EXACT RESULTS FOR DISORDERED SYSTEMS I: Theorem of Inclusions & the Weak Disorder Limit

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National Science Foundation
WHERE DISCOVERIES BEGIN

AFOSR
MURI

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Exact results with illustrations

- Theorem of inclusions

- Phase transitions out of fully gapped states have to be of the Griffith type

- phase transitions between fully gapped and conducting phases are prohibited

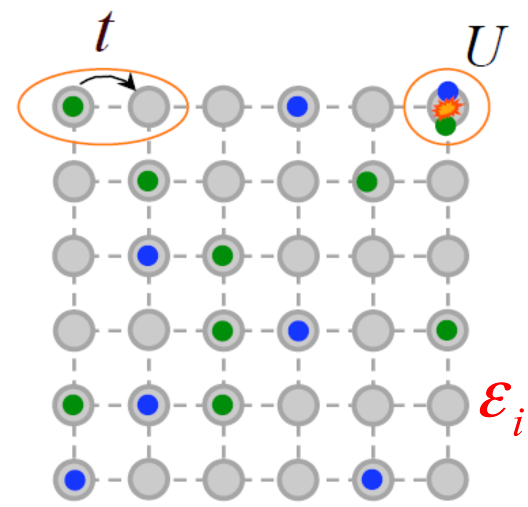
- Disordered lattice superfluids

- Weak disorder limit

- Universal Gaussian distribution

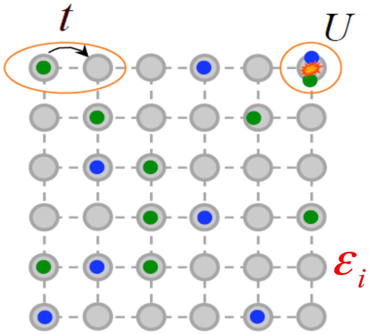
- critical disorder strength $\propto U^{(4-d)/4}$

Model to keep in mind



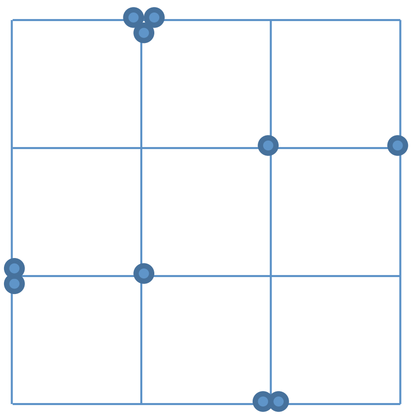
Bose-Hubbard:
Bosons on a lattice with on-site repulsion U
and nearest-neighbor hopping amplitude t.
Disorder is added as random on-site potential ε_i

$$H = -t\sum_{<ij>} b_i^\dagger b_j + \frac{U}{2}\sum_i n_i(n_i - 1) - \sum_i (\mu - \varepsilon_i) n_i$$



$$H = -t\sum_{<ij>} b_i^{\dagger} b_j + \frac{U}{2}\sum_i n_i(n_i - 1) - \sum_i (\mu - \varepsilon_i) n_i$$

Ground-state phases

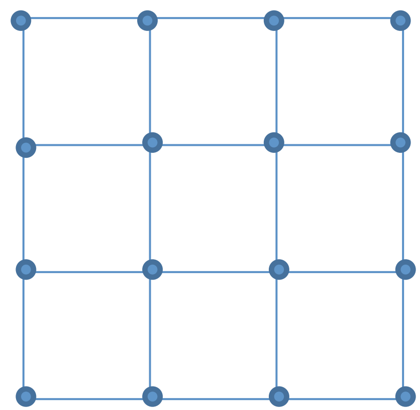


Superfluid

$\kappa \neq 0$ compressibility

$\Lambda \neq 0$ superfluid stiffness

$\Delta = 0, \quad t >> U$

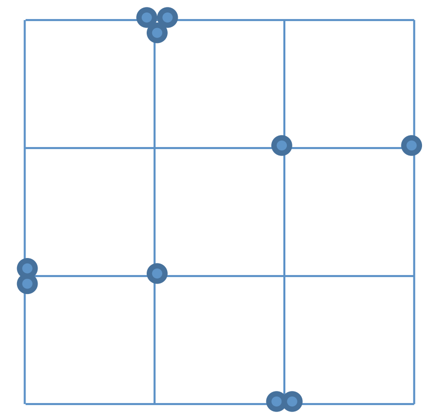


Mott-insulator $\langle n \rangle = 1$

$\kappa = 0$ Incompressible

$\Lambda = 0$ insulator

$\Delta = 0, \quad U >> t$



Bose Glass

$\kappa \neq 0$ Compressible

$\Lambda = 0$ insulator

$U, t << \Delta$

Theorem of Inclusions

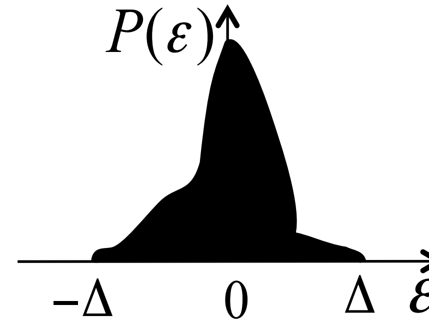
Mathematical background required

Lemma:

Product of a finite number of non-zero numbers is non-zero

(Anything is possible within the allowed bounds)

