Cours au Collège de France - Février 2016

Towards the ultimate precision limits in parameter estimation: An introduction to quantum metrology

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Deuxième Leçon: Métrologie quantique, intrication et amplification de valeurs faibles

But de cette leçon

Dans cette leçon, on discute l'extension quantique de la théorie de Cramér-Rao-Fisher. Le rôle de l'intrication pour accroître la précision de l'estimation est discuté. La théorie générale, ainsi développée, est appliqué à l'interférométrie optique et atomique, et aussi à l'analyse de la méthode connue comme "amplification de valeurs faibles" (weak-value amplification).

Téléchargement de la première leçon: http://www.college-de-france.fr/site/jean-dalibard/guestlecturer-2016-02-04-11h00.htm

Rappel de la prémiere leçon: théorie classique de l'estimation de paramètres

H. Cramér

C. R. Rao

R.A. Fisher

Cramér-Rao bound for unbiased estimators:

$$\Delta X \geq 1/\sqrt{N\,F(X)\big|_{X=X_{\mathrm{true}}}}\,,\quad F(X) \equiv \sum_j P_j(X)\left(\frac{d\,\ln[P_j(X)]}{dX}\right)^2$$

Fisher information

$N \rightarrow$ Number of repetitions of the experiment

$P_j(X) \rightarrow$ probability of getting an experimental result j

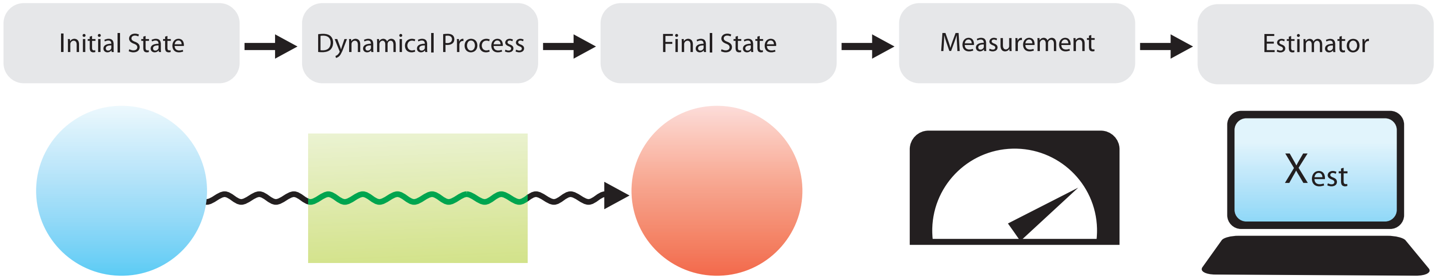
or yet, for continuous measurements: $F(X) \equiv \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2$

where ξ are the measurement results

(Average over all experimental results)

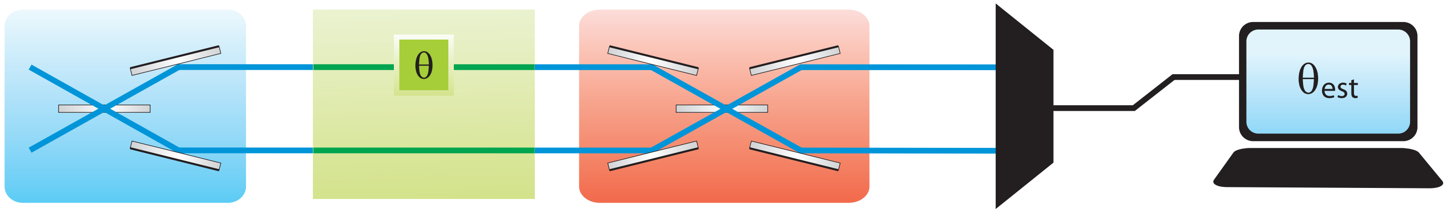
I.2 - Quantum parameter estimation

Quantum parameter estimation



The general idea is the same as before: one sends a probe through a parameter-dependent dynamical process and one measures the final state to determine the parameter. The precision in the determination of the parameter depends now on the distinguishability between quantum states corresponding to nearby values of the parameter.

Example: Optical interferometry



$$\left|\langle\alpha|\alpha e^{i\delta\theta}\rangle\right|^2 = \exp\left(-\left|\alpha\left(1-e^{i\delta\theta}\right)\right|^2\right)$$
$$\approx \exp\left[-\langle n\rangle(\delta\theta)^2\right] \Rightarrow \delta\theta \approx 1/\sqrt{\langle n\rangle}$$

→ Standard limit (shot noise)

(Coincides with the limit calculated in Lecture 1)

Possible method to increase precision for the same average number of photons: Use NOON states [J. J. Bolinguer et al., PRA **54**, R4649 (1996); J. P. Dowling, PRA 57, 4736 (1998)]

$$|\psi(N)\rangle = \left(|N,0\rangle + |0,N\rangle\right)/\sqrt{2} \to |\psi(N,\theta)\rangle = \left(|N,0\rangle + e^{iN\theta}|0,N\rangle\right)/\sqrt{2},\ \ (\langle n\rangle = N)$$

$$\left|\langle\psi(N)|\psi(N,\delta\theta)\rangle\right|^2 = \cos^2(N\delta\theta/2) \Rightarrow \delta\theta \approx 1/N$$

$$\left[\cos^2(N\delta\theta/2) = 0 \Rightarrow \delta\theta = \pi/N\right]$$

HEISENBERG LIMIT — Precision is better, for the same amount of resources (average number of photons)!

Quantum Cramér-Rao bound

The derivation of the Cramér-Rao bound is as before. But now the probability density for output ξ, given that the value of the parameter is X and the probe is in the state $|\psi(X)\rangle$, is

$$p(\xi|X) = \langle\psi(X)|\hat{E}(\xi)|\psi(X)\rangle$$

where the non-negative Hermitian operators $\hat{E}(\xi)$ describe a (generalized) measurement, that is, they are members of a **positive-operator valued measure** (POVM) — a set $\{\hat{E}(\xi)\}$ of Hermitian positive semi-definite operators in Hilbert space such that $\int d\xi\, \hat{E}(\xi) = \mathbf{1}$, implying that $\int d\xi\, p(\xi|X) = 1$, as expected. The elements of a POVM are not necessarily orthogonal, as is the case for Von Neumann projectors, so that the number of elements may be larger than the dimensionality of the space. One has, as before, for a given set $\{\hat{E}(\xi)\}$:

$$\sqrt{\langle(\Delta X_{\mathrm{est}})^2\rangle} \geq \frac{1}{\sqrt{NF(X)}}$$

with

$$F[X;\{\hat{E}(\xi)\}] = \int d\xi\, p(\xi|X)\left[\frac{\partial \ln p(\xi|X)}{\partial X}\right]^2 = \int d\xi \frac{1}{p(\xi|X)}\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2$$

Quantum Cramér-Rao bound (2)

The above bound corresponds to an optimization over estimators for a given quantum measurement. In order to get the ultimate lower bound for $\langle(\Delta X_{\rm est})^2\rangle$ one should still optimize over all quantum measurements. One gets then the Quantum Fisher Information:

$$\mathcal{F}_Q(X) = \max_{\{\hat{E}(\xi)\}} F[X;\{\hat{E}(\xi)\}]$$

so that

$$\sqrt{\langle(\Delta X_{\rm est})^2\rangle} \geq 1/\sqrt{N\mathcal{F}_Q(X)}$$

We show now that, for pure states, this maximization can actually be carried out analytically, yielding a simple expression for the quantum Fisher information.

Quantum Fisher information for pure states (1)

Consider a unitary process, the initial state of the probe is $|\psi(0)\rangle$, and the final X-dependent state is $|\psi(X)\rangle = \hat{U}(X)|\psi(0)\rangle$, where $\hat{U}(x)$ is a unitary operator. Define the auxiliary operator

$$\hat{h}(X) = -i\frac{d\hat{U}(X)}{dX}\hat{U}^\dagger(X)$$

so that

Like Schrödinger equation, with Hamiltonian $-\hat{h}(X)$

$$\frac{d|\psi(X)\rangle}{dX} = \frac{d\hat{U}(X)}{dX}|\psi(0)\rangle = \frac{d\hat{U}(X)}{dX}\hat{U}^\dagger(X)|\psi(X)\rangle = i\hat{h}(X)|\psi(X)\rangle$$

$\hat{h}(X)$—> generator of $\hat{U}(X)$

Let $p(\xi|X) = \langle\psi(X)|\hat{E}(\xi)|\psi(X)\rangle, \int d\xi\hat{E}(\xi) = \mathbf{1}$. Then

$$\frac{\partial p(\xi|X)}{\partial X} = \left[\frac{d}{dX}\langle\psi(X)|\right]\hat{E}(\xi)|\psi(X)\rangle + \langle\psi(X)|\hat{E}(\xi)\left[\frac{d}{dX}|\psi(X)\rangle\right]$$

$$= i\langle\psi(X)|[\hat{E}(X),\hat{h}(X)]|\psi(X)\rangle = -2\mathrm{Im}\left[\langle\psi(X)|\hat{E}(X)\hat{h}(X)|\psi(X)\rangle\right]$$

which may also be written as [with g(X) a real function]:

$$\frac{\partial p(\xi|X)}{\partial X} = -2\mathrm{Im}\left\{\langle\psi(X)|\hat{E}(X)\left[\hat{h}(X) - g(X)\right]|\psi(X)\rangle\right\}$$

Quantum Fisher information for pure states (2)

