^{(M-2)}, -\widetilde{\alpha}^{(M-2)}]$, but disjoint from $[\widetilde{\alpha}^{(M-1)}, -\widetilde{\alpha}^{(M-1)}] = A(1)$, hence it is contained in C^+_{M-1} or C^-_{M-1}. Its image J^+ is therefore contained in B_0. The image $P_c^{M-2}(J^+)$ is bounded by two consecutive points of Δ_{n-M+1} in A. If $P_c^{M-2}(J^+)$ is $(n-M+1)$-singular, we take $J^* := P_c^{M-2}(J^+)$ and have $J^+ = g_{B_0}(J^*)$.

In the other case, $P_c^{M-2}(J^+)$ is contained in some $J_0 \in \mathcal{J}$. The only possibility is in fact $J_0 = C_2^-$: otherwise the points $P_c^j(0)$, $0 \leqslant j \leqslant M-1$ do not belong to $\widehat{J}_0$, and the component of $P_c^{-1}(g_{B_0}(J_0))$ containing J would be a regular interval containing J. Recalling that J does bot intersect $\mathrm{int}\breve{A}(1)$, $P_c^{M-2}(J^+)$ is in fact contained in $[-\widetilde{\alpha}^{(2)}, -\widehat{\alpha}^{(4)}]$, hence $J^* := P_c^M(J^+)$ is contained in $[\widetilde{\alpha}^{(2)}, -\alpha]$.

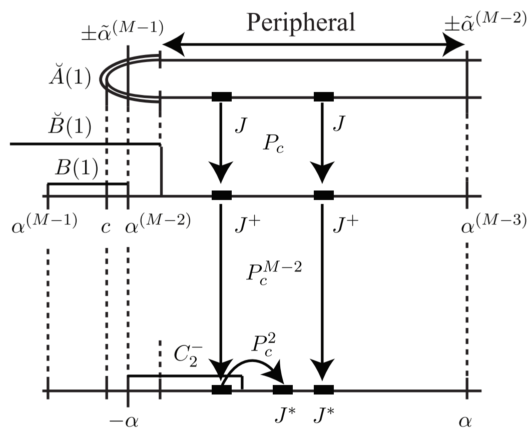

FIGURE 4

We claim that J^* is $(n-M-1)$-singular: indeed it is bounded by two consecutive points of Δ_{n-M-1} in A, it is not contained in C_2^+, and for any $J_0 \in \mathfrak{J}$ distinct from C_2^+ the image $g_{B_0} \circ g_{C_2^-}(\widehat{J}_0)$ does not contain the critical value; hence the two components of $P_c^{-1}(g_{B_0} \circ g_{C_2^-}(J_0))$ are regular intervals. This shows that J^* is not contained in J_0 and proves the claim. Therefore $J^+ = g_{B_0} \circ g_{C_2^-}(J^*)$ has the required form. □

Consider now a lateral interval $J \in \mathcal{E}(n)$. For each $1 \leqslant k \leqslant K$, the endpoints of $\breve A(k)$ belong to Δ_{N_k+3}, with $N_k + 3 \leqslant n$. Therefore either J is contained in $\breve A(k)$ or it is disjoint from $\operatorname{int} \breve A(k)$. Let $k = k(J)$ be the largest integer such that $J \subset \breve A(k)$. As J is neither central nor peripheral, we have $1 \leqslant k < K$ and $J \cap \operatorname{int} \breve A(k+1) = \emptyset$.

Definition 4.4. The integer $k(J)$ is the *level* of J. The level is *stationary* if $A(k) = A(k+1)$.

Remark 4.5. One never has $B(k) = B(k+1)$, but these two intervals may have the same right endpoint, in which case one has $A(k) = A(k+1)$. This happens exactly when $J(k) = C_2^-$ if $g_{B(k)}$ preserves the orientation, or $J(k) = C_2^+$ if $g_{B(k)}$ reverses the orientation.

In the next proposition, D^+ is the interval $[\widehat\alpha^{(3)}, \alpha]$, which is regular of order 3.

Proposition 4.6. *Assume that $J \in \mathcal{E}(n)$ is lateral of stationary level k, and that $g_{B(k)}$ reverses the orientation. Then there is an interval $J^* \in \mathcal{E}(n - N_k - 3)$, contained in $[\widetilde\alpha^{(2)}, -\alpha]$, such that $J^+ := P_c(J)$ is equal to $g_{B(k)} \circ g_{D^+}(J^*)$.*

There is a similar statement, replacing D^+ by $D^- := -D^+$, when $g_{B(k)}$ preserves the orientation. The precise formulation of this statement and its proof are left to the reader.

Proof. In view of Remark 4.5, we can write $J^+ = g_{B(k)}(J')$ for some interval $J' \subset [\widehat\alpha^{(3)}, \widehat\alpha^{(5)}]$ bounded by two consecutive points of Δ_{n-N_k}. We claim that $J^* := P_c^3(J') \subset [\widetilde\alpha^{(2)}, -\alpha]$ is $(n-N_k-3)$-singular. Indeed, otherwise J^* is contained in an interval $J_0 \in \mathfrak{J}$

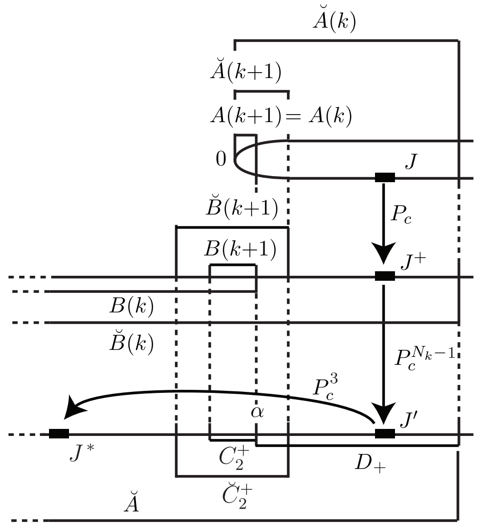

FIGURE 5

distinct from C_2^+. The image $g_{B(k)} \circ g_{D^+}(\widehat{J}_0)$ does not contain the critical value. Therefore one of the two components of $P_c^{-1}(g_{B(k)} \circ g_{D^+}(J_0)$ is a regular interval containing J. This proves the claim and the proposition. □

Proposition 4.7. *Assume that $J \in \mathcal{E}(n)$ is lateral of non stationary level k, and that $g_{B(k)}$ reverses (resp. preserves) the orientation. Then*

- *either there is an interval $J^* \in \mathcal{E}(n - N_k)$ such that $J^+ := P_c(J)$ is equal to $g_{B(k)}(J^*)$;*
- *or there exist an integer $n_0 \in [2, \mathrm{ord}(J(k))+1]$ and an interval $J^* \in \mathcal{E}(n - N_k - n_0)$ such that $J^+ = g_{B(k)} \circ g_{J_0}(J^*)$, where J_0 is the regular interval of order n_0 whose right (resp. left) endpoint is immediately to the left (resp. right) of $P_c^{N_k}(0)$ in Δ_{n_0}.*

When $J(k)$ is simple, the only possible occurrence of the second case is that J_0 is the simple regular interval adjacent to the left (resp. right) endpoint of $J(k)$.

Proof. We assume that $g_{B(k)}$ reverses the orientation, the other case being similar. As the level k is non stationary, the interval $J(k)$ is distinct from C_2^+. The interval $J' := P_c^{N_k-1}(J^+)$ is contained in the left component of $\widehat{A} \setminus \widehat{J}(k)$. We claim that J' is actually contained in A.