Theorem of Inclusions A

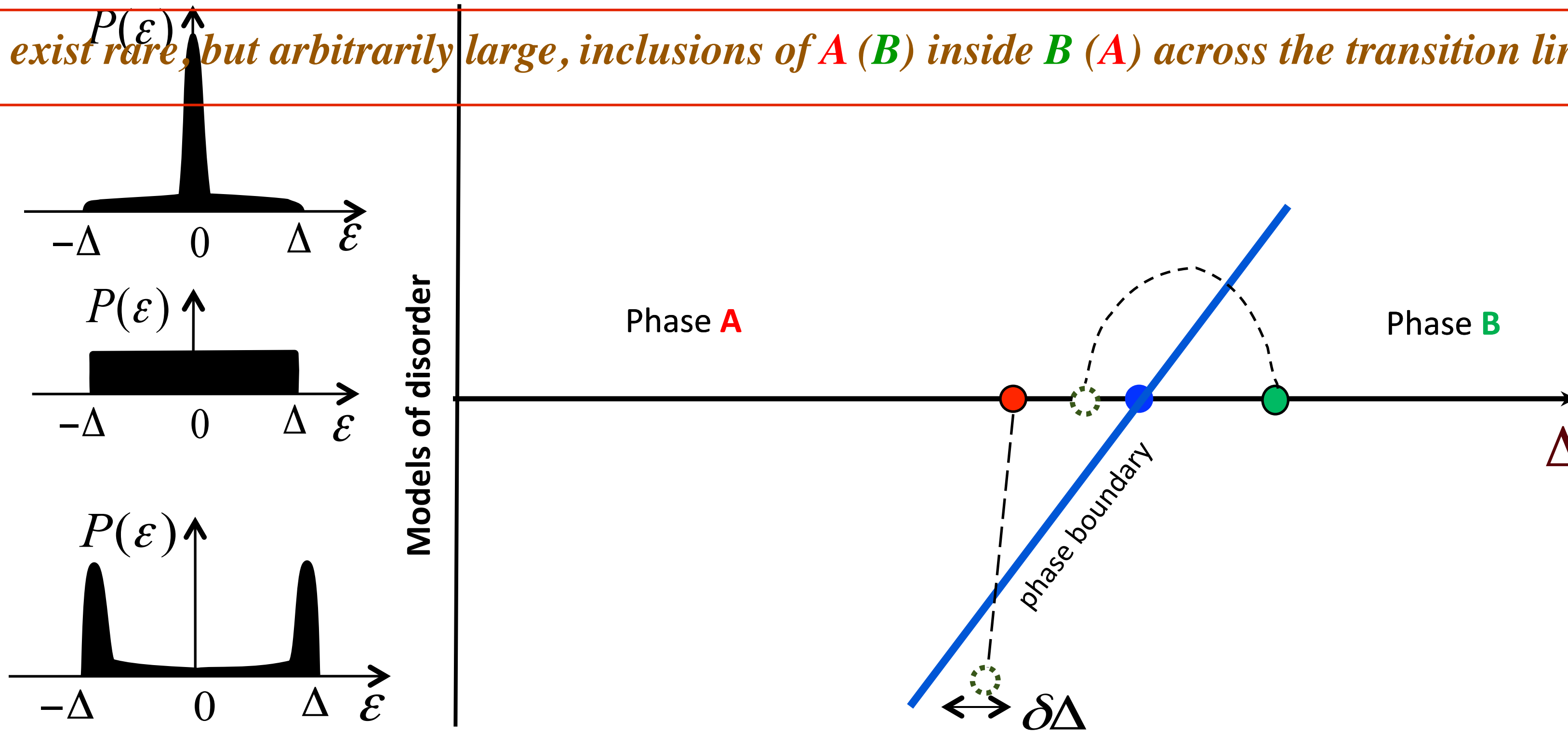


Def: Generic disorder = non-zero probability density for any

$$\varepsilon_i \in (-\Delta, \Delta)$$

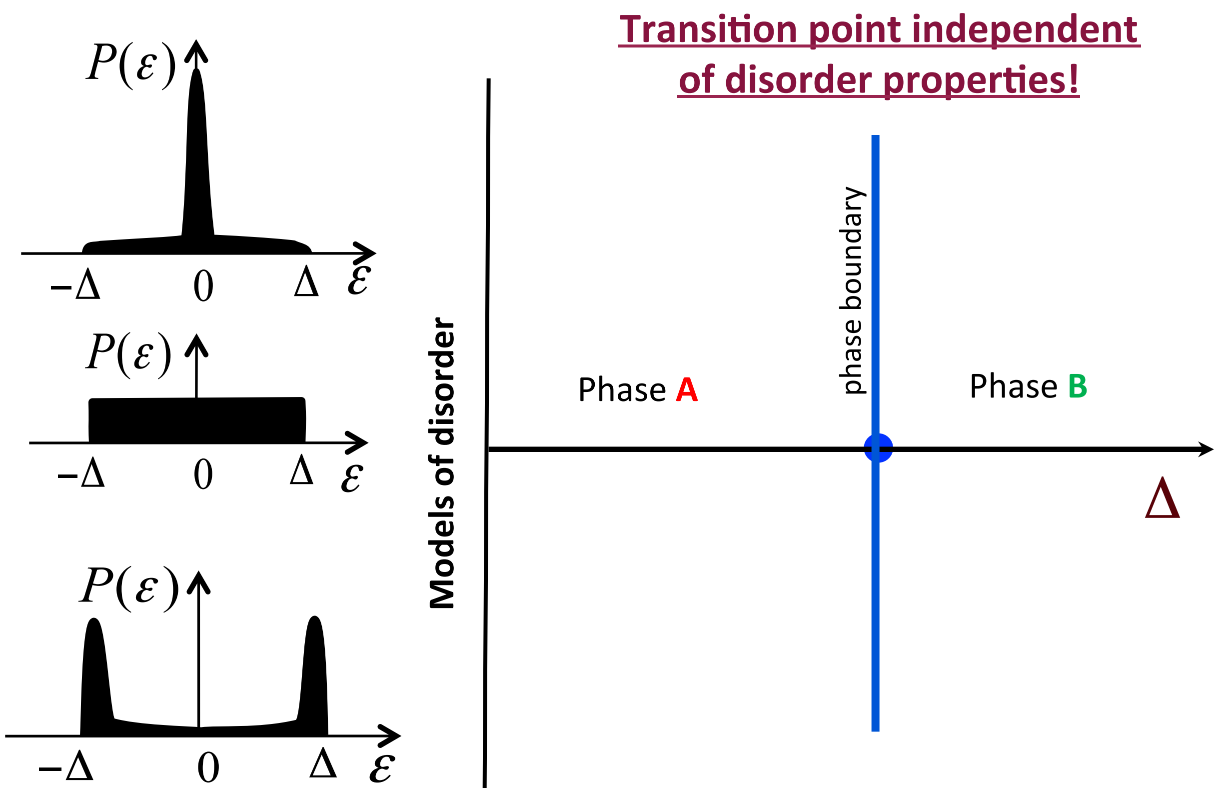
For an arbitrary transition in a system with generic disorder
<u>with transition point</u> Δ_C <u>depending on disorder properties</u>

there exist rare, but arbitrarily large, inclusions of A (B) inside B (A) across the transition line.