Squaring $\dfrac{\partial p(\xi|X)}{\partial X} = -2\mathrm{Im}\left\{\langle\psi(X)|\hat{E}(X)\left[\hat{h}(X) - g(X)\right]|\psi(X)\rangle\right\}$

one gets

$$\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2 = 4\mathrm{Im}^2\left\{\langle\psi(X)|\hat{E}(\xi)\left[\hat{h}(X) - g(X)\right]|\psi(X)\rangle\right\}$$
$$\leq 4\left|\langle\psi(X)|\hat{E}^{1/2}(\xi)|\hat{E}^{1/2}(\xi)\left[\hat{h}(X) - g(X)\right]|\psi(X)\rangle\right|^2$$
$$\leq \langle\psi(X)|\hat{E}(\xi)|\psi(X)\rangle\langle\psi(X)|\hat{E}(\xi)\left[\hat{h}(X) - g(X)\right]^2|\psi(X)\rangle$$

where in the last step we have used the Schwarz inequality. Therefore

$$\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2 \leq p(\xi|X)\langle\psi(X)|\hat{E}(\xi)\left[\hat{h}(X) - g(X)\right]^2|\psi(X)\rangle$$

Dividing by $p(\xi|X)$ and integrating with respect to ξ:

$$F(X) = \int d\xi\,\frac{1}{p(\xi|X)}\left[\frac{\partial p(\xi|X)}{\partial X}\right]^2 \leq 4\int d\xi\langle\psi(X)|\hat{E}(\xi)\left[\hat{h}(X) - g(X)\right]^2|\psi(X)\rangle$$
$$= 4\langle\psi(X)|\left[\hat{h}(X) - g(X)\right]^2|\psi(X)\rangle$$

since $\int d\xi\,\hat{E}(\xi) = \mathbf{1}$.

Quantum Fisher information for pure states (3)

The right-hand side of the expression $F(X) \leq 4\langle\psi(X)|\left[\hat{h}(X) - g(X)\right]^2|\psi(X)\rangle$ can be written in terms of the initial state $|\psi(0)\rangle$ by defining

$$\hat{H}(X) \equiv \hat{U}^\dagger(X)\hat{h}(X)\hat{U}(X) = i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$$

This looks like Hamiltonian in the Heisenberg picture

so that $F(X) \leq 4\langle\psi(0)|\left[\hat{H}(X) - g(X)\right]^2|\psi(0)\rangle$.

Note that, if $\hat{U}(X) = \exp(i\hat{\mathcal{O}}X)$, $\hat{\mathcal{O}}$ constant, then $\hat{H}(X) = \hat{\mathcal{O}}$. If $\hat{\mathcal{O}}$ is a Hamiltonian, then X is a time displacement, and $\hat{U}(X)$ is the evolution operator.

This bound attains its minimum value when $g(X) = \langle\psi(0)|\hat{H}(X)|\psi(0)\rangle \equiv \langle\hat{H}(X)\rangle_0$
Therefore, we find finally the upper bound for the Fisher information:

$$F(X) \leq 4\langle(\Delta\hat{H})^2\rangle_0\,, \qquad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

We show now that this upper bound is actually attained by a proper measurement, and therefore it coincides with the quantum Fisher information.

Quantum Fisher information for pure states (4)

We consider that the outgoing state is $|\psi(X')\rangle$, and the measurement defined by

$$E_1 = |\psi(X)\rangle\langle\psi(X)|\,, \quad E_2 = 1 - |\psi(X)\rangle\langle\psi(X)|$$

and show that the corresponding Fisher information attains the upper bound derived in the last slide when $X' \to X$. We have, in this case:

$$F_X(X') = \frac{1}{p_1(X')}\left[\frac{dp_1(X')}{dX'}\right]^2 + \frac{1}{p_2(X')}\left[\frac{dp_2(X')}{dX'}\right]^2,$$

$$p_1(X') = |\langle\psi(X')|\psi(X)\rangle|^2\,, \quad p_2(X') = 1 - p_1(X')\,.$$

Therefore, $F_X(X') = \dfrac{1}{p_1(X')[1-p_1(X')]}\left[\dfrac{dp_1(X')}{dX'}\right]^2$

Since $\lim_{X'\to X} p_1(X) = 1$ and $\lim_{X'\to X}[dp_1(X)/dX] = 0$, the limit $X' \to X$ of this expression is indeterminate.

Using l'Hôpital's rule, one gets:

$$\lim_{X'\to X} F_X(X') = -2\left[\frac{d^2 p_1(X')}{dX'^2}\right]_{X'\to X} = 4\langle\psi(0)|(\Delta\hat{H})^2|\psi(0)\rangle$$

where, as before, $\hat{H}(X) \equiv i\frac{d\hat{U}^\dagger(X)}{dX}\hat{U}(X)$.

This is precisely the upper bound found before!

Quantum Fisher information for pure states (5)

Therefore, for pure states,

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{H})^2\rangle_0\,, \quad \langle(\Delta\hat{H})^2\rangle_0 \equiv \langle\psi(0)|\left[\hat{H}(X) - \langle\hat{H}(X)\rangle_0\right]^2|\psi(0)\rangle$$

From the definition of $\hat{H}(X)$ and from the above expression, it follows that the quantum Fisher information can also be written as

$$\mathcal{F}_Q(X) = 4\left[\frac{d\langle\psi(X)|}{dX}\frac{d|\psi(X)\rangle}{dX} - \left|\frac{d\langle\psi(X)|}{dX}|\psi(X)\rangle\right|^2\right]$$

This expression is very useful, and it will be used a few times in these lectures.

Example 1: Optical interferometry

$\hat{n} = \hat{a}^\dagger a \rightarrow$ Generator of phase displacements $|\alpha\rangle \rightarrow |\alpha \exp(i\theta)\rangle$

$\Rightarrow \mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0$ where $\langle(\Delta\hat{n})^2\rangle_0$ is the photon-number variance in the upper arm.

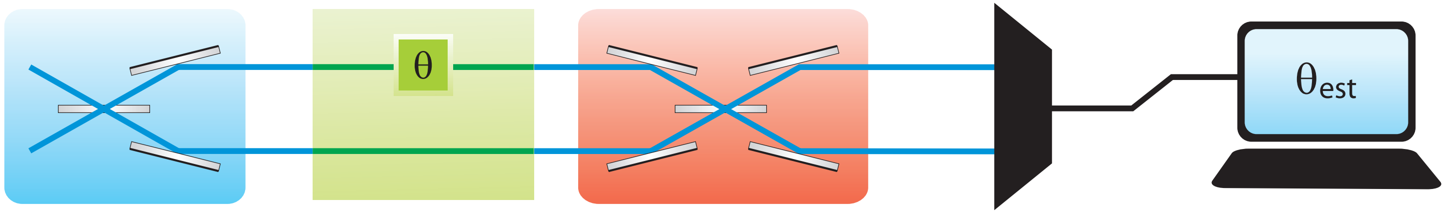
$\Rightarrow \delta\theta \geq \dfrac{1}{2\sqrt{\langle(\Delta\hat{n})^2\rangle}} \quad (\nu = 1)$ $\quad \nu \rightarrow$ Number of repetitions

Standard limit: coherent states

$$\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 = 4\langle\hat{n}\rangle \Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle n\rangle}}$$

This lower bound is better by a factor of two than the bound found before, which was $\delta\theta_{\min} = 1/\sqrt{\langle n\rangle}$. This earlier bound corresponds to comparing the displaced-phase coherent state in the upper arm of an interferometer with an undisplaced coherent state with the same amplitude in the other arm. The result found here indicates that a better measurement of the phase is possible: indeed, a homodyne measurement allows the comparison of the displaced coherent state with a classical reference field (local oscillator), which is just a coherent state with a number of photons much larger than that of the measured state — this yields a better precision in the estimation of the phase.

Example 1: Optical interferometry



Increasing the precision: maximize variance with NOON states:

$$|\psi(N)\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$$ —> entangled state

$$\mathcal{F}_Q(\theta) = 4\langle(\Delta\hat{n})^2\rangle_0 \Rightarrow \delta\theta \geq \frac{1}{2\sqrt{\langle(\Delta\hat{n})^2\rangle}} \quad (\nu = 1)$$

$$\left\langle(\Delta\hat{n})^2\right\rangle_0 = \frac{N^2}{4} \Rightarrow \delta\theta \geq \frac{1}{N}$$

Example 2: Spatial displacement

$$|\psi(X)\rangle = e^{iX\hat{P}}|\psi(0)\rangle \Rightarrow \hat{H} = i\frac{d\hat{U}^\dagger}{dX}\hat{U}(X) = \hat{P}$$

$$\mathcal{F}_Q(X) = 4\langle(\Delta\hat{P})^2\rangle_0 \Rightarrow \langle(\Delta X)^2\rangle \geq \frac{1}{4\langle(\Delta\hat{P})^2\rangle}$$

Coherent state: $\langle(\Delta\hat{P})^2\rangle_0 = 1/2 \Rightarrow \langle(\Delta X)^2\rangle = 1/2$ —> standard quantum limit — coherent state saturates Cramér-Rao bound

Maximizing variance of P for better precision: e.g., squeezed states —> Also saturate the bound (Gaussian states)

Looks like Heisenberg uncertainty relation, but X is a parameter, not an operator!