Otherwise, J' would be contained in $D^+ = [\widehat{\alpha}^{(3)}, \alpha]$, which is regular of order 3. The image $g_{B(k)}(\widehat{D}^+)$ does not contain the critical value, hence one of the two components of $P_c^{-1}(g_{B(k)}(D^+))$ would be a regular interval containing J. The claim is proved.

The interval J' is bounded by two consecutive points of Δ_{n-N_k}. If J' is $(n - N_k)$-singular, we define $J^* := J'$ and the first case of the proposition occurs.

For the rest of the proof, we assume that J' is not singular. It is therefore contained in regular intervals of positive order. We choose the smallest such interval J_0, with largest order n_0 and define $J^* := P_c^{n_0}(J') \subset A$.

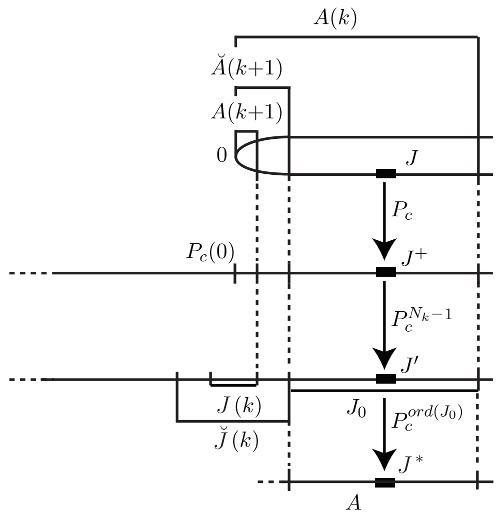

FIGURE 6

The interval $J^\star$ is $(n - N_k - n_0)$-singular: it is bounded by two consecutive points of $\Delta_{n-N_k-n_0}$, and for any regular interval of positive order J_1 containing J^* , the image $g_{J_0}(J_1)$ would contain J', be regular of positive order and strictly smaller than J_0.

We have shown that J^+ has the required form, except for the properties of J_0 and n_0.

The point $P_c^{N_k}(0)$ belongs to $\widehat{J_0} - J_0$: if $P_c^{N_k}(0)$ was not in $\widehat{J_0}$, the two components of $P_c^{-1}(g_{B(k)}(J_0))$ would be regular and J would not be singular. On the other hand, $P_c^{N_k}(0)$ belongs to $J(k)$, which is disjoint from J'. If we had $P_c^{N_k}(0) \in J_0$, $J(k)$ would be contained in and strictly smaller than J_0, contradicting the maximality property of elements of $\mathfrak{J}$.

Write $J_0 = [\gamma^-, \gamma^+]$, $\widehat{J_0} = (\widehat{\gamma}^-, \widehat{\gamma}^+)$. The points $\gamma^+, \widehat{\gamma}^+$ are consecutive points of Δ_{n_0+1} (Proposition 2.5, part (1)) and $P_c^{N_k}(0)$ belongs to $(\gamma^+, \widehat{\gamma}^+)$. This proves the assertion about the right endpoint of J_0.

We have $n_0 \geqslant 2$ because 2 is the minimal order of a regular interval contained in A. Assume that $n_0 \geqslant \operatorname{ord}(J(k))$. As J_0 is not contained in $J(k)$, these two intervals have disjoint interior. Moreover, the left endpoint of $J(k)$ is equal to the right endpoint of J_0. The left endpoint of $\breve{J}(k)$ belongs to Δ_m with $m = \operatorname{ord}(J(k)) + 3$. The same point must belong to the interior of J_0, because $J' \subset J_0$ is disjoint from the interior of $\breve{J}(k)$. As the interior of J_0 does not intersect Δ_{n_0+1} (Proposition 2.5, part (2)), we have $n_0 + 1 < m$.

Finally, assume that $J(k)$ is a simple regular interval. From the properties of n_0, J_0 seen above, it is clear that we must have $J_0 = C_n^+$ if $J(k) = C_{n+1}^+$ (for $2 \leqslant n < M - 2$) and $J_0 = C_n^-$ if $J(k) = C_{n-1}^-$ (for $2 < n \leqslant M - 2$).

□

The next result about the number central n-singular intervals will be useful in the next section (Proposition 5.9).

Proposition 4.8. (1) *When $J(K)$ is simple, there are at most 9 central n-singular intervals.*

(2) *When $J(K)$ is not simple, there are at most $2^{\,C\,2^{-\sqrt{M}}N_K}$ central n-singular intervals.*

Proof. Recall that $3 \leqslant n - N_K \leqslant \operatorname{ord} J(K) + 2$.

(1) The proof is by direct inspection of the different cases. The worst case, assuming that $g_{B(K)}$ preserves orientation, is when $J(K) = C^+_{M-3}$ or C^+_{M-2} and $n = \operatorname{ord} J(K) + 2$. Details are left to the reader.

(2) The images by $P_c^{N_K}$ (a 2-to-1 map from $\breve{A}(K)$ into $\breve{A}$) of the endpoints of the central n-singular intervals belong to Δ_{n-N_K}. The cardinality of Δ_m is at most 2^{m+1}. Therefore, when $J(K)$ is not simple, the cardinality of Δ_{n-N_K} is at most $2^{\,C\,2^{-\sqrt{M}}N_K}$. This is also a bound (with a different constant C) for the number of central n-singular intervals.

$\square$

4.3. **Measure estimates.** From the combinatorial structure of the set of n-singular intervals described in the previous subsection, we derive inductive estimates on the sum of the lengths of n-singular intervals, which will be denoted by $E(n)$.

The assumptions in this subsection are the same than in the previous one: n is an integer $\geqslant M+3$, c is a parameter in $(c^{(M)}, c^{(M-1)})$ which is strongly regular up to a level K such that $N_K + 3 \leqslant n < N_{K+1} + 3$.

The first result will control the total measure of central n-singular intervals.

Proposition 4.9. *For $1 \leqslant k \leqslant K+1$, the length of the interval $\breve{A}(k)$ satisfies*

$$|\log|\breve{A}(k)| + \frac{1}{2}\,(N_k + M)\,\log 2| \leqslant C\,M^{-1}\,N_k.$$

Proof. From general univalent function theory ([P]), the length of $\breve{B}(k)$, and the length of each component $\mathcal{C}$ of $\breve{B}(k) \setminus B(k)$, are controlled by the length of $B(k)$:

$$|\breve{B}(k)| \leqslant C|B(k)|, \qquad |\mathcal{C}| \geqslant C^{-1}|B(k)|.$$

It follows that

$$C^{-1}|B(k)|^{1/2} \leqslant |\breve{A}(k)| \leqslant C^{-1}|B(k)|^{1/2}.$$

The length of $|B(k)|$ is controlled by the estimate of Proposition 3.10: for $x \in A$

$$|\log(|Dg_{B(k)}(x)|\,\frac{h_c(g_{B(k)}(x))}{h_c(x)}) + (N_k - 1)\log 2| \leqslant C\,M^{-1}\,N_k.$$

The function h_c satisfies $C^{-1} \leqslant h_c(x) \leqslant C$ for $x \in A$ and $C^{-1}2^M \leqslant h_c(y) \leqslant C\,2^M$ for $y \in B(k)$ (cf. Proposition 3.4). We obtain

$$|\log|B(k)| + (N_k + M)\,\log 2| \leqslant C\,M^{-1}\,N_k, \tag{4.1}$$

which implies the estimate of the proposition. $\square$

We next deal with the sum $E_p(n)$ of the lengths of peripheral n-singular intervals.