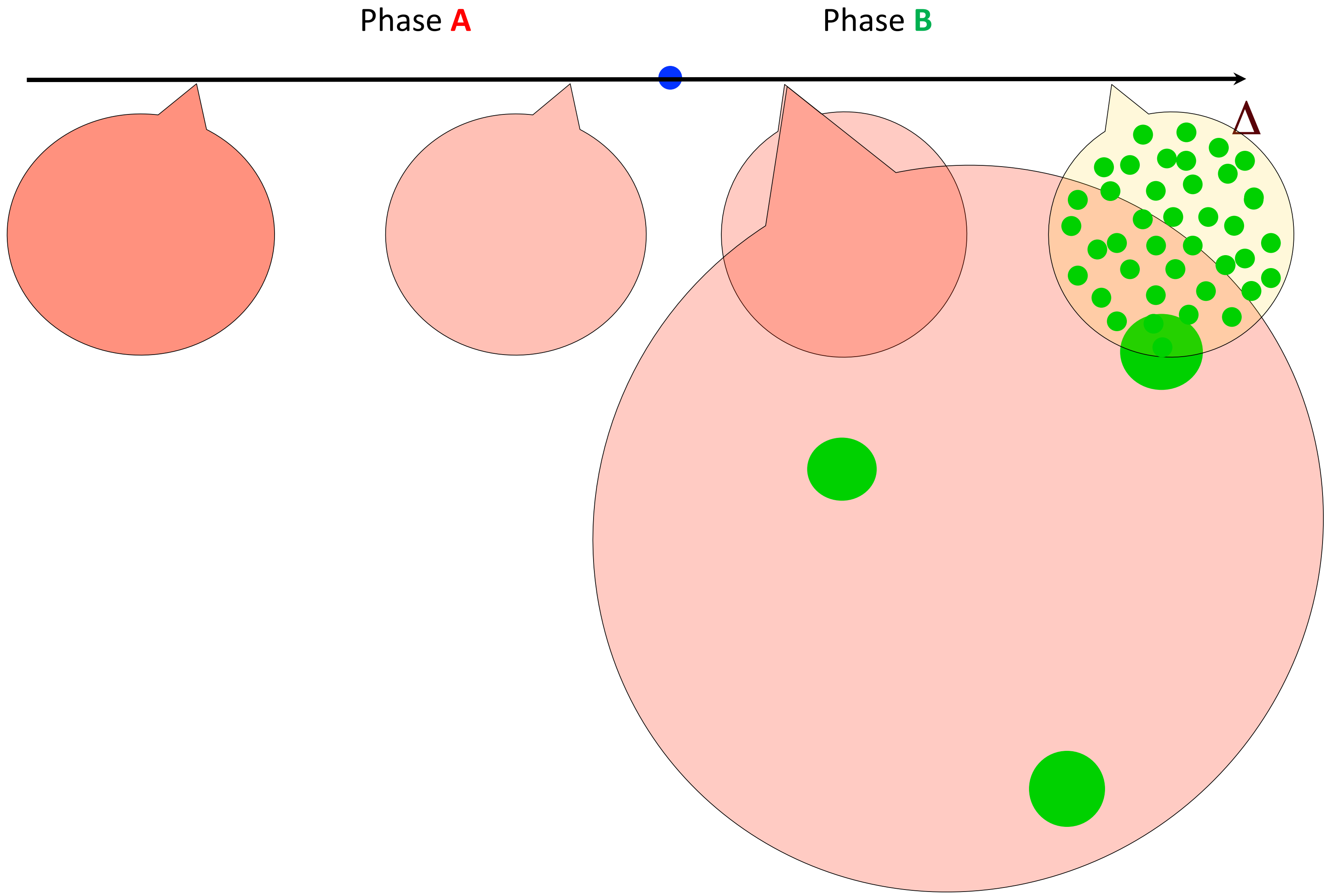
There exist rare, but arbitrarily large, inclusions of A (B) inside B (A) across the transition line.

Theorem of Inclusions B



Transition is driven by statistically rare fluctuations when locally disorder emulates a regular external field with amplitude Δ *, e.g.* $\varepsilon_i = \Delta = E_{MI\ GAP} / 2$