Rappel sur l'intrication

Consider a multipartite system S of N particles. The state of the system is defined in a Hilbert space resulting from the tensor product of the N individual Hilbert spaces of the subsystems:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$$

A pure state describing a system with many parts is said to be separable if and only if it can be written as the product of the states of each part:

$$|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$$

This means that it is possible to assign a state vector to each subsystem: this implies that one has full information about each part. Otherwise, the state is said to be entangled. The most general state in this space can be written as

$$|\Psi\rangle = \sum_{j_1 \cdots j_N} a_{j_1 \cdots j_N} |j_1\rangle \otimes \cdots \otimes |j_N\rangle \equiv \sum_{j_1 \cdots j_N} a_{j_1 \cdots j_N} |j_1 \cdots j_N\rangle$$

where $|j_i\rangle$, with $0 \leq j_i \leq d_i - 1$, is an orthonormal basis of $\mathcal{H}_i$ (dimension d_i). This is not necessarily a product of vectors belonging to the subspaces $\mathcal{H}_i$.

Examples of entangled states (two qubits): Bell states

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \qquad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

In this example, one has maximal ignorance on the state of each qubit - these are maximally entangled states.

Schrödinger on entanglement

Naturwissenschaften
23, 807 (1935)

"This is the reason that knowledge of the individual systems can decline to the scantiest, even zero, while that of the combined system remains continually maximal. Best possible knowledge of a whole does not include best possible knowledge of its parts – and that is what keeps coming back to haunt us."

Possible strategies for quantum-enhanced metrology (1)

Single probe

Recall that $\mathcal{F}_Q(|\psi\rangle) = 4\langle(\Delta\hat{H})^2\rangle$ so in order to increase the precision one needs to choose a state $|\psi\rangle$ that maximizes the variance $\langle(\Delta\hat{H})^2\rangle$. If $\hat{H}$ has a discrete and bounded spectrum, this is accomplished by letting

$$|\psi\rangle_{\mathrm{opt}} = \frac{1}{\sqrt{2}}\left(|\lambda_{\max}\rangle + |\lambda_{\min}\rangle\right)$$

where $|\lambda_{\max}\rangle$ and $|\lambda_{\min}\rangle$ are eigenstates of $\hat{H}$ corresponding to the maximum and minimum eigenvalues.

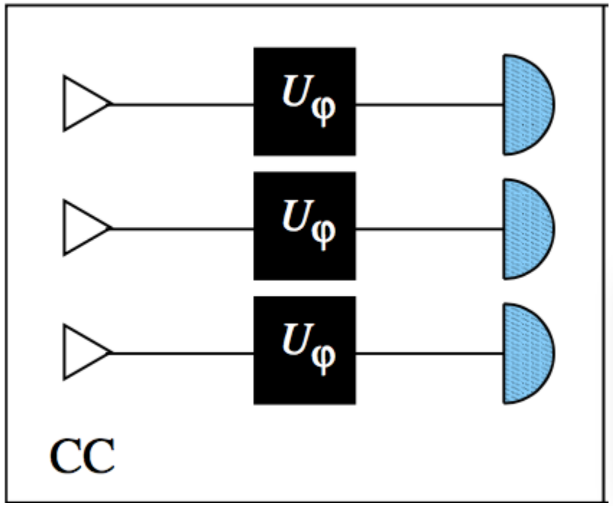
Then $\langle(\Delta\hat{H})^2\rangle = (\lambda_{\max} - \lambda_{\min})^2/4$ and

$$\Delta X_{(1)} \geq \frac{1}{\sqrt{\nu}\left(\lambda_{\max} - \lambda_{\min}\right)}$$

(ν —> number of repetitions of single probe experiment)

Question: What is the best strategy if one has N probes?

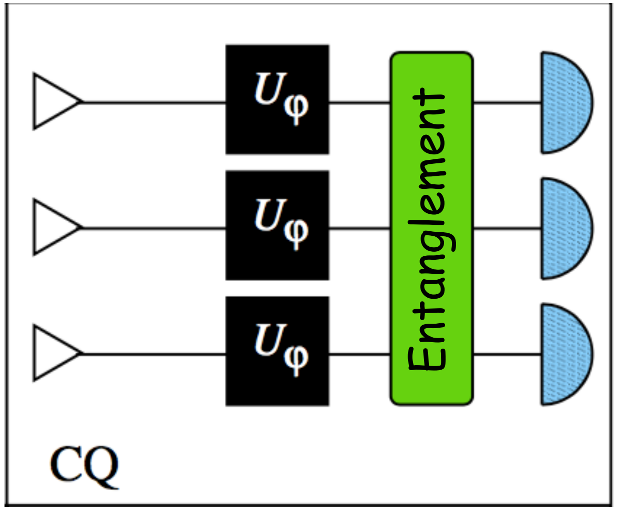
Possible strategies for quantum-enhanced metrology (2)



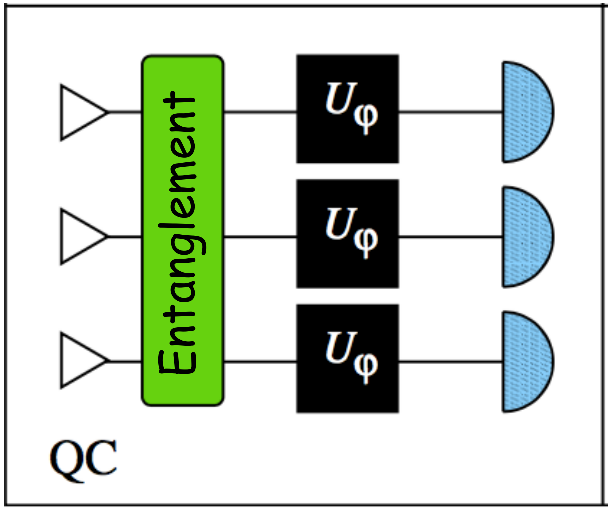
Separable input states, separable measurements

N probes

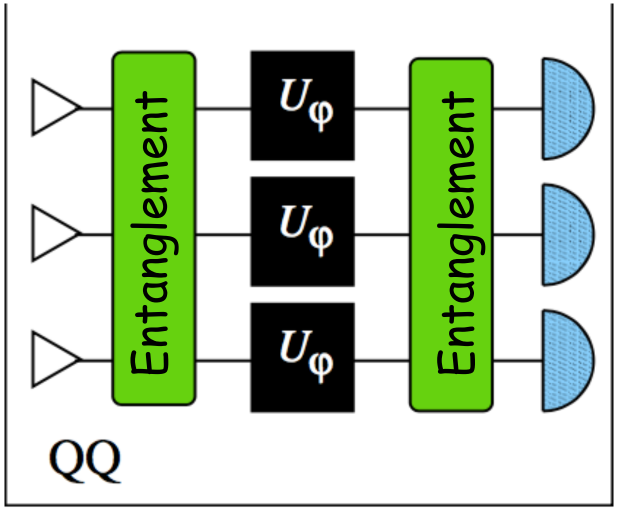
V. Giovannetti, S. Lloyd, and L. Macone, PRL **96**, 010401 (2006)



Separable input states, general measurement schemes (with entanglement)

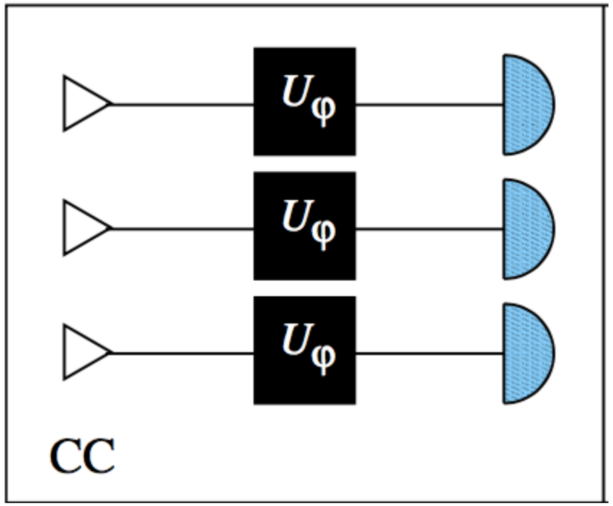


General input states (with entanglement), separable measurements



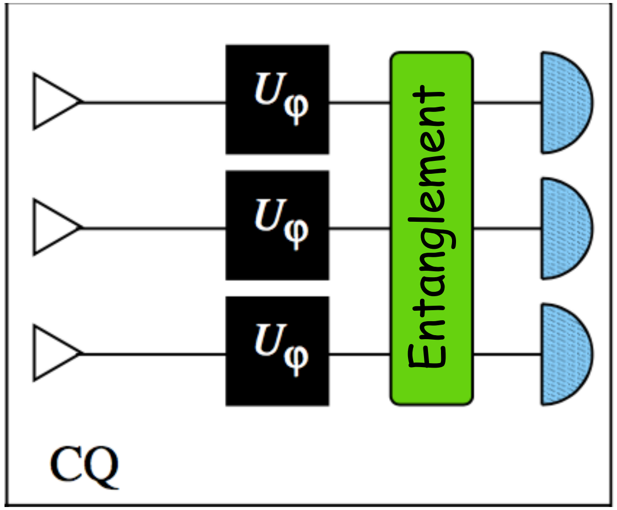
General input states, general measurement schemes (with entanglement)

Possible strategies for quantum-enhanced metrology (3)



Separable input states, separable measurements

N probes



Separable input states, general measurement schemes (including entanglement)

$$\hat{U}_{(N)}(X) = \hat{U}(X)^{\otimes N} \quad \hat{\mathcal{H}} = \sum_{j=1}^{N} \hat{H}_j \longrightarrow \text{generators of } \hat{U}(X)$$

Product initial state: $\left\langle \Delta \hat{\mathcal{H}}^2 \right\rangle = \sum_{j=1}^{N} \left\langle \Delta \hat{H}_j^2 \right\rangle_{|\psi_j\rangle}$