Proposition 4.10. *The total length of peripheral n-singular intervals satisfies*

$$E_p(n) \leqslant C\,2^{-M}E(n - M - 1).$$

Proof. Let J be a peripheral n-singular interval. According to Proposition 4.3, $J^+ := P_c(J)$ is either of the form $g_{B_0}(J^*)$, for some $J^* \in \mathcal{E}(n-M+1)$, or $g_{B_0} \circ g_{C_2^-}(J^*)$, for some $J^* \in \mathcal{E}(n-M-1)$. Here, B_0 is the interval $[\alpha^{(M-2)}, \alpha^{(M-3)}]$. As J is peripheral, any $x \in J$ satisfies $|x| \geqslant C^{-1}2^{-M}$ (Proposition 4.9, with $k=1$). We have therefore $|J| \leqslant C\,2^M|J^+|$. On the other hand, by Proposition 3.5, the derivatives of the inverse branches involved satisfy, for $x \in J^+$

$$|Dg_{B_0}(P_c^{M-2}(x))| \leqslant C\,4^{-M},$$

in the first case, and

$$|D(g_{B_0} \circ g_{C_2^-})(P_c^M(x))| \leqslant C\,4^{-M}$$

in the second case. We obtain

$$E_p(n) \leqslant C\,2^{-M}(E(n-M+1) + E(n-M-1)) \leqslant C\,2^{-M}E(n-M-1)$$

. □

Let $1 \leqslant k < K$. We denote by $E_\ell(n,k)$ the sum of the lengths of those n-singular intervals which are lateral of level k.

The next proposition deals with the case of stationary level.

Proposition 4.11. *Assume that k is stationary. The quantity $E_\ell(n,k)$ satisfies*

$$E_\ell(n,k) \leqslant C\,2^{-1/2(N_k+M)}2^{CM^{-1}N_k}E(n-N_k-3).$$

Proof. The proof is very similar to the proof of the previous proposition. We assume for instance that $g_{B(k)}$ reverses orientation. Let J be a n-singular interval which is lateral of stationary level k. According to Proposition 4.6, the image $J^+ = P_c(J)$ is equal to $g_{B(k)} \circ g_{D^+}(J^*)$, for some $J^* \in \mathcal{E}(n-N_k-3)$. Moreover, J^* is contained in $[\widetilde{\alpha}^{(2)}, -\alpha]$ hence $g_{D^+}(J^*)$ is contained in $[\widehat{\alpha}^{(3)}, \widehat{\alpha}^{(5)}]$, while $P_c^{N_k}(0)$ belongs to $[\alpha, \widetilde{\alpha}^{(2)}]$. This implies (using the bounded distortion property of univalent functions) that $x - \beta \geqslant x - P_c(0) \geqslant C^{-1}|B(k)|$ for $x \in J^+$. We obtain $|J| \leqslant C|B(k)|^{1/2}|J^*|$. The length $|B(k)|$ has been estimated in (4.1). This estimate leads to the inequality of the proposition.

□

Finally we consider the case of a non stationary level.

Proposition 4.12. *Assume that k is non stationary. The quantity $E_\ell(n,k)$ satisfies*

$$E_\ell(n,k) \leqslant C \operatorname{ord} J(k)\, 2^{\operatorname{ord} J(k)}2^{-1/2(N_k+M)}2^{CM^{-1}N_k}E(n-N_{k+1}-1).$$

When $\operatorname{ord} J(k) < M$*, we get the better estimate*

$$E_\ell(n,k) \leqslant C\,2^{-1/2(N_k+M)}2^{CM^{-1}N_k}(2^{\frac{1}{2}\operatorname{ord} J(k)}E(n-N_k)+2^{-\frac{1}{2}\operatorname{ord} J(k)}E(n-N_{k+1}-1)).$$

Proof. Let J be a n-singular interval which is lateral of non stationary level k. According to Proposition 4.7, one may write $J^+ := P_c(J) = g_{B(k)}(J')$, where either $J' =: J^*$ is $(n-N_k)$-singular or J' is of the form $g_{J_0}(J^*)$ for some regular interval J_0 uniquely determined by its order $n_0 \in [2, \operatorname{ord} J(k)+1]$, and some $(n-N_k-n_0)$-singular interval J^*.

As $|Dg_{J_0}| \leqslant C$, the length of J^+ satisfies in all cases $|J^+| \leqslant C\max_A |Dg_{B(k)}|\,|J^*|$. On the other hand, as J does not intersect the interior of $\breve{A}(k+1)$, one has, for $x \in J^+$

$$x - P_c(0) \geqslant C^{-1}|J(k)|\,|B(k)|.$$

This gives $|J| \leqslant C|J(k)|^{-1/2}|B(k)|^{1/2}|J^*|$. As $|DP_c| \leqslant 4$ we have $|J(k)| \geqslant 4^{-\mathrm{ord}\, J(k)}|A|$. Recalling the estimate for $Dg_{B(k)}$ in Proposition 3.10, we obtain

$$|J| \leqslant C\, 2^{\mathrm{ord}\, J(k)} 2^{-1/2(N_k+M)} 2^{CM^{-1}N_k}|J^*|.$$

To deduce the inequality of the proposition, we have just to observe that the sequence $E(m)$ is non-increasing, and that there is only one choice of J_0 for each order $n_0 \in [2, N_{k+1} - N_k + 1]$.

When $J(k)$ is simple, we use the better estimates $|J(k)| \geqslant C^{-1}2^{-\mathrm{ord}\, J(k)}$, $|Dg_{J_0}| \leqslant C2^{-\mathrm{ord}\, J(k)}$ (Proposition 3.5). From Proposition 4.7 there is only one possibility for (J_0, n_0). This implies the second inequality. □

4.4. **Measure of non n-regular points.** The size of all types of n-singular intervals is now under control.

Proposition 4.13. (1) *For $2 \leqslant n \leqslant M-2$, one has, for all $c \in (c^{(M)}, c^{(M-1)})$*

$$E(n) = |[\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n)}]| \leqslant 4 \sin \frac{\pi}{3.2^n}.$$

(2) *Assume that $c \in (c^{(M)}, c^{(M-1)})$ is strongly regular up to level K. Let $\theta \in (0, \frac{1}{2})$. If $M \geqslant M_0(\theta)$ is large enough, one has, for all $M-2 < n < N_{K+1}+3$*

$$E(n) \leqslant 2^{-\theta n}.$$

Proof. (1) For $2 \leqslant n \leqslant M-2$, the only n-singular interval is $[\widetilde{\alpha}^{(n)}, -\widetilde{\alpha}^{(n)}]$. Its length for $c = -2$ is $4 \sin \frac{\pi}{3.2^n}$ (subsection 2.2). By Proposition 3.1, part (1), the map $c \to \alpha^{(n-1)} - P_c(0)$ is decreasing on $[-2, \frac{3}{2}]$, hence the same is true of $c \to |\widehat{\alpha}^{(n)}|$.

(2) For $2 \leqslant n \leqslant M-2$, by the first part of the proposition, one obtains

$$E(n) \leqslant \frac{4\pi}{3} 2^{-n} \leqslant 3.2^{-n/2}.$$

For $M-2 < n \leqslant 2M-9$, one gets

$$E(n) \leqslant E(M-2) \leqslant \frac{16\pi}{3} 2^{-M} \leqslant 2^{\frac{9}{2}-M} \leqslant 2^{-n/2}.$$

We may now assume that $n \geqslant 2M-8$. Replacing if necessary K by a smaller integer, we may also assume that $N_K + 3 \leqslant n < N_{K+1} + 3$.