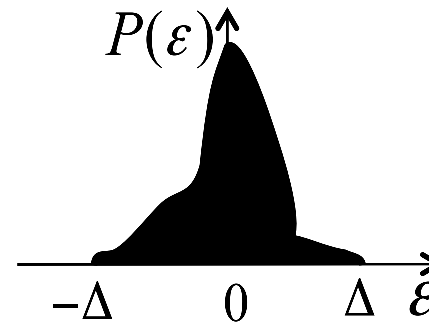
= Griffiths type transition


Phase A
Phase B
Δ

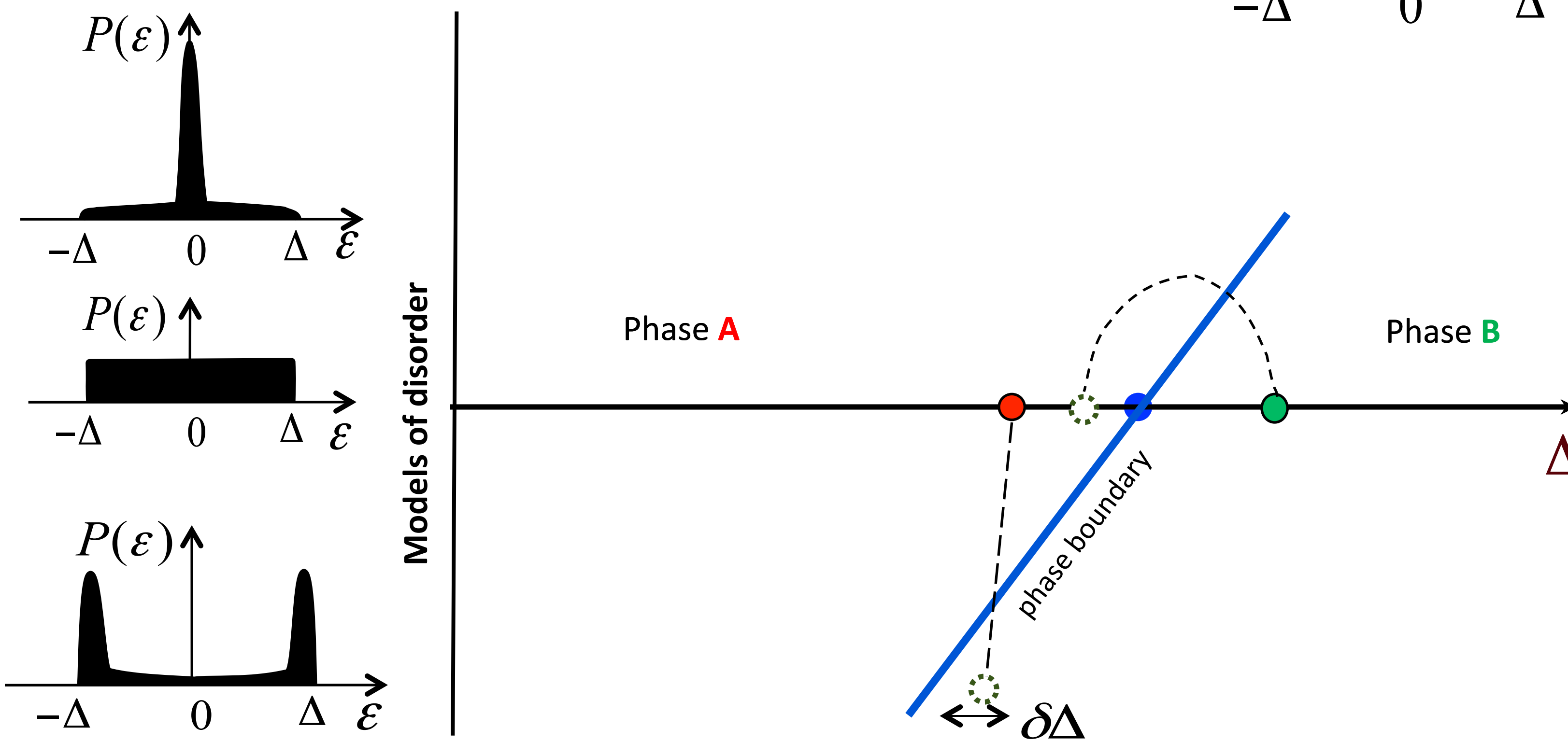
Theorem of Inclusions

Def: Generic disorder = non-zero probability density for any

$$\varepsilon_i \in (-\Delta, \Delta)$$



For an arbitrary transition in a system with generic disorder <u>with transition point</u> Δ_C <u>depending on disorder properties</u>



There exist rare, but arbitrarily large, inclusions of A (B) inside B (A) across the transition line.

Consequences:

- For **generic transitions**: if A is **gapless** then B is **gapless** too and vise versa .

- All transitions between **gapfull** and **gapless** phases are of the **Griffiths type.**

- All phases next to the **gapfull** one are **insulating**

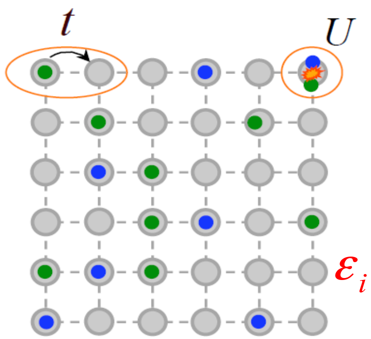
→ **SF-to-Mott insulator** transition is forbidden (any D) .
Same for (fully gapped) – (metallic) transitions in any system.

→ an intermediate phase (BG) must separate the two

- For **generic transitions**: if A is **superfluid** then B is

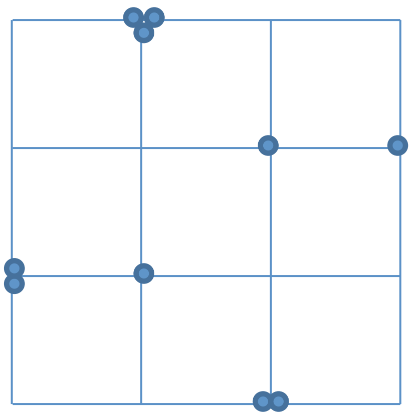
→ **compressible** (in the absence of particle-hole symmetry)

→ **gapless (possibly incompressible** in particle-hole symmetric case)



$$H = -t\sum_{<ij>} b_i^{\dagger} b_j + \frac{U}{2}\sum_i n_i (n_i - 1) - \sum_i (\mu - \varepsilon_i) n_i$$