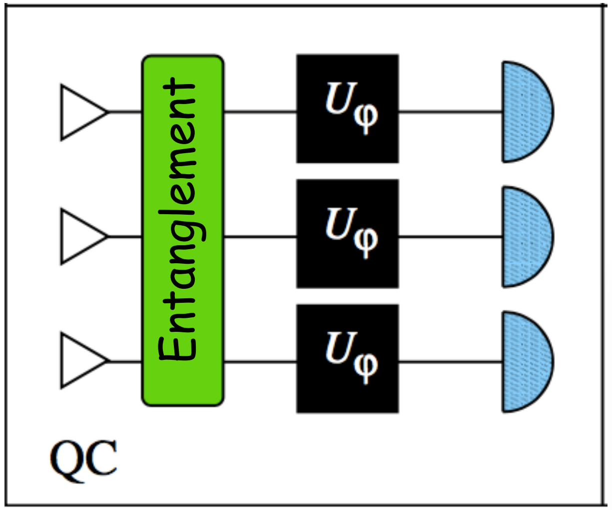
$$\left|\Psi\right\rangle_{\rm opt} = \left|\psi\right\rangle_{\rm opt}^{(1)} \otimes \left|\psi\right\rangle_{\rm opt}^{(2)} \otimes \cdots \otimes \left|\psi\right\rangle_{\rm opt}^{(N)} \rightarrow \left\langle \Delta \hat{\mathcal{H}}^2 \right\rangle = N(\lambda_{\max} - \lambda_{\min})^2/4$$

Therefore

$$\Delta X_{(N)} \geq \frac{1}{\sqrt{\nu N}\,(\lambda_{\max} - \lambda_{\min})} = \frac{\Delta X_{(1)}}{\sqrt{N}}$$

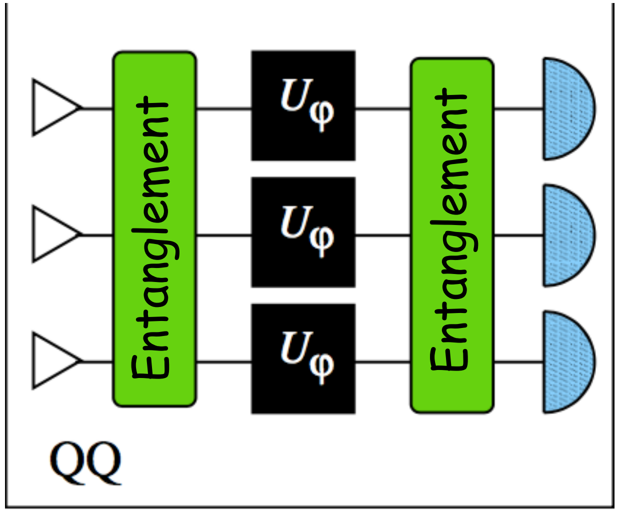
Possible strategies for quantum-enhanced metrology (4)

N probes



General input states, separable measurements

Entanglement of initial state is necessary for going beyond shot-noise scaling.



General input states, general measurement schemes

$$\hat{U}_{(N)}(X) = \hat{U}(X)^{\otimes N} \quad \hat{\mathcal{H}} = \sum_{j=1}^{N} \hat{H}_j$$

Maximization of variance $\langle(\Delta\hat{\mathcal{H}})^2\rangle$:

$$|\Psi\rangle_{\rm opt} = \frac{1}{\sqrt{2}}\left(|\lambda_{\max}\rangle_1 \otimes |\lambda_{\max}\rangle_2 \otimes \cdots \otimes |\lambda_{\max}\rangle_N + |\lambda_{\min}\rangle_1 \otimes |\lambda_{\min}\rangle_2 \otimes \cdots \otimes |\lambda_{\min}\rangle_N\right)$$

$$\langle(\Delta\hat{\mathcal{H}})^2\rangle = N^2(\lambda_{\max} - \lambda_{\min})^2/4$$

Therefore: $$\Delta X_{(N)} \geq \frac{1}{N\sqrt{\nu}\,(\lambda_{\max} - \lambda_{\min})} = \frac{\Delta X_{(1)}}{N}$$

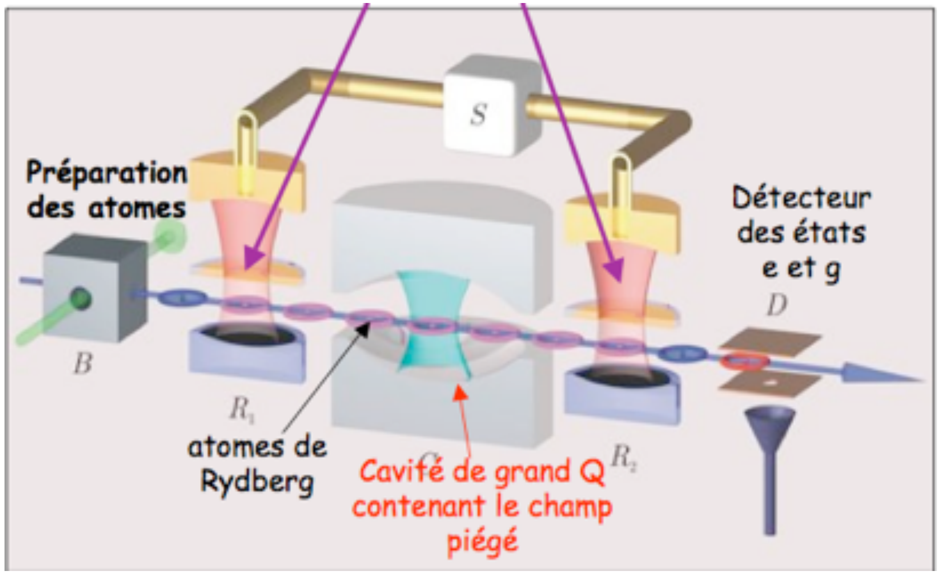
$1/\sqrt{N}$ gain!

—> Heisenberg limit

Entanglement-assisted parameter estimation: atomic spectroscopy

1. Separable qubits. Prepare N qubits in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The evolution of each qubit is given by $|0\rangle \to |0\rangle\,,\ |1\rangle \to \exp(i\phi)|1\rangle$. Therefore, the state $|+\rangle$ evolves into

$$|+\rangle \to |\phi\rangle \equiv (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$$



We must now choose a proper measurement to estimate ϕ. We choose the one associated with the Pauli $\hat{\sigma}_x$ operator (and show that this is the best one!). The measurement of $\hat{\sigma}_x$ has two possible outcomes, ± 1 , with probabilities

$$p(\pm 1|\phi) = |\langle \pm|\phi\rangle|^2 = (1 \pm \cos\phi)/2\,,\ \ |\pm\rangle = (|0\rangle \pm |1\rangle/)\sqrt{2}$$

The Fisher information for this measurement is thus given by

$$F(\phi) = \sum_{\pm 1} p^{-1}(\pm 1|\phi)\,[\partial p(\pm 1|\phi)/\partial\phi]^2 = 1.$$

However, we know that for the best measurement $\mathcal{F}_Q(\phi) = 4\langle(\Delta\hat{H})^2\rangle_0$, where $\hat{H}$ here is the generator of phase displacements: $\hat{H} = (1 + \hat{\sigma}_z)/2$. Since for the initial state $|+\rangle$ we have $\langle(\Delta\hat{H})^2\rangle_0 = 1/4$, it follows that the measurement of $\hat{\sigma}_x$ maximizes the Fisher information, leading to the corresponding Cramér-Rao bound in $\delta\phi \geq 1/\sqrt{N\mathcal{F}_Q(\phi)} = 1/\sqrt{N}$, the so-called standard limit.

Entanglement-assisted parameter estimation: atomic spectroscopy (2)

2. Entangled qubits. Now N qubits form a GHZ-like state, with the same evolution as before, $|0\rangle \to |0\rangle\,,\ |1\rangle \to \exp(i\phi)|1\rangle$,
$|\psi_N(0)\rangle = |+_N\rangle \equiv (|\bar{0}\rangle + |\bar{1}\rangle)/\sqrt{2}$, where $|\bar{0}\rangle = |0, 0 \cdots, 0\rangle,\ |\bar{1}\rangle = |1, 1 \cdots, 1\rangle$,
and we define also $|-_N\rangle \equiv (|\bar{0}\rangle - |\bar{1}\rangle)/\sqrt{2}$. After the evolution, the initial state becomes $|\psi_N(\phi)\rangle = [|\bar{0}\rangle + \exp(iN\phi)|\bar{1}\rangle]/\sqrt{2}$.

In order to estimate the phase, we choose the observable

$$\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} \cdots \otimes \hat{\sigma}_x^{(N)}$$

with eigenvectors $|\pm_N\rangle$ corresponding to the eigenvalues ± 1, so that
$p(\pm 1|\psi(\phi)\rangle = |\langle \pm_N|\psi(\phi)\rangle|^2 = (1 \pm \cos N\phi)/2$
which leads to the Fisher information $F(\phi) = \sum_{\pm 1} \frac{1}{p(\pm 1|\phi)} \left[\frac{\partial p(\pm 1|\phi)}{\partial \phi}\right]^2 = N^2$.

The generator of phase displacements is $\hat{H} = \sum_{i=1}^N \left(1 + \hat{\sigma}_z^{(i)}\right)/2$, so that $\langle\psi(0)|(\Delta\hat{H})^2|\psi(0)\rangle = N^2/4$, which means that the above measurement leads to the maximum value of the Fisher information and to the Cramér-Rao bound in $\delta\phi \geq 1/\sqrt{\mathcal{F}_Q(\phi)} = 1/N$, the Heisenberg limit. Note that the higher precision for the same N was obtained by entangling the qubits and making local measurements of $\sigma_x^{(i)}$ on the outgoing state.