From Proposition 4.9, the sum $E_c(n)$ of the length of central n-singular intervals satisfies

$$E_c(n) \leqslant |\breve{A}(k)| \leqslant 2^{-1/2(N_K+M)} 2^{C\,M^{-1}N_K}.$$

The right hand side is $\leqslant \frac{1}{4}2^{-\theta(N_{K+1}+2)} \leqslant \frac{1}{4}2^{-\theta n}$ iff

$$\theta(N_{K+1} - N_K) \leqslant (1/2 - \theta - CM^{-1})N_K + M/2 - 2 - 2\theta.$$

If $N_{K+1} - N_K \leqslant M-2$, one has $\theta(N_{K+1} - N_K) \leqslant M/2 - 2 - 2\theta$ when M is large enough. If $N_{K+1} - N_K > M$, one has $\theta(N_{K+1} - N_K) \leqslant (1/2 - \theta - CM^{-1})N_K$ from the strong regularity assumption (cf. Remark 3.9) if M is large enough. We have shown that $E_c(n) \leqslant \frac{1}{4}\, 2^{-\theta n}$.

From Proposition 4.10 and the induction hypothesis, the sum of the length of peripheral n-singular intervals satisfies

$$E_p(n) \leqslant C\, 2^{-M} E(n-M-1) \leqslant C\, 2^{-M}\, 2^{\theta(M+1)} 2^{-\theta n},$$

which is $\leqslant \frac{1}{4}\, 2^{-\theta n}$ for large M.

Let $1 \leqslant k < K$. Comparing Propositions 4.11 and 4.12 , we observe that the bound for $E_\ell(n,k)$ in the non stationary case is worse than the bound in the stationary case, hence may be used in all cases.

We estimate $E(n-N_k)$ or $E(n-N_{k+1})$ by the induction hypothesis. When $\operatorname{ord} J(k) > M$, we obtain

$$2^{\theta n} E_\ell(n,k) \leqslant C\, 2^{\theta}\, 2^{-\frac{M}{2}\operatorname{ord} J(k)}\, 2^{\operatorname{ord} J(k)} 2^{(\theta + CM^{-1} - 1/2)N_k}.$$

For M large enough, we have $\theta + CM{-}1 - 1/2 < \frac{1}{2}(\theta - 1/2)$. The strong regularity assumption (cf. Remark 3.9) implies $\operatorname{ord} J(k) \leqslant 2^{-\sqrt{M}}(1-2^{-\sqrt{M}})^{-1} N_k$. We obtain, for M large enough

$$\sum_{1 \leqslant k < K, \operatorname{ord} J(k) > M} E_\ell(n,k) \leqslant \frac{1}{4} 2^{-\theta n}.$$

When $J(k)$ is simple, we use the second inequality in Proposition 4.12. From the induction hypothesis, we have

$$2^{\frac{1}{2}\operatorname{ord} J(k)} E(n-N_k) + 2^{-\frac{1}{2}\operatorname{ord} J(k)} E(n-N_{k+1}-1) \leqslant 3\, 2^{-\theta n}\, 2^{\frac{M}{2}}\, 2^{\theta N_k}.$$

This implies

$$2^{\theta n}\, E_\ell(n,k) \leqslant C\, 2^{\,(\theta + CM^{-1} - 1/2)N_k},$$

$$\sum_{1 \leqslant k < K, \operatorname{ord} J(k) < M} E_\ell(n,k) \leqslant \frac{1}{4} 2^{-\theta n}.$$

The proof of the proposition is complete.

□

Corollary 4.14. *For M large enough, strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ are regular.*

5. The parameter space

5.1. **A natural partition of parameter space.** For $n > 1$, define

$$\widetilde{\Delta}_n = \{c \in [-2, c^{(2)}],\, P_c^n(0) = \pm\alpha(c)\} = \{c \in [-2, c^{(2)}],\, P_c^{n+1}(0) = \alpha(c)\}.$$

Each $\widetilde{\Delta}_n$ is a finite set, with $\widetilde{\Delta}_n \subset \widetilde{\Delta}_{n+1}$. From Proposition 3.1, part (2), one obtains

$$\widetilde{\Delta}_2 = \{c^{(2)}\}, \qquad \widetilde{\Delta}_3 = \{c^{(2)}, c^{(3)}\}.$$

Moreover, $c^{(n)}$ is the lowest element of $\widetilde{\Delta}_n$ and $c^{(n-1)}$ is the next lowest (for $n > 2$).

In particular, $(c^{(n)}, c^{(n-1)})$ is a component of $\mathbb{R} \setminus \widetilde{\Delta}_n$.

In the next proposition, n is an integer > 2, U denotes a bounded component of $\mathbb{R} \setminus \widetilde{\Delta}_n$, and c_0 is a parameter in U.

Proposition 5.1. (1) *There is a unique continuous map $(\delta, c) \mapsto \delta(c)$ from $\Delta_n(c_0) \times U$ to $\mathbb{R}$ which satisfies $\delta(c_0) = \delta$ for all $\delta \in \Delta_n(c_0)$ and $P_c(\delta(c)) = \delta_1(c)$ whenever $P_{c_0}(\delta) = \delta_1$.*

(2) *For each $\delta \in \Delta_n(c_0)$, the function $c \mapsto \delta(c)$ is real-analytic on U, and extends continuously on $\bar{U}$.*

(3) *For $c \in U$, $\delta \in \Delta_n(c_0)$, $\delta(c)$ is distinct from the critical point 0. If moreover $\delta \in \Delta_{n-1}(c_0)$, $\delta(c)$ is distinct from the critical value $P_c(0)$.*

(4) *For each $c \in U$, the map $\delta \mapsto \delta(c)$ is injective, and its image is $\Delta_n(c)$.*

(5) *Let $J = [\gamma^-, \gamma^+]$ be a regular interval of order $k < n$ for P_{c_0}. Then, for any $c \in U$, the interval $[\gamma^-(c), \gamma^+(c)]$ is regular of order k for P_c.*

Proof. (1) The functions $c \mapsto \delta(c)$, for $\delta \in \Delta_k(c_0)$, $0 \leqslant k \leqslant n$ are constructed by induction on k. For $k = 0$, the functions $c \mapsto \pm\alpha(c)$ have the required properties. Assume that the functions $c \mapsto \delta(c)$ have been constructed for $\delta \in \Delta_k(c_0)$, for some $k < n$. Let $\delta \in \Delta_{k+1}(c_0)$. Consider $\delta_1 := P_{c_0}(\delta) \in \Delta_k(c_0)$. One has $\delta_1(c_0) \geqslant P_{c_0}(0)$ since $\delta^2 + c_0 = \delta_1$. As U is connected and does not intersect $\widetilde{\Delta}_{k+1}$, the inequality $\delta_1(c) > P_c(0)$ holds for all $c \in U$. Then $\delta(c)$ is uniquely determined by

$$\delta(c_0) = \delta, \qquad \delta^2(c) = \delta_1(c) - P_c(0).$$

(2) Real-analyticity on U and continuity on $\bar{U}$ are obvious from the inductive construction of the functions $c \mapsto \delta(c)$.

(3) If one had $\delta(c) = 0$ (resp. $\delta(c) = P_c(0)$) for some $c \in U$ and some δ in $\Delta_n(c_0)$ (resp. in $\Delta_{n-1}(c_0)$), one would have $P_c^n(0) = \pm\alpha(c)$ and $c \in \widetilde{\Delta}_n$.

(4) Let $c \in U$. If the map $\delta \mapsto \delta(c)$ from $\Delta_n(c_0)$ to $\mathbb{R}$ is not injective, there is a smaller integer k such that the restriction to $\Delta_k(c_0)$ is not injective. As $\alpha(c) < 0$ for $c < 0$, the integer k is positive. Let $\delta, \delta' \in \Delta_k(c_0)$ be distinct points such that $\delta(c) = \delta'(c)$. Then, $P_c(\delta(c)) = P_c(\delta'(c))$. By the minimality of k, we have $P_{c_0}(\delta) = P_{c_0}(\delta')$, hence $\delta' = -\delta$. But, by part (3), δ and δ' do not vanish in U, hence $\delta(c) = \delta'(c)$ is impossible.