Ground-state phases



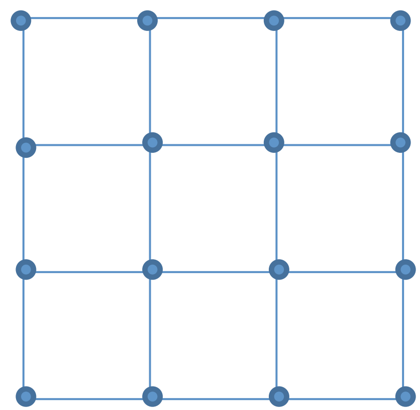
Superfluid

$\kappa \neq 0$ compressibility

$\Lambda \neq 0$ superfluid stiffness

$\Delta = 0, \quad t >> U$

gapless



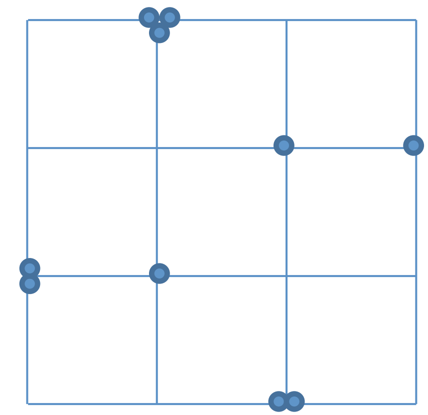
Mott-insulator

$\kappa = 0$ Incompressible

$\Lambda = 0$ Insulator

$\Delta = 0, \quad U >> t$

gapped



Bose Glass

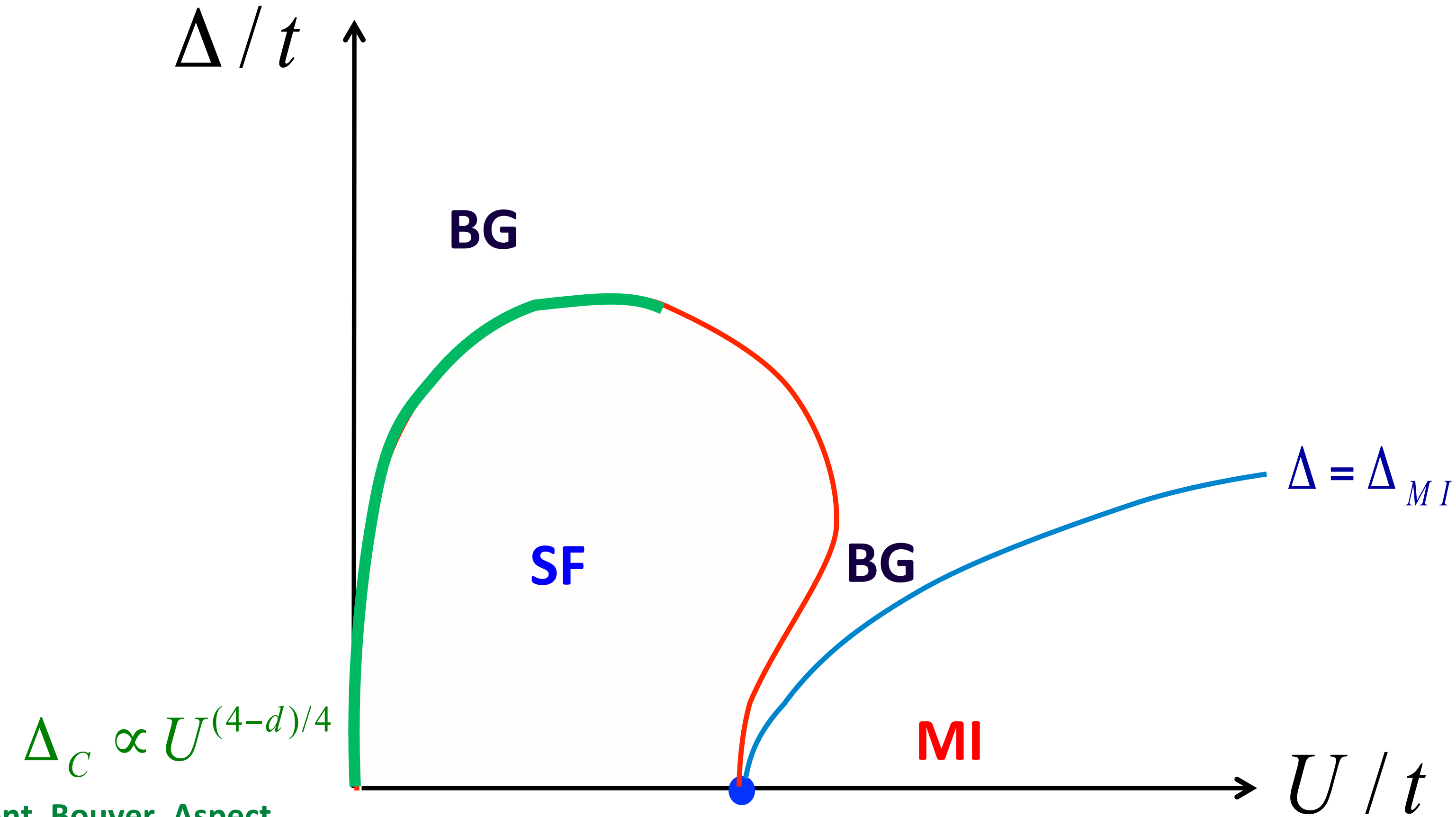
$\kappa \neq 0$ Compressible

$\Lambda = 0$ Insulator

$U, t << \Delta$

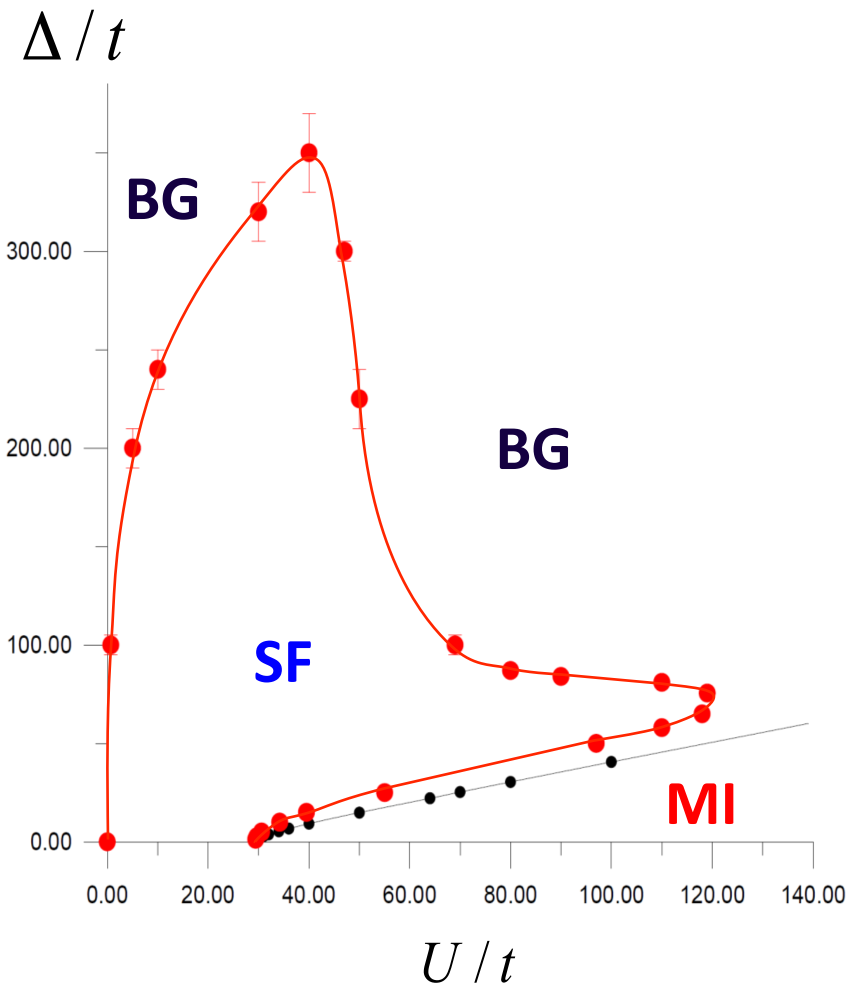
as it should be!

Bose Hubbard model with $\varepsilon_i \in (-\Delta, \Delta)$ at $\langle n_i \rangle = 1$ (or other integer filling)