Recent experimental result

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Measurement noise 100 times lower than the quantum-projection limit using entangled atoms

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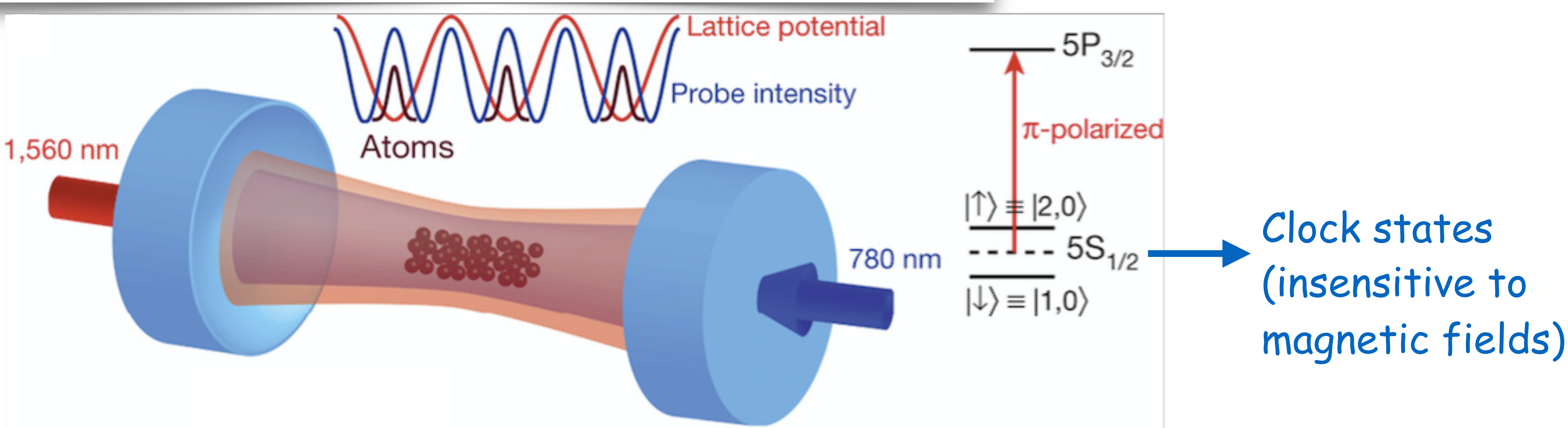
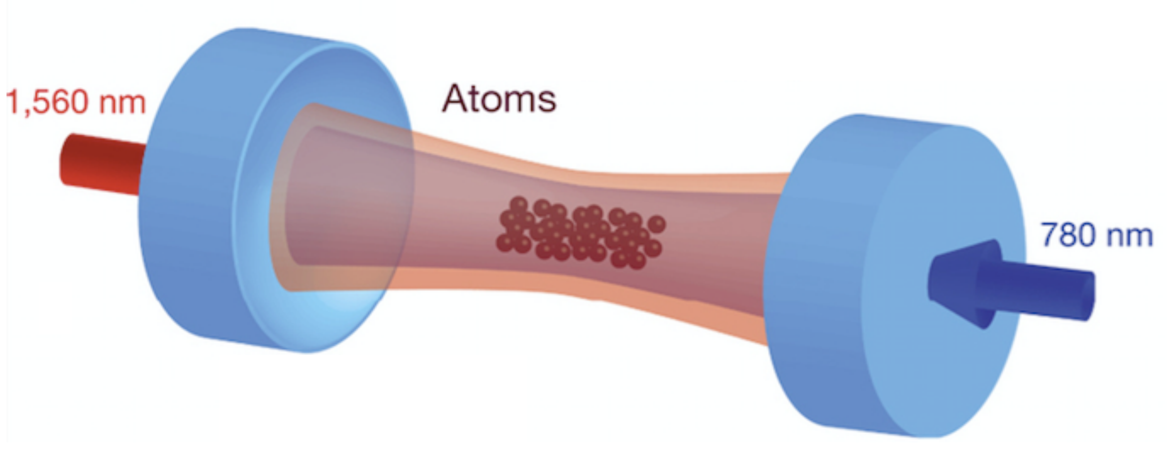
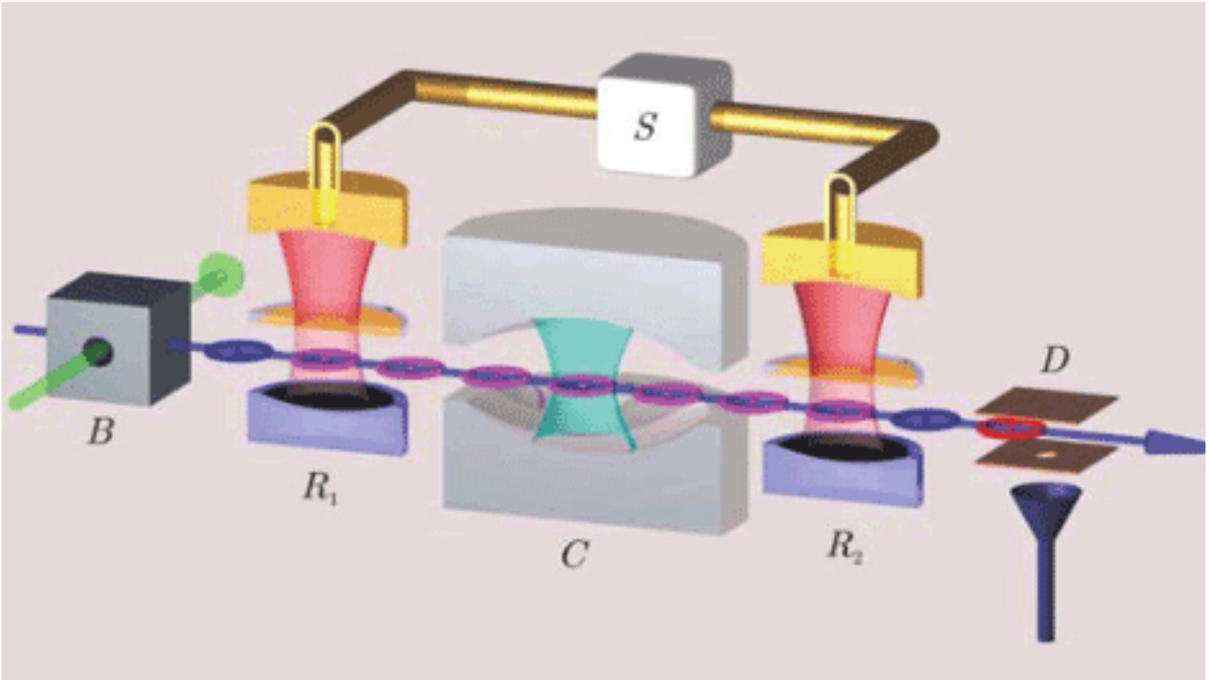


Rb^{87} atoms are trapped at the maxima of the probe intensity profile by the 1,560 nm lattice. The 780 nm probe light, which is uniformly coupled to the atoms, is detuned by equal and opposite amounts from the two clock states. Change in the frequency of the probe field allows a collective population difference measurement on the atom — the frequency shift of the cavity resonance is a direct predictor of J_z. This is a quantum non-demolition (QND) measurement of J_z (no atomic transitions, since the coupling is dispersive), which projects the quantum state into one with a narrower distribution of J_z than that of a coherent spin state.

QND measurements of atoms and fields

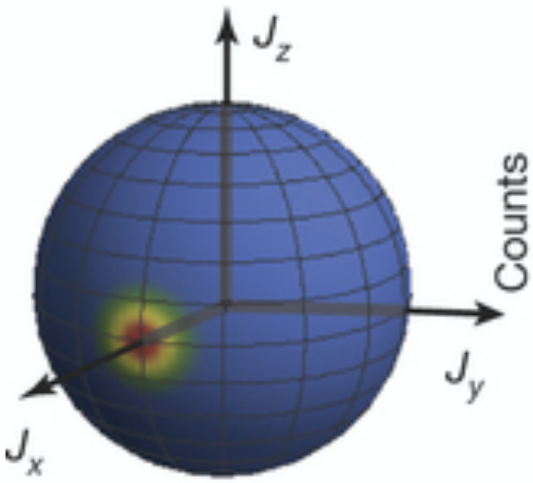


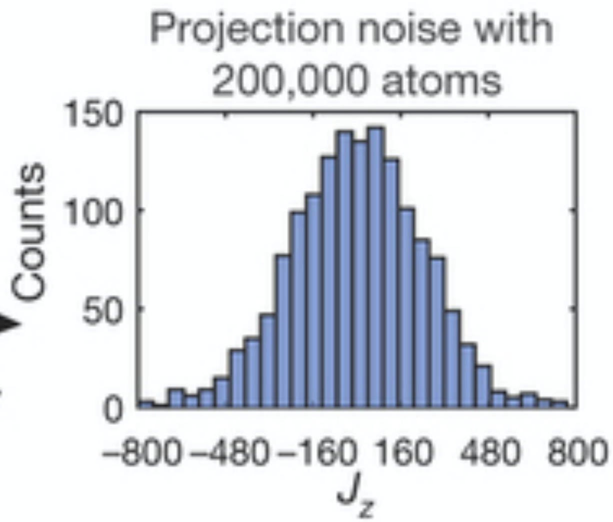
Using a field to make a QND measurement of the collective atomic state —> leads to squeezed atomic state



Using atoms to make a QND measurement of the field (ENS) —> leads to sub-Poissonian field, eventually to a Fock state of the field.