Obviously, for any $\delta \in \Delta_n(c_0)$ and any $c \in U$, the point $\delta(c)$ belongs to $\Delta_n(c)$. Therefore the cardinality of $\Delta_n(c)$ is at least equal to the cardinality of $\Delta_n(c_0)$. As c_0 was an arbitrary point of U, we conclude that this cardinality is constant in U and that the image of the map $\delta \mapsto \delta(c)$ is equal to $\Delta_n(c)$.

(5) Write $\widehat{J} = (\widehat{\gamma}^-, \widehat{\gamma}^+)$. The points $\widehat{\gamma}^\pm$ belong to $\Delta_{k+1}(c_0)$. For any $0 \leqslant j < k$, the interval $P_{c_0}^j(\widehat{J})$ does not contain the critical point, hence by part (3) of this proposition, the same is true for $P_c^j((\widehat{\gamma}^-(c), \widehat{\gamma}^+(c)))$. Moreover the endpoints of this interval are $\pm\alpha^{(1)}(c)$. This proves that $[\gamma^-(c), \gamma^+(c)]$ is regular of order k for P_c.

□

5.2. **Strongly regular parameter intervals.** Let M be a large integer, let n be an integer $\geqslant M$, let U be a component of $\mathbb{R} - \widetilde{\Delta}_n$ contained in $(c^{(M)}, c^{(M-1)})$, and let $c_0 \in U$.

Suppose that there exist integers $M = N_1 < \cdots < N_{K+1} \leqslant n$, and , for each $1 \leqslant k \leqslant K$, a regular interval $J(k) = [\gamma_k^-, \gamma_k^+]$ of order $N_{k+1} - N_k$ in $\mathfrak{J}(c_0)$ such that $P_{c_0}^{N_k}(0)$ belongs to the interior of $J(k)$. Then, for all $c \in U$, the interval $[\gamma_k^-(c), \gamma_k^+(c)]$ is regular of the same order $N_{k+1} - N_k$, belongs to $\mathfrak{J}(c)$ and contains $P_c^{N_k}(0)$ in its interior.

If moreover c_0 is strongly regular up to level K, the same is true for c.

Definition 5.2. Let $c_0 \in (c^{(M)}, c^{(M-1)})$ be a parameter which is strongly regular up to level K. The component of c_0 in $\mathbb{R} \setminus \widetilde{\Delta}_{N_{K+1}}$, denoted by $U(K)$, is a *strongly regular parameter interval of level K*.

The parameter interval $(c^{(M)}, c^{(M-1)}) = U(0)$ is strongly regular of level 0. Strongly regular parameter intervals of the same level have disjoint interiors.

Let $U(K)$ be a strongly regular parameter of level K. The sequence $N_1 = M < \cdots < N_{K+1}$ is the same for all $c \in U(K)$. For $k \leqslant K$, there is exactly one strongly regular parameter interval of level k containing $U(K)$, namely the component $U(k)$ of $\mathbb{R} - \widetilde{\Delta}_{N_{k+1}}$ containing $U(K)$.

We explain how a strongly regular parameter interval $U(K-1)$ of level $(K-1)$ splits into strongly regular parameter intervals of level K and bad parameter intervals.

Let $N_1 = M < \cdots < N_K$ be the sequence associated to $U(K-1)$. If some parameter $c \in U(K-1)$ is strongly regular up to level K, then, according to Remark 3.9 the point $T_c^K(0) = P_c^{N_K}(0)$ belongs to some regular interval $J(K) \in \mathcal{J}(c)$ of order $\leqslant N_K^\sharp$ with

$$N_K^\sharp = \begin{cases} M-2 & \text{if } N_K < 2^{\sqrt{M}}M, \\ \lfloor 2^{-\sqrt{M}}(1-2^{-\sqrt{M}})^{-1}\ (N_K - M)\rfloor & \text{if } N_K \geqslant 2^{\sqrt{M}}M. \end{cases}$$

Fix some $c_0 \in U(K-1)$. Consider the partition of A into regular intervals in $\mathcal{J}(c_0)$ of order $\leqslant N_K^\sharp$, and $N_K^\sharp$-singular intervals. Denote by

$$\Delta^\sharp(N_K) = \{\alpha < \widetilde{\alpha}^{(2)} < \cdots < \widetilde{\alpha}^{(M-2)} < \cdots < -\widetilde{\alpha}^{(M-2)} < \cdots < -\widetilde{\alpha}^{(2)} < -\alpha\}$$

the endpoints of the intervals in this partition. From Proposition 5.1, there is a real-valued continuous map $(\gamma, c) \mapsto \gamma(c)$ on $\Delta^\sharp(N_K) \times U(K-1)$ such that, for each $c \in U(K-1)$, the points $\{\gamma(c), \gamma \in \Delta^\sharp(N_K)\}$ split A into regular intervals in $\mathcal{J}(c)$ of order $\leqslant N_K^\sharp$, and $N_K^\sharp$-singular intervals (for P_c).

Define

$$\widetilde{\Delta}(U(K-1)) = \{c \in U(K-1),\ \exists \gamma \in \Delta^\sharp(N_K) \text{ s.t } P_c^{N_K}(0) = \gamma(c)\}.$$

Let V be a component of $U(K-1) \setminus \widetilde{\Delta}(U(K-1))$. When c varies in V, the point $P_c^{N_k}(0)$ stays in the same element J of the partition of A cut by $\Delta^\sharp(N_K)$.

Definition 5.3. When J is $N_K^\sharp$-singular, V is a *bad* component of $U(K-1)\setminus\widetilde{\Delta}(U(K-1))$. No point of V is strongly regular up to level K.

When J is simple regular, V is a strongly regular parameter interval of level K, with $N_{K+1} = N_K + \operatorname{ord} J$.

When J is regular non simple, V is a *candidate* interval. This case can only occur when $N_K^\sharp > M$, i.e $N_K > 2^{\sqrt{M}}M$.

When V is a candidate interval, one has also $N_{K+1} = N_K + \operatorname{ord}(J)$. The parameter interval V is strongly regular of level K iff

$$(1-2^{-\sqrt{M}}) \operatorname{ord} J + \sum_{\substack{1 \leqslant k < K \\ N_{k+1}-N_k > M}} (N_{k+1} - N_k) \leqslant 2^{-\sqrt{M}}(N_K - M).$$

5.3. **Bounds for the variations w.r.t. the parameter.** The first estimates extend those of Proposition 3.1. Let c be a parameter in $(c^{(M)}, c^{(M-1)})$.

Proposition 5.4. (1) *For* $0 < n < M$, $x \in \breve{A}$

$$|\frac{\partial}{\partial c}\, \alpha^{(n)}(c) - 1/3\,| \leqslant Cn\, 4^{-n}, \qquad |\frac{\partial}{\partial c}\, g_{B(1)}(x) - 1/3\,| \leqslant CM\, 4^{-M}.$$

(2) *For* $2 \leqslant n \leqslant M-2$, $x \in \breve{A}$

$$|\frac{\partial}{\partial c}\, \widetilde{\alpha}^{(n)}(c)| \leqslant C\, 2^n, \qquad |\frac{\partial}{\partial c}\, g_{C_n^\pm}(x)| \leqslant C\, 2^n.$$

Proof. (1) Consider an inverse branch $x_0, \cdots, x_n, \cdots$ for P_c with $x_{n+1}^2 + c = x_n$, and $x_{n+1} > 0$. We take for x_0 either $x_0 = -\alpha(c)$ or $x_0 \in \breve{A}$ independent of c. We estimate $\frac{\partial x_n}{\partial c}$ from the recurrence relation

$$\frac{\partial x_{n+1}}{\partial c} = \frac{1}{2\, x_{n+1}} \left(\frac{\partial x_n}{\partial c} - 1\right). \tag{5.1}$$

From this relation, we get

$$\frac{\partial x_{n+1}}{\partial c} + \frac{1}{3} = \frac{1}{2x_{n+1}} \left(\frac{\partial x_n}{\partial c} + \frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{2} - \frac{1}{x_{n+1}}\right).$$

From Propositions 3.1 and 3.4, we get $|x_n - 2| \asymp 4^{-n}$ for $0 \leqslant n < M$. This implies the first two inequalities in the proposition by induction on n.