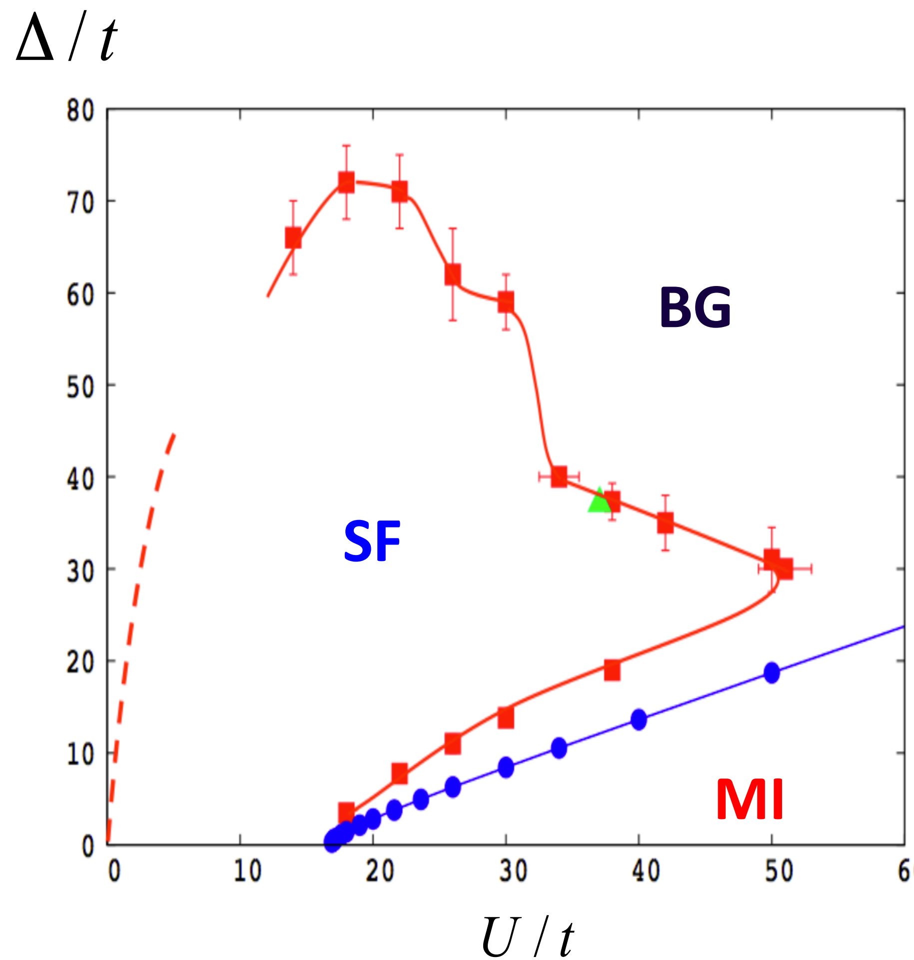


Lugan, Clement, Bouyer, Aspect, Lewenstein, Sanchez-Pelencia '07

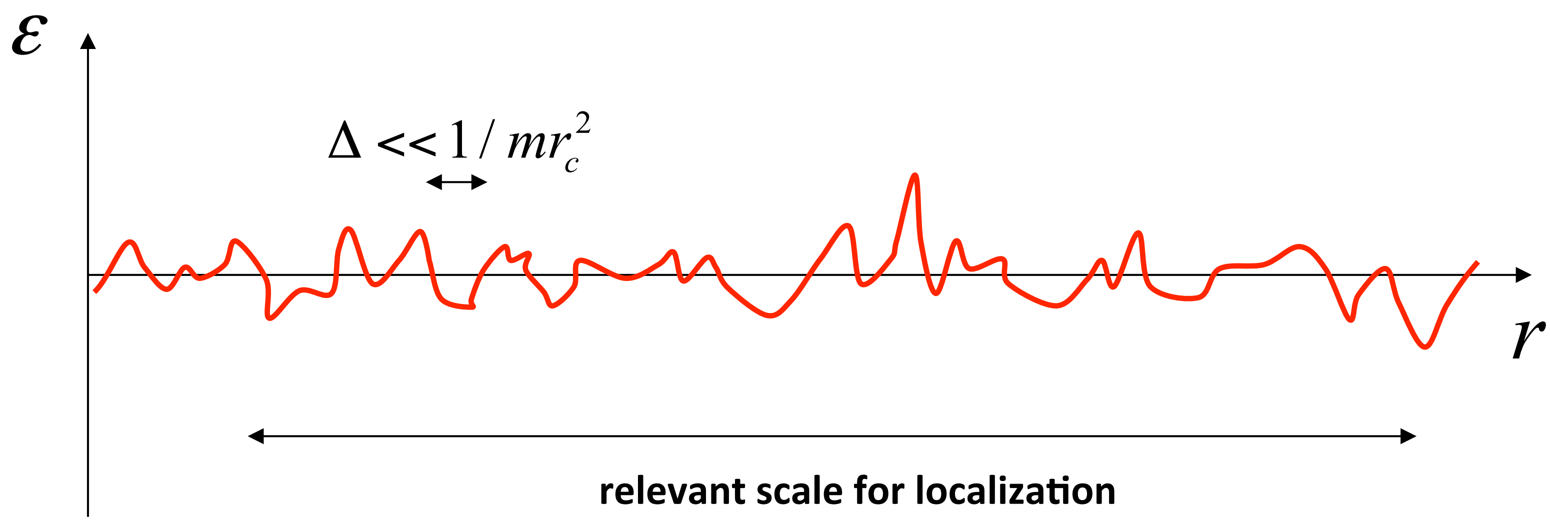
3D


Δ / t
BG
BG
SF
MI
300.00
200.00
100.00
0.00
0.00
20.00
40.00
60.00
80.00
100.00
120.00
140.00
U / t

2D


Δ / t
BG
SF
MI
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
U / t

Universal Gaussian distribution for weak disorder:



$$\lambda >> r_c$$

$$\varepsilon_\lambda = \lambda^{-d} \int_{\lambda^d} d^d r \, \varepsilon(r)$$ **(average over volume λ^d)**

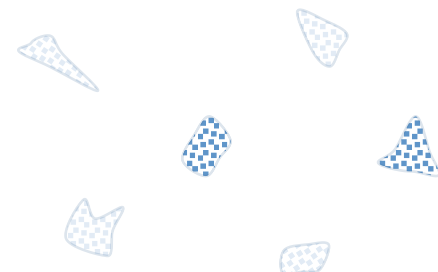
Central Limit Theorem $$P(\varepsilon_\lambda) \propto \exp\left\{-\varepsilon_\lambda^2 / 2\sigma_\lambda^2\right\}$$

$$\sigma_\lambda = \sigma_c (r_c / \lambda)^{d/2} \; : \; \Delta(a/\lambda)^{d/2}$$

Universal Gaussian distribution for weak disorder:

!!! Ideal gas is a "pathological" because all particles go to the deepest available well; density inside this well is infinite.

Finite interaction U prevents infinite densities and leads to a picture of isolated (locally "superfluid") lakes, or Bose Glass; not superfluid on a global scale.