Preparation of an atomic coherent state





Apply a $\pi/2$ microwave pulse to the atoms, initially in the ground state. Resulting state is not entangled:

$$\left(\frac{|e\rangle + |g\rangle}{\sqrt{2}}\right)^{\otimes N}$$

(eigenstate of $\hat{J}_x$)

For this state, $\langle \hat{J}_x \rangle = N/2$, since $\hat{J}_x = \sum_{i=1}^{N} \hat{S}_{ix}$ and $\langle \hat{S}_{ix} \rangle = 1/2$.

From $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$, it follows the uncertainty relation $\Delta\hat{J}_z \cdot \Delta\hat{J}_y \geq |\langle \hat{J}_x \rangle/2|$.

For the above state, $\langle \hat{J}_z \rangle = \langle \hat{J}_y \rangle = 0$, and $\langle \hat{J}_z^2 \rangle = \langle \hat{J}_y^2 \rangle = \sum_{i=1}^{N} \langle \hat{S}_{iy}^{\,2} \rangle = N/4$, so that $\Delta\hat{J}_z = \Delta\hat{J}_y = \sqrt{N}/2$. We have then a minimal uncertainty state: $\Delta\hat{J}_z \cdot \Delta\hat{J}_y = |\langle \hat{J}_x \rangle/2|$. Since $\Delta\hat{J}_z = \Delta\hat{J}_y$, it corresponds to a **coherent spin state**, and the value of these variances is the **projection noise** (equivalent to the shot noise for the electromagnetic field). Bound on uncertainty in the measurement of a phase displacement is $\Delta\varphi_{\min} = \Delta\hat{J}_z/|\langle \hat{J}_x \rangle| = 1/\sqrt{N}$.

This uncertainty can be reduced by 10 by multiplying N by 100.

Recent experimental result (2)

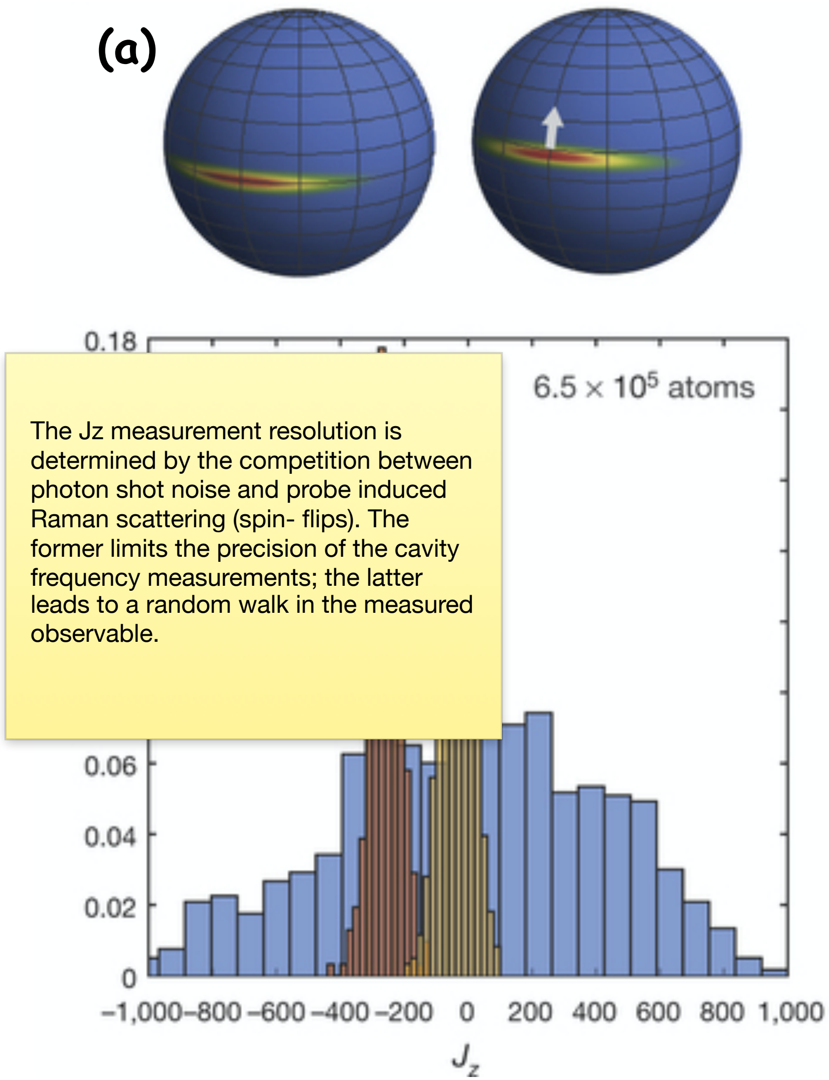
Metrological improvement provided by squeezing is quantified by

$$\chi^2 = \left(\frac{\sqrt{N}/2}{\Delta \hat{J}_z} \cdot \frac{|\langle \hat{J}_x \rangle|}{N/2} \right)^2$$

where first factor on the r.h.s corresponds to noise reduction, and second factor represents coherence loss. For a coherent state, the two factors are equal to one, and $\chi^2 = 1$. In the experiment, $\chi^2 = 100$ (20 dB) was attained, equivalent to increasing 100 times the number of atoms in a coherent state.

Owing to systematic errors arising from collisions between atoms, there is typically an upper bound to the number of atoms that can be employed in state-of-the-art cold atom sensors. In this experiment, up to 7X10^5 atoms are used.

The single-shot phase resolution of 147 microradians achieved by the apparatus is better than that achieved by the best engineered cold atom sensors despite lower atom numbers.



The Jz measurement resolution is determined by the competition between photon shot noise and probe induced Raman scattering (spin- flips). The former limits the precision of the cavity frequency measurements; the latter leads to a random walk in the measured observable.

(a) Two squeezed spin states, one rotated by 660 μrad in the direction of the white arrow, by a weak microwave pulse. **(b)** The corresponding measured squeezed distributions compared to the un-squeezed distribution.

Quantum metrology and weak-value amplification

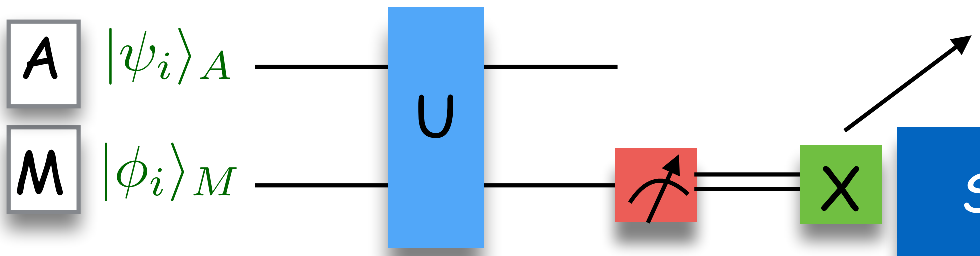
Usual framework: Start with Von Neumann measurement scheme

$\hat{H}_I(t) = \hbar g \delta(t - t_0)\hat{A} \otimes \hat{M} \ \Rightarrow \hat{U}(g) = \exp(-ig\hat{A} \otimes \hat{M})$ Free-evolution neglected

$\hat{A} \rightarrow$ System observable (assume discrete non-degenerate spectrum: $\hat{A}|a_i\rangle = a_i|a_i\rangle$)

$\hat{M} \rightarrow$ Meter observable (assume continuous spectrum)

Initial state of A+M: $|\Psi_i\rangle = |\psi_i\rangle_A \otimes |\phi_i\rangle_M$

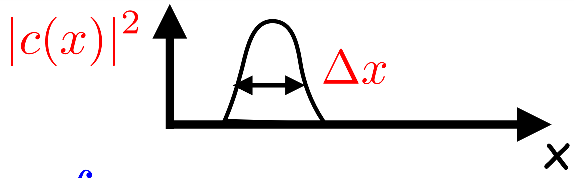


Value of pointer observable - canonically conjugate to $\hat{M}$

Strong measurement: $|g|\delta a \gg \Delta x$

Weak measurement: $|g|\delta a \lesssim \Delta x$

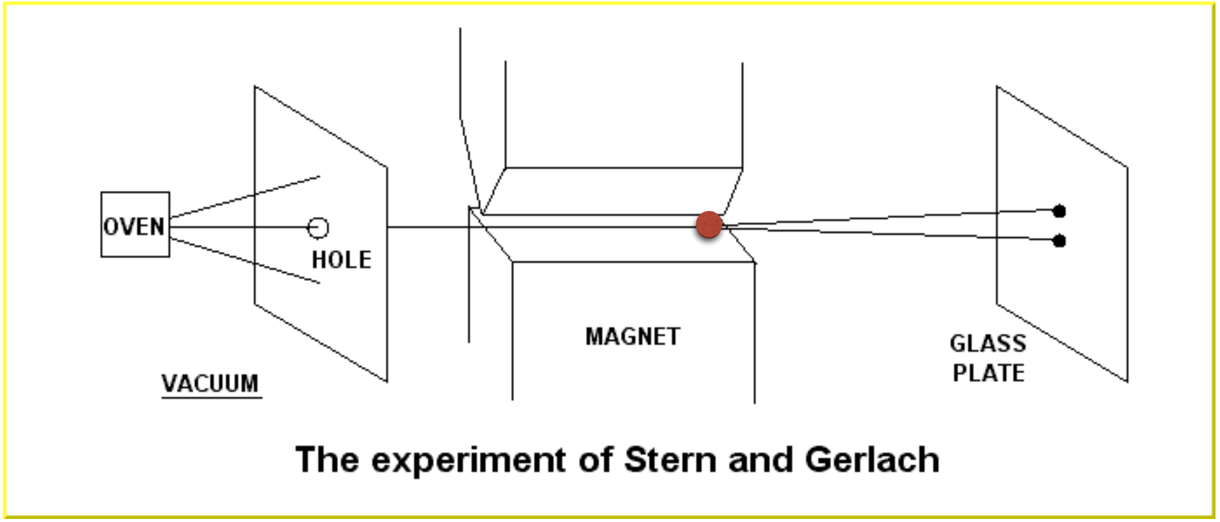
Example: If $\hat{M}$ = momentum, $\hat{X}$ = position