(2) We have $|\widetilde{\alpha}^{(n)}| \asymp 2^{-n}$ for $2 \leqslant n \leqslant M - 2$ by Proposition 3.4. The last two inequalities of the proposition are then immediately deduced from relation (5.1). □

For the inverse branch associated to a general regular interval, the estimate is slightly worse than in the simple case.

Proposition 5.5. *Let J be a regular interval. The associated inverse branch satisfies, for $x \in \breve{A}$*

$$\left|\frac{\partial}{\partial c} g_J(x)\right| \leqslant C\, 4^{\operatorname{ord} J}.$$

Proof. Let $x \in \breve{A}$. Write $n := \operatorname{ord} J$. We take the derivative w.r.t. the parameter of the identity $P_c^n \circ g_J(x) = x$. We obtain

$$D(P_c^n) \circ g_J(x). \frac{\partial}{\partial c} g_J(x) + \left(\frac{\partial}{\partial c} P_c^n\right) \circ g_J(x) = 0.$$

From general univalent function theory ([P]), we have $|D(P_c^n)| \geqslant C^{-1} |J|^{-1} \geqslant C^{-1}$ on $\breve{J}$. On the other hand, the differential of the map $(x, c) \mapsto (P_c(x), c)$ has norm $\leqslant 4$ (in the operator norm associated to the sup-norm on $\mathbb{R}^2$) on $[-2, 2] \times \mathbb{R}$. It follows that $|\frac{\partial}{\partial c} P_c^n| \leqslant 4^n$ on $\breve{J}$. This gives the estimate of the proposition. □

We are now able to get the crucial estimate.

Proposition 5.6. *Let $U \subset (c^{(M)}, c^{(M-1)})$ be a strongly regular parameter interval of level $K - 1$. For any regular interval J of order $\leqslant N_K^\sharp$, the endpoints $\gamma^\pm$ of $g_{B(K)}(J)$ satisfy, for $c \in U$*

$$\left|\frac{\partial}{\partial c} \gamma^\pm(c) - \frac{1}{3}\right| \leqslant C\, 2^{-M}.$$

Proof. Define $J(K) := J$, $g_{B(K+1)} := g_{B(K)} \circ g_J$, $N_{K+1} = N_K + \operatorname{ord} J$, so that $\gamma^\pm = g_{B(K+1)}(\pm\alpha)$.

We first get rid of the dependence of α on c. We have

$$\frac{\partial}{\partial c} \left[g_{B(K+1)}(\pm\alpha(c))\right] = \left(\frac{\partial}{\partial c}\, g_{B(K+1)}\right)(\pm\alpha) \pm D g_{B(K+1)}(\pm\alpha)\, \frac{\partial \alpha}{\partial c}.$$

Here, $\frac{\partial \alpha}{\partial c}$ is bounded and the derivative of $g_{B(K+1)}$ satisfies

$$|D g_{B(K+1)}(\pm\alpha)| \leqslant C\, 2^{-(N_{K+1} + M)}\, 2^{C M^{-1} N_{K+1}}.$$

The proof is exactly the same as for Proposition 3.10, using (3.4) when J is simple and (3.5) when it is not.

We now estimate $\frac{\partial}{\partial c} g_{B(k)}$ by induction on k; for $k = 1$ this has been done in Proposition 5.4, part (1). The recurrence relation is

$$\frac{\partial}{\partial c}\, g_{B(k+1)} = \Big(\frac{\partial}{\partial c}\, g_{B(k)}\Big) \circ g_{J(k)} + Dg_{B(k)} \circ g_{J(k)} . \frac{\partial}{\partial c}\, g_{J(k)} .$$

From Proposition 3.10, the derivative of $g_{B(k)}$ is controlled in A by

$$|Dg_{B(k)}| \leqslant 2^{-(N_k+M)}\, 2^{CM^{-1}N_k} .$$

When $J(k)$ is simple, we have from Proposition 5.4, part (2)

$$|\frac{\partial}{\partial c}\, g_{J(k)}| \leqslant C\, 2^M .$$

When $J(k)$ is not simple, we have from Proposition 5.5

$$|\frac{\partial}{\partial c}\, g_{J(k)}| \leqslant C\, 4^{\operatorname{ord} J(k)} ,$$

with $\operatorname{ord} J(k) \leqslant 2^{1-\sqrt{m}} N_k$.

In both cases, we obtain

$$|Dg_{B(k)} \circ g_{J(k)} . \frac{\partial}{\partial c}\, g_{J(k)}| \leqslant C\, 2^{-N_k(1-CM^{-1})} .$$

Plugging this estimate in the recurrence relation gives the estimate of the proposition. $\square$

5.4. **Measure estimate in parameter space.** We give more information on the parameter space structure described in Subsection 5.2.

In the next proposition, K is a positive integer, $U = (c^-, c^+)$ is a strongly regular parameter interval of level $K-1$ contained in $(c^{(M)}, c^{(M-1)})$, c_0 is a point in U, $B(K) = (\gamma^-, \gamma^+)$ is the regular interval of order $(N_K - 1)$ which contains the critical value $P_{c_0}(0)$. We still use the notation $N_K^\sharp$ for the integer defined in Subsection 5.2. Finally, $\mathfrak{J}(c_0)$ is as above the set of regular intervals (for P_{c_0}) of positive order contained in A which are maximal with this property.

Proposition 5.7. (1) *One has* $\gamma^-(c^-) = P_{c^-}(0)$, $\gamma^+(c^+) = P_{c^+}(0)$.

(2) *Let* $J \in \mathfrak{J}(c_0)$ *be a regular interval of order* $\leqslant N_K^\sharp$. *Write* $g_{B(K)}(J) = [\gamma_J^-, \gamma_J^+]$. *There is an interval* $[c_J^-, c_J^+] \subset U$, *either candidate or strongly regular of level* K, *such that the critical value* $P_c(0)$ *satisfies*

$$\begin{aligned}
P_c(0) &< \gamma_J^-(c) && \textit{for } c^- \leqslant c < c_J^- ,\\
P_c(0) &= \gamma_J^-(c) && \textit{for } c = c_J^- ,\\
\gamma_J^-(c) < P_c(0) &< \gamma_J^+(c) && \textit{for } c_J^- < c < c_J^+ ,\\
P_c(0) &= \gamma_J^+(c) && \textit{for } c = c_J^+ ,\\
P_c(0) &> \gamma_J^+(c) && \textit{for } c^+ \geqslant c > c_J^+ .
\end{aligned}$$

Proof. By induction on K. For $K = 1$, one has $U = (c^{(M)}, c^{(M-1)})$, $B(1) = [\alpha^{(M-1)}, \alpha^{(M-2)}]$. From Proposition 3.1, part (2), the critical value is equal to $\alpha^{(M-1)}$ for $c = c^{(M)}$, equal to $\alpha^{(M-2)}$ for $c = c^{(M-1)}$, and belongs to $(\alpha^{(M-1)}, \alpha^{(M-2)})$ for $c \in (c^{(M)}, c^{(M-1)})$. Assume now that $K \geqslant 1$, and that U is a strongly regular parameter interval of level $K-1$ contained in $(c^{(M)}, c^{(M-1)})$ satisfying $\gamma^-(c^-) = P_{c^-}(0)$, $\gamma^+(c^+) = P_{c^+}(0)$. Let $J \in \mathfrak{J}(c_0)$ be a regular interval of order $\leqslant N_K^\sharp$. From Proposition 5.6, we have $|\frac{\partial}{\partial c}\, \gamma_J^\pm(c) - \frac{1}{3}| \leqslant C\, 2^{-M}$. For large M, this implies the existence of c_J^-, c_J^+ with the properties of part (2) of the proposition. These properties imply that $[c_J^-, c_J^+]$ is either a candidate interval or a strongly regular parameter interval of level K. Finally, the first part of the proposition at level K is implied by the second part at level $K-1$. This completes the induction step. $\square$