Counting: $N = N_{lakes} \times N_{per\ lake}$

Global superfluidity condition: typical lakes localizing particles start to overlap at the Bose-glass --- superfluid quantum phase transition

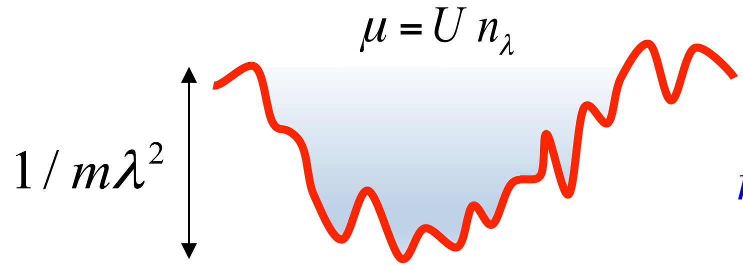
Universal Gaussian distribution for weak disorder:

$\lambda -$ wells with bound states inside them need to be deep enough : $\varepsilon_\lambda : \; 1/m\lambda^2 \quad (1)$

They occur with probability $\propto \exp\left\{-\varepsilon_\lambda^2 / 2\sigma_\lambda^2\right\} \propto \exp\left\{-\frac{\#}{m^2\Delta^2 a^d \lambda^{4-d}}\right\}$

They start to overlap when $\lambda \sim (m^2\Delta^2 a^d)^{1/(d-4)} \quad (2)$

Particle density in $\lambda -$ lakes:

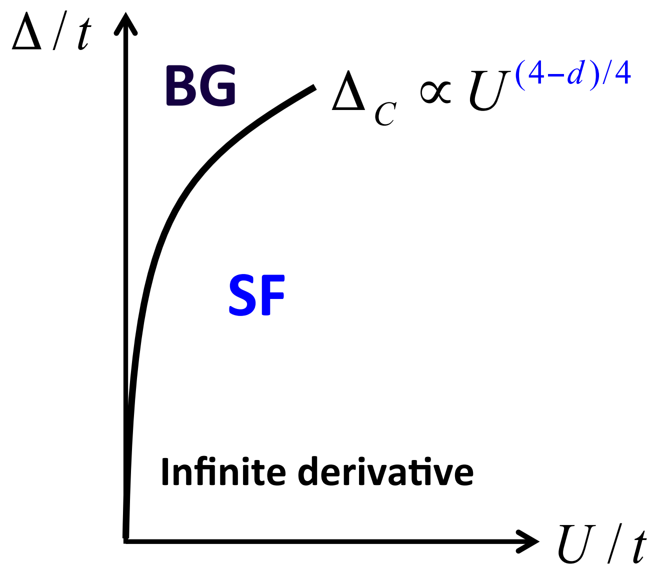


$n_\lambda \sim 1/mU\lambda^2 \sim n \quad (3)$

Global superfluidity condition: $n \sim \dfrac{(m^2\Delta^2 a^d)^{2/(4-d)}}{mU} \propto \boxed{\dfrac{\Delta^{4/(4-d)}}{U}}$

Bose Hubbard model with $\varepsilon_i \in (-\Delta, \Delta)$

$$H = -t\sum_{<ij>} b_i^\dagger b_j + \frac{U}{2}\sum_i n_i(n_i - 1) - \sum_i (\mu - \varepsilon_i) n_i$$