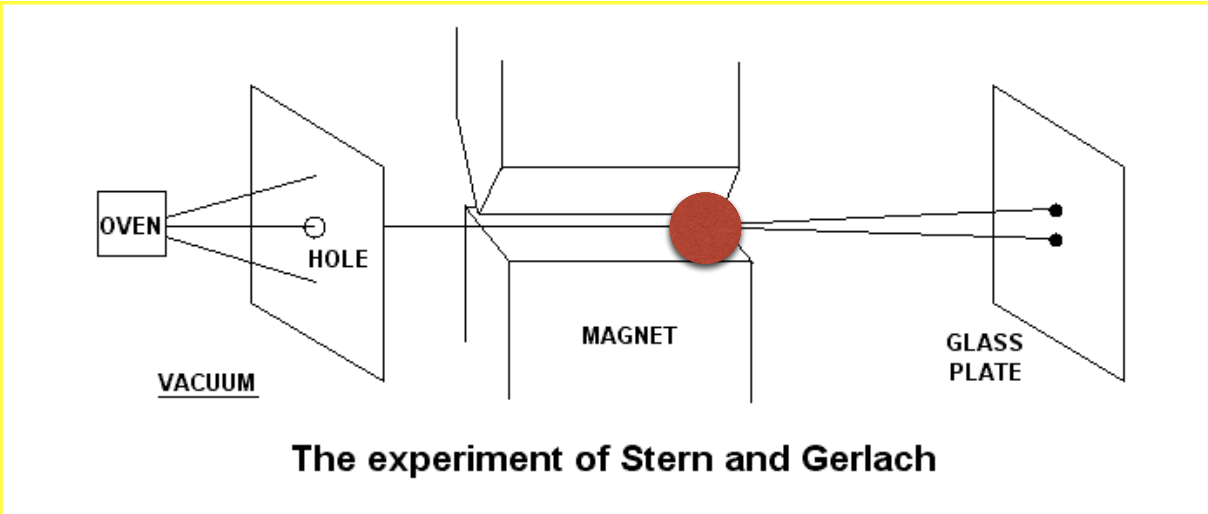
$$|\psi_i\rangle_A = \sum_i c_i|a_i\rangle\,,\ |\phi_i\rangle_M = \int dx\, c(x)|x\rangle$$

$$\Rightarrow |\Psi_f\rangle = \exp(-ig\hat{A} \otimes \hat{p})|\psi_i\rangle_A \otimes |\phi_i\rangle_M = \sum_i c_i|a_i\rangle \otimes \int dx\, c(x)|x - ga_i\rangle_M$$

Quantum metrology and weak-value amplification

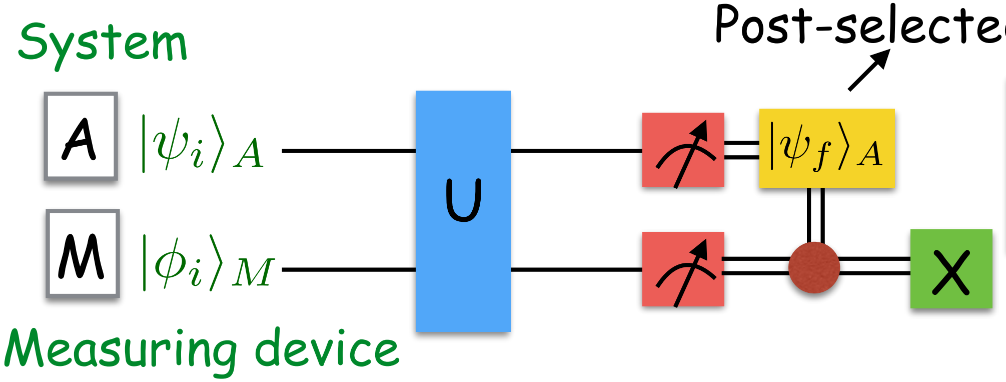


Strong



Weak

Pre- and post-selected measurements



Measurement of X is conditioned on measurement of A in state $|\psi_f\rangle_A$

$$|\Psi_f\rangle = \exp(-ig\hat{A}\otimes\hat{M})|\psi_i\rangle_A \otimes |\phi_i\rangle_M$$

Unnormalized meter state after post-selection (assuming weak interaction):

$$\begin{aligned}|\tilde{\phi}_f(g)\rangle_M &= {}_A\langle\psi_f|\exp(-ig\hat{A}\otimes\hat{M})|\psi_i\rangle_A \otimes |\phi_i\rangle_M \\ &\approx {}_A\langle\psi_f|1 - ig\hat{A}\otimes\hat{M}|\psi_i\rangle_A \otimes |\phi_i\rangle_M \\ &= {}_A\langle\psi_f|\psi_i\rangle_A(1 - igA_w\hat{M})|\phi_i\rangle_M\end{aligned}$$

$$A_w = \frac{{}_A\langle\psi_f|\hat{A}|\psi_i\rangle_A}{{}_A\langle\psi_f|\psi_i\rangle_A} \rightarrow \text{Weak value}$$

Could be much larger than $\langle\hat{A}\rangle$, by choosing $\delta = {}_A\langle\psi_f|\psi_i\rangle_A$ sufficiently small

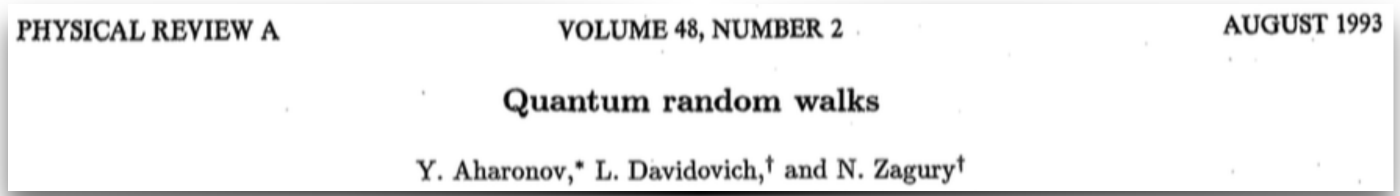
Must have, however, $|gA_w|\Delta M \ll 1$, where $\Delta M \rightarrow$ width of $|\phi_i\rangle_M$.

Then, probability of post-selection is very small:

$$p_f(g) = |\langle\tilde{\phi}_f(g)|\tilde{\phi}_f(g)\rangle|^2 = |\langle\psi_f|\hat{U}(g)|\Psi_i\rangle|^2 \approx |\langle\psi_f|\psi_i\rangle|^2 + O(g^2)$$

Note that $|\phi_f\rangle = |\tilde{\phi}_f(g)\rangle/\sqrt{p_f} = \langle\psi_f|\Psi_f\rangle/\sqrt{p_f}$

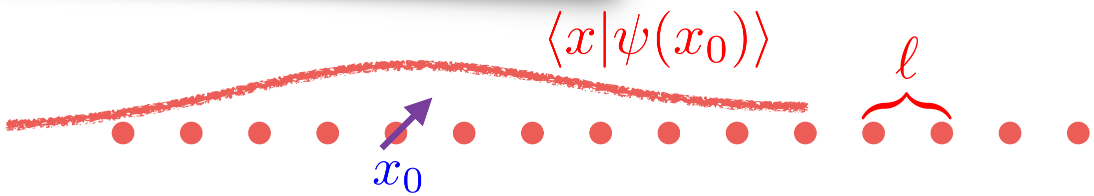
Example: Quantum version of random walks



PHYSICAL REVIEW A VOLUME 48, NUMBER 2 AUGUST 1993

Quantum random walks

Y. Aharonov,* L. Davidovich,† and N. Zagury†



Consider a particle with spin 1/2 moving on a one-dimensional lattice, with the width of the wave-packet in position space much larger than the lattice parameter, and centered around x_0 .

The spin works as a "quantum coin" for the movement of the particle: if the spin is up, the particle moves right, if it is down it moves left. This dynamics can be described by the evolution operator $\hat{U} = \exp(-i\hat{S}_z\hat{P}\ell/\hbar)$.