With U as above, let n be an integer in $[2, N_K^\sharp]$.Let

$$\gamma_0^- = \alpha < \gamma_0^+ = \widetilde{\alpha}^{(2)} \leqslant \gamma_1^- < \gamma_1^+ \leqslant \ldots \leqslant \gamma_r^- = -\widetilde{\alpha}^{(2)} < \gamma_r^+ = -\alpha$$

be the points in Δ_n such that $[\gamma_i^-, \gamma_i^+]$, $0 \leqslant i \leqslant r$ are exactly the regular intervals in $\mathfrak{J}$ of order $\leqslant n$. For $0 \leqslant i < r$, either $\gamma_i^+ = \gamma_{i+1}^-$ (with adjacent intervals in $\mathfrak{J}$) or $[\gamma_i^+, \gamma_{i+1}^-]$ is a union of consecutive n-singular intervals.

Assume for instance that $g_{B(K)}$ preserves the orientation. From Proposition 5.7, there are parameter values

$$c_0^- = c^- < c_0^+ \leqslant c_1^- < c_1^+ \leqslant \ldots \leqslant c_r^- < c_r^+ = c^+$$

such that, for any $c \in U$, $0 \leqslant i \leqslant r$

$$\begin{array}{rcl} P_c^{N_k}(0) = \gamma_i^\pm(c) & \Longleftrightarrow & c = c_i^\pm, \\ P_c^{N_k}(0) > \gamma_i^\pm(c) & \Longleftrightarrow & c > c_i^\pm, \\ P_c^{N_k}(0) < \gamma_i^\pm(c) & \Longleftrightarrow & c < c_i^\pm. \end{array}$$

Lemma 5.8. *The size of the gaps (c_i^+, c_{i+1}^-) is controlled by*

$$C^{-1} \max_{[c_i^+, c_{i+1}^-]} (\gamma_{i+1}^- - \gamma_i^+) \leqslant \frac{c_{i+1}^- - c_i^+}{c^+ - c^-} \leqslant C \min_{[c_i^+, c_{i+1}^-]} (\gamma_{i+1}^- - \gamma_i^+),$$

Proof. From Proposition 5.6, the endpoints $\gamma^\pm$ of $B(K)$ satisfy $1/4 \leqslant \frac{\partial}{\partial c}\gamma^\pm \leqslant 1/2$ for $c \in U$. This implies, for any $c^* \in U$

$$\begin{array}{rcl} c^+ - c^- & = & \gamma^+(c^+) - \gamma^-(c^-) \\ & = & (\gamma^+(c^+) - \gamma^+(c^*)) + (\gamma^+(c^*) - \gamma^-(c^*)) + (\gamma^-(c^*) - \gamma^-(c^-)) \\ & \leqslant & \dfrac{1}{2}(c^+ - c^-) + (\gamma^+(c^*) - \gamma^-(c^*)) \end{array}$$

We obtain $c^+ - c^- \leqslant 2(\gamma^+ - \gamma^-)(c^*)$, and similarly $c^+ - c^- \geqslant \frac{4}{3}(\gamma^+ - \gamma^-)(c^*)$. In the same way, we have, for any $c^* \in (c_i^+, c_{i+1}^-)$

$$\frac{4}{3}(g_{B(K)} \circ \gamma_{i+1}^- - g_{B(K)} \circ \gamma_i^+)(c^*) \leqslant c_{i+1}^- - c_i^+ \leqslant 2(g_{B(K)} \circ \gamma_{i+1}^- - g_{B(K)} \circ \gamma_i^+)(c^*).$$

The bounded distortion property of $g_{B(K)}$ on A (for any $c^* \in (c_i^+, c_{i+1}^-)$) implies the inequalities of the lemma. $\square$

To estimate the **sum** of the size of the gaps, we need a slight improvement on Proposition 4.13.

Let U be a parameter interval which is strongly regular of some level $K \geqslant 1$. Let n be an integer such that $M - 2 < n < N_{K+1} + 3$.

Let $J(c_0) = (\gamma^-(c_0), \gamma^+(c_0))$ be a n-singular interval for P_{c_0}, for some $c_0 \in U$. Then $J(c) = (\gamma^-(c), \gamma^+(c))$ is n-singular for P_c, for all $c \in U$. Define

$$||J||_U := \max_{c \in U} |J(c)|, \qquad E(n, U) := \sum_{J\ n-\text{singular}} ||J||_U.$$

Proposition 5.9. *Let $\theta \in (0, \frac{1}{2})$. If M is large enough, one has $E(n,U) \leqslant 2^{-\theta n}$.*

Remark 5.10. It is assumed in the proposition that $n > M-2$. For $2 \leqslant n \leqslant M-2$, there is only one n-singular interval and the first part of Proposition 4.13 gives $E(n,U) \leqslant 4\sin\frac{\pi}{3.2^n}$.

Proof. We may assume that $n \leqslant M+3$ as the number of n-singular intervals for $n < M+3$ is bounded and each of them has length $\leqslant 4\sin\frac{\pi}{3.2^{M-2}}$.

We divide as before n-singular intervals into central, peripheral and lateral. We introduce

$$E_c(n,U) := \sum_{J \text{ central } n-\text{singular}} ||J||_U,$$

and similarly $E_p(n,U)$, $E_\ell(n,k,U)$. The level k here varies from 1 to the largest integer $k_{\max}$ such that $n \geqslant N_k + 3$.

Copying word for word the proof of Proposition 4.10, one obtains

$$E_p(n,U) \leqslant C\, 2^{-M} E(n-M-1,U). \tag{5.2}$$

Similarly the proofs of Propositions 4.11 and 4.12 give

$$E_\ell(n,k,U) \leqslant C\, 2^{-1/2(N_k+M)} 2^{CM^{-1}N_k} E(n-N_k-3,U). \tag{5.3}$$

for a stationary level k,

(5.4)

$$E_\ell(n,k,U) \leqslant C\, 2^{-1/2(N_k+M)} 2^{CM^{-1}N_k} (2^{\frac{1}{2}\operatorname{ord} J(k)} E(n-N_k,U) + 2^{-\frac{1}{2}\operatorname{ord} J(k)} E(n-N_{k+1}-1,U)).$$

for a non stationary level k with $J(k)$ simple, and

$$E_\ell(n,k,U) \leqslant C \operatorname{ord} J(k)\, 2^{\operatorname{ord} J(k)} 2^{-1/2(N_k+M)} 2^{CM^{-1}N_k} E(n-N_{k+1}-1,U). \tag{5.5}$$

for a general non stationary level k.

The case of central n-singular intervals requires a little more care. From Proposition 4.9, any such interval J satisfies

$$||J||_U \leqslant 2^{-1/2(N_{k_{\max}}+M)} 2^{CM^{-1}N_{k_{\max}}}.$$

Proposition 4.8 provides a bound for the number of central n-singular intervals which can be absorbed into the constant C in the last inequality, giving

$$E_c(n,U) \leqslant 2^{-1/2(N_{k_{\max}}+M)} 2^{CM^{-1}N_{k_{\max}}}. \tag{5.6}$$

From (5.2)-(5.6), we derive the estimate of the proposition exactly as in the proof of Proposition 4.13.