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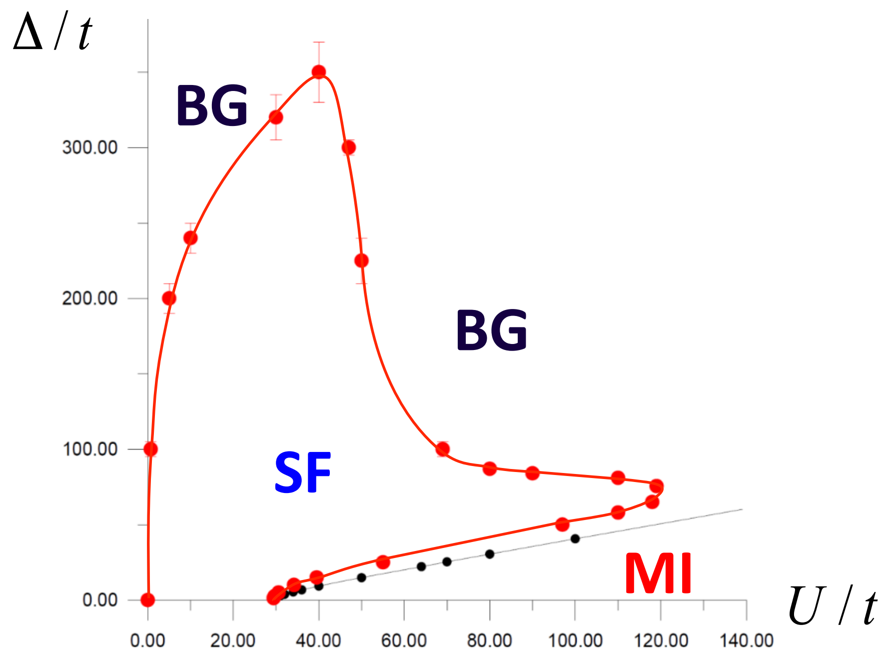
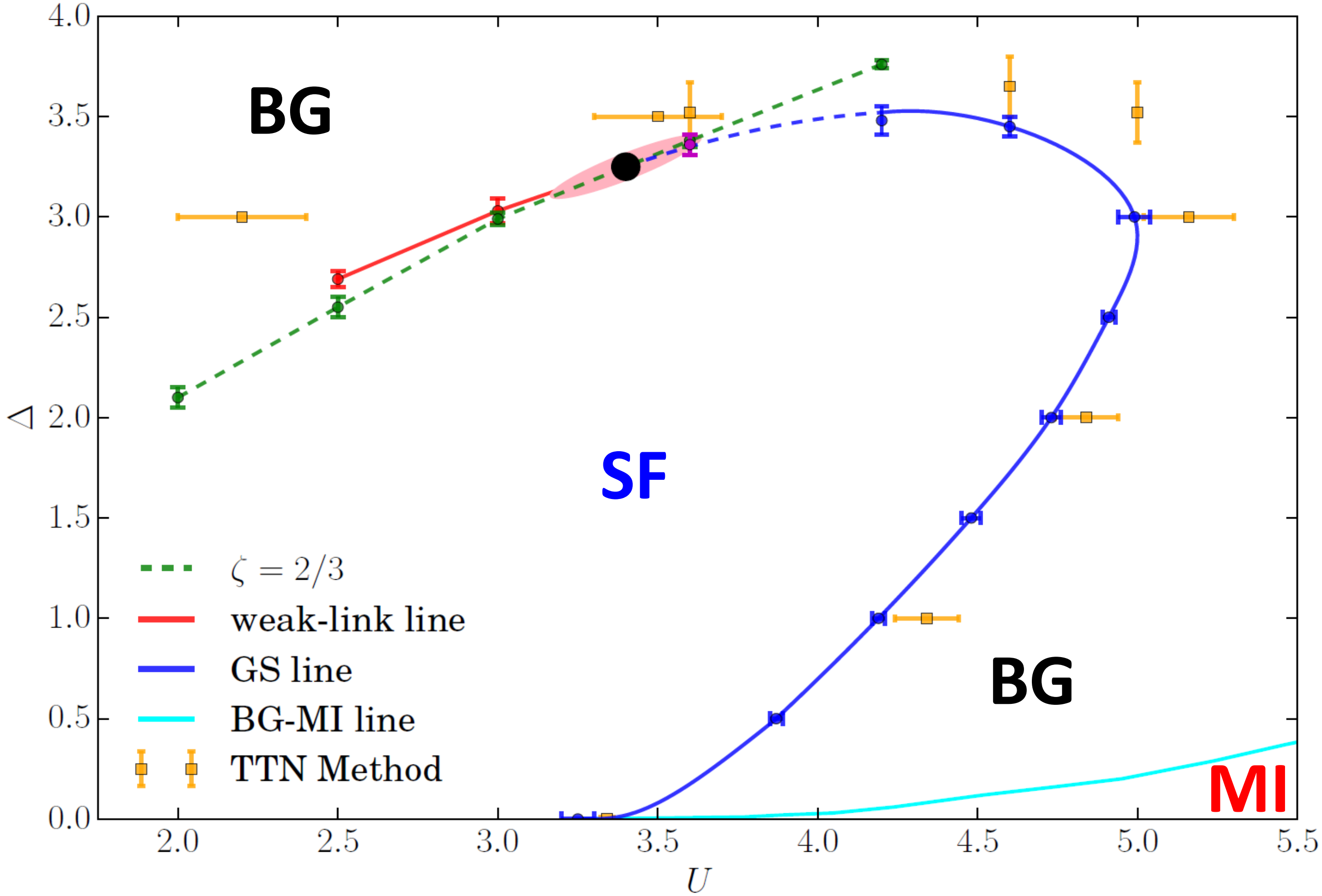


Interacting bosonic superfluids are extremely robust against disorder!

[Recall Anderson localization at the single-particle level: $\Delta_c / t \approx 2z = 16$, not 350]

Next lecture: 1D



● Tri-critical point separating sXY and GS lines