□

From Lemma 5.8 and the proposition, we deduce

Corollary 5.11. *Let $U \subset (c^{(M)}, c^{(M-1)})$ a strongly regular parameter interval of level $K-1$. For $2 \leqslant n \leqslant N_K^\sharp$, the relative measure of parameters $c \in U$ such that $P_c^{N_K}(0)$ belongs to a regular interval in $\mathfrak{J}$ of order $\leqslant n$ is at least $1 - C\,2^{-\theta n}$. In particular, the sum of the lengths of the bad components contained in U (cf. Definition 5.3) is at most $C\,2^{-\theta N_K^\sharp}|U|$.*

5.5. **The large deviation argument.** From Corollary 5.11, we will deduce, through a standard large deviation argument, that a large subset of $(c^{(M)}, c^{(M-1)})$ is formed of strongly regular parameters. Strongly regular parameters are regular (Corollary 4.14). Regular parameters exhibit a dynamical behavior described in Section 2, having in particular an absolutely continuous ergodic invariant probability measure with positive Lyapunov exponents. Therefore this argument will complete the proof of Jakobson's theorem.

Proposition 5.12. *Let $\theta^\star$ be any constant in $(0, 1/2)$. When M is large enough, the relative measure in $(c^{(M)}, c^{(M-1)})$ of the subset of strongly regular parameters is at least $1 - 2^{-\theta^\star M}$.*

Proof. We define on $(c^{(M)}, c^{(M-1)})$ a sequence $(X_k)_{k\geqslant 1}$, of integer-valued "random" variables by

- If c is **not** strongly regular up to level $(k-1)$, $X_k(c) = 0$.
- If c is strongly regular up to level $(k-1)$, but $P_c^{N_k}(0)$ does not belong to a regular interval in $\mathcal{J}$ of order $\leqslant N_k^\sharp$, $X_k(c) = N_k^\sharp + 3$ $(> M)$.
- If c is strongly regular up to level $(k-1)$, and $P_c^{N_k}(0)$ belongs to a regular interval $J(k) \in \mathcal{J}$ of order $\leqslant N_k^\sharp$, $X_k(c) = \operatorname{ord} J(k)$.

If c is strongly regular up to level $K \geqslant 0$, the formula $N_{K+1} = M + \sum_{k=1}^K X_k$ holds.

On the other hand, If c is strongly regular up to level $(K-1)$, but is not strongly regular up to level K, one has

$$\sum_{1\leqslant k\leqslant K,\, X_k > M} X_k(c) > 2^{-\sqrt{M}} \sum_{1\leqslant k\leqslant K} X_k(c),$$

and also $\sum_{1\leqslant k\leqslant K} X_k(c) \geqslant 2K$.

Define $Y_k := \mathbf{1}_{\{X_k > M\}} X_k$ and $S_K = \sum_1^K Y_k$. We will bound the measure of $\{S_K \geqslant 2^{1-\sqrt{M}} K\}$.

Let $0 < \bar\theta < \theta < 1/2$ be constants such that $\theta - \bar\theta > \theta^\star$. We assume that M is large enough for the estimate of Corollary 5.11 to be valid. Define

$$I(K) := \int_{c^{(M)}}^{c^{(M-1)}} 2^{\bar\theta\, S_K(c)} dc$$

On an interval V of parameters which are not strongly regular up to level $(K-1)$, we have $X_K(c) = Y_K(c) = 0$ hence

$$\int_V 2^{\bar\theta\, S_K(c)} dc = \int_V 2^{\bar\theta\, S_{K-1}(c)} dc.$$

On a strongly regular parameter interval U of level $(K-1)$, we have, from Corollary 5.11

$$\begin{aligned}
\int_U 2^{\bar\theta\, Y_K(c)} dc &\leqslant |U| + \sum_{m>M} 2^{\bar\theta m} \int_U \mathbf{1}_{\{X_K = m\}}\, dc \\
&\leqslant (1 + C \sum_{m>M} 2^{(\bar\theta - \theta)m})\, |U| \\
&\leqslant (1 + C\, (\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M})\, |U|\,.
\end{aligned}$$

As S_{K-1} is constant on U, we also obtain

$$\int_U 2^{\bar\theta\, S_K(c)} dc \leqslant (1 + C\, (\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M}) \int_U 2^{\bar\theta\, S_{K-1}(c)} dc.$$

We conclude that

$$I(K) \leqslant (1 + C\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M})^K (c^{(M-1)} - c^{(M)}).$$

It follows that

$$\frac{\mathrm{Leb}\{c \in (c^{(M)}, c^{(M-1)}),\ S_K(c) \geqslant 2^{1-\sqrt{M}}K\}}{c^{(M-1)} - c^{(M)}} \leqslant u_K,$$

with

$$u_K := \frac{(1 + C\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M})^K - 1}{2^{\,\bar\theta\, 2^{1-\sqrt{M}}K} - 1}.$$

The relative measure of the complement of strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ is therefore bounded by $\sum_{K\geqslant 1} u_K$. To bound this series, we observe that the quantities u_K satisfy, when M is large enough

- For $K \leqslant \bar\theta^{-1}\, 2^{\sqrt{M}}$,

  $$u_K \leqslant C\,\bar\theta^{-1}\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M+\sqrt{M}},$$

  as numerator and denominator of u_K are controlled in this range by linear functions of K.
- For $\bar\theta^{-1}\, 2^{\sqrt{M}} \leqslant K \leqslant (\theta - \bar\theta)2^{(\theta-\bar\theta)M}$,

  $$u_K \leqslant C\,K\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M} 2^{\,-\bar\theta\, 2^{1-\sqrt{M}}K},$$

  as the numerator is still in the linear mode, while the denominator is in exponential mode.
- For $K \geqslant (\theta - \bar\theta)2^{(\theta-\bar\theta)M}$,

  $$u_K \leqslant C\,\rho^K, \qquad \rho := (1 + C\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M})2^{\,-\bar\theta\, 2^{1-\sqrt{M}}},$$

  as numerator and denominator are in exponential mode.

The sum of the u_K in the first range is less than $C\,\bar\theta^{-2}\,(\theta - \bar\theta)^{-1}\, 2^{(\bar\theta-\theta)M+2\sqrt{M}}$. As $\theta^\star < \theta - \bar\theta$, this is smaller than $\frac{1}{3}2^{-\theta^\star M}$ when M is large enough.

The sum of the u_K in the third range is less than $C\,(1-\rho)^{-1}\rho^{(\theta-\bar\theta)2^{(\theta-\bar\theta)M}}$. As $1-\rho \asymp \bar\theta 2^{-\sqrt{M}}$, this is smaller than $2^{2^{-\theta^\star M}} \leqslant \frac{1}{3}2^{-\theta^\star M}$ when M is large enough.

In the second range, we group together for each $j \geqslant 1$ the terms such that $j\,\bar\theta^{-1}\, 2^{\sqrt{M}} \leqslant K \leqslant (j+1)\,\bar\theta^{-1}\, 2^{\sqrt{M}}$. The sum of the u_K in the second range is less than

$$C\,\bar\theta^{-2}\,(\theta - \bar\theta)^{-1}\, 2^{2\sqrt{M}}\, 2^{(\bar\theta-\theta)M} \sum_{j\geqslant 1} j\, 2^{-2j}.$$

This is smaller than $\frac{1}{3}2^{-\theta^\star M}$ when M is large enough.

Thus the sum of the series $\sum_{K\geqslant 1} u_K$ is $\leqslant 2^{-\theta^\star M}$. The proof of the proposition is complete.

□

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