We have therefore

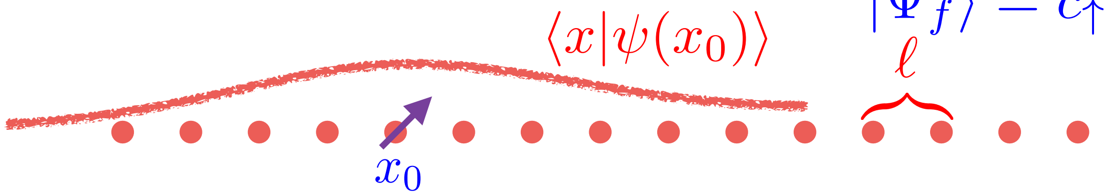
Initial state $|\Psi_i\rangle = |\psi(x_0)\rangle(c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle)$

—> Final state $|\Psi_f\rangle = c_\uparrow|\uparrow\rangle|\psi(x_0+\ell)\rangle + c_\downarrow|\downarrow\rangle|\psi(x_0-\ell)\rangle$

where $|\psi(x_0 \pm \ell)\rangle$ is centered around $x_0 \pm \ell$

Example: Quantum version of random walks

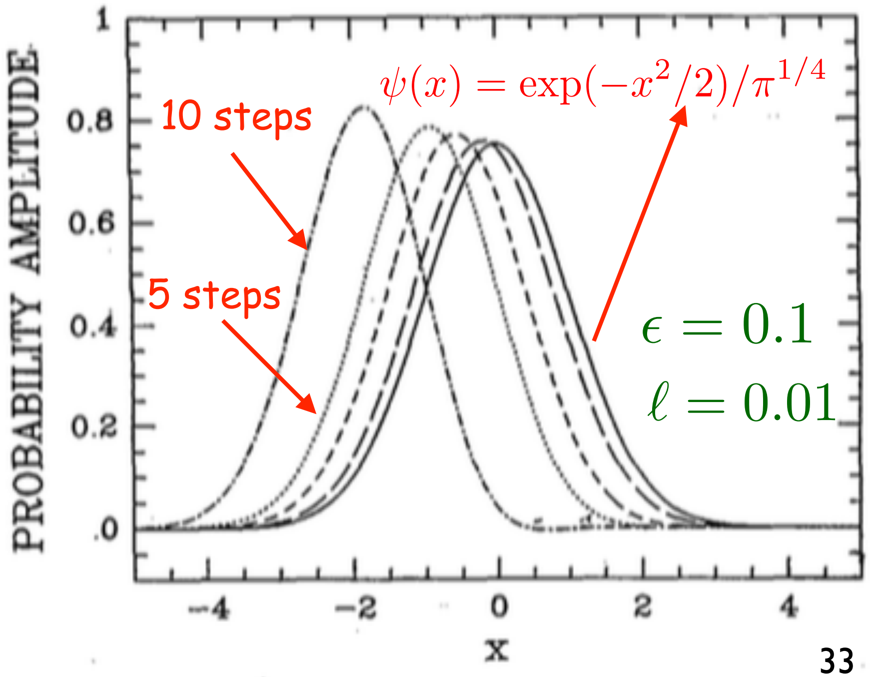
$$|\Psi_f\rangle = c_\uparrow|\uparrow\rangle|\psi(x_0+\ell)\rangle + c_\downarrow|\downarrow\rangle|\psi(x_0-\ell)\rangle$$



Suppose one measures the spin component along a direction (θ, ϕ). The state in configuration space after the measurement is then a coherent superposition of $|\psi(x_0+\ell)\rangle$ and $|\psi(x_0-\ell)\rangle$. Assuming the spin $-\hbar/2$ is found and choosing ϕ as the argument of $c_\downarrow/c_\uparrow$ and

$$\tan(\theta/2) = |c_\downarrow/c_\uparrow|(1+\epsilon)$$

with $0 < \epsilon \ll 1$, the direction (θ, ϕ) becomes almost orthogonal to the initial direction of the spin. The resulting wave-packet is shown in the picture. The interference between the two wave-packets produces, after a few steps, a displacement much larger than the elementary step in the lattice.



What about the precision?

Quantum Fisher information corresponding to g (averages in initial state):

$$\hat{U}(g) = \exp(-ig\hat{A}\otimes\hat{M}) \Rightarrow \mathcal{F}(g) = 4\left[\langle\hat{A}^2\rangle\langle\hat{M}^2\rangle - \langle\hat{A}\rangle^2\langle\hat{M}\rangle^2\right] \to \mathcal{F}(g) = 4\langle\hat{A}^2\rangle\langle\hat{M}^2\rangle$$

(Assume ${}_M\langle\phi_i|\hat{M}|\phi_i\rangle_M = 0$)

$$|\Psi_i\rangle = |\psi_i\rangle_A \otimes |\phi_i\rangle_M$$

Set of measurement operators (POVM's) corresponding to post-selection procedure:

$$\{|\psi_f\rangle\langle\psi_f|\otimes\hat{E}_i, (\hat{\mathbb{1}}_{\mathcal{A}} - |\psi_f\rangle\langle\psi_f|)\otimes\hat{\mathbb{1}}_{\mathcal{M}}\} \qquad i = 1\ldots n$$

where the operators $\{\hat{E}_i\}$, with $\sum_i \hat{E}_i = \hat{1}_{\mathcal{M}}$, act on the Hilbert space of $\mathcal{M}$.

This set leads to the corresponding probabilities (averages of the measurement operators on the final state):

$$\{P_i(g), 1 - p_f(g)\} \equiv \{\langle\Psi_f|\psi_f\rangle\langle\psi_f|\hat{E}_i|\Psi_f\rangle,\ (1 - |\langle\Psi_f|\psi_f\rangle|^2)\}$$

where $p_f(g)$ is the probability of post-selection and, according to the expression $|\phi_f\rangle = \langle\psi_f|\Psi_f\rangle/\sqrt{p_f}$, one has $P_i(g) = p_f(g)P_i^{\mathcal{M}}(g)$, where $P_i^{\mathcal{M}}(g) = \langle\phi_f(g)|\hat{E}_i|\phi_f(g)\rangle$ is the probability of getting the result associated with the operator $\hat{E}_i$ after the proper state is selected.

What about the precision?

Fisher information with post-selection procedure

$$F_{ps}(g) = \frac{1}{1-p_f(g)}\left(\frac{d[1-p_f(g)]}{dg}\right)^2 + \sum_i \frac{1}{P_i(g)}\left(\frac{dP_i(g)}{dg}\right)^2$$

$$p_f(g) = |\langle\psi_f|\hat{U}(g)|\Psi_i\rangle|^2$$

$$P_i(g) = p_f(g)P_i^{\mathcal{M}}(g)$$

$$P_i^{\mathcal{M}}(g) = \langle\phi_f(g)|\hat{E}_i|\phi_f(g)\rangle$$

This can be rewritten as

$$F_{ps}(g) = \underbrace{p_f(g)\sum_{i=1}^{n}\frac{1}{P_i^{\mathcal{M}}(g)}\left[\frac{dP_i^{\mathcal{M}}(g)}{dg}\right]^2}_{} + \underbrace{\frac{1}{p_f(g)[1-p_f(g)]}\left[\frac{dp_f(g)}{dg}\right]^2}_{}$$

Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

$|\psi_f\rangle \rightarrow$ Post-selected state of A

$\hat{E}_j \rightarrow$ Generalized measurements on M

What about the precision?

$$F_{ps}(g) = \underbrace{p_f(g) \sum_{i=1}^{n} \frac{1}{P_i^{\mathcal{M}}(g)} \left[\frac{dP_i^{\mathcal{M}}(g)}{dg} \right]^2}_{F_M(g)} + \underbrace{\frac{1}{p_f(g)[1-p_f(g)]} \left[\frac{dp_f(g)}{dg} \right]^2}_{F_{p_f}(g)}$$

Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

The quantum Fisher information for the meter, corresponding to the best possible measurement, is given by the expression

$$\mathcal{F}_M(g) = 4 \left[\frac{d\langle\phi(g)|}{dg} \frac{d|\phi(g)\rangle}{dg} - \left| \frac{d\langle\phi(g)|}{dg} |\phi(g)\rangle \right|^2 \right]$$

multiplied by the post-selection probability $p_f(g)$.

What about the precision?

$$F_{ps} = \underbrace{\mathcal{F}_M(g)} + \underbrace{F_{p_f}(g)}$$

Quantum Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

Perturbation theory is tricky, since there are two small parameters: g and $|{}_A\langle\psi_f|\psi_i\rangle_A|$. So, must consider two regions separately:

$|gA_w|\Delta M \ll 1 \;\Rightarrow$ Region of validity of weak-value theory

$|gA_w|\Delta M \gg 1 \;\Rightarrow$ Attained if $|{}_A\langle\psi_f|\psi_i\rangle_A| \ll 1$

PHYSICAL REVIEW A **91**, 062107 (2015)

Weak-value amplification as an optimal metrological protocol

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What about the precision?

$$F_{ps} = \underbrace{\mathcal{F}_M(g)} + \underbrace{F_{p_f}(g)}$$

Quantum Fisher information corresponding to measurements on the meter after post-selection, degraded by loss of statistical data

Information on g encoded in $p_f(g)$

In PRA 91, 062107 (2015) it is shown that, if the post-selected state is given by

$$|\psi_f^{\text{opt}}\rangle = \frac{\hat{A}|\psi_i\rangle}{\langle \hat{A}^2\rangle^{1/2}} \Rightarrow \lim_{g\to 0} F_{ps}(g) \to \mathcal{F}(g) = 4\langle \hat{A}^2\rangle\langle \hat{M}^2\rangle$$ Quantum Fisher information

However, the contributions to F_{ps} depend on the region of parameters:

$|gA_w|\Delta M \ll 1 \Rightarrow \mathcal{F}_M(g) \to \mathcal{F}(g)$ Region of validity of weak-value theory

$|gA_w|\Delta M \gg 1 \Rightarrow F_{p_f}(g) \to \mathcal{F}(g)$ Region $|\langle\psi_f|\psi_i\rangle| \ll 1$

For this optimal choice of post-selected state, one has

Weak value: $$A_w = \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = \frac{\langle\psi_i|\hat{A}^2|\psi_i\rangle}{\langle\psi_i|\hat{A}|\psi_i\rangle} \geq \langle\psi_i|\hat{A}|\psi_i\rangle$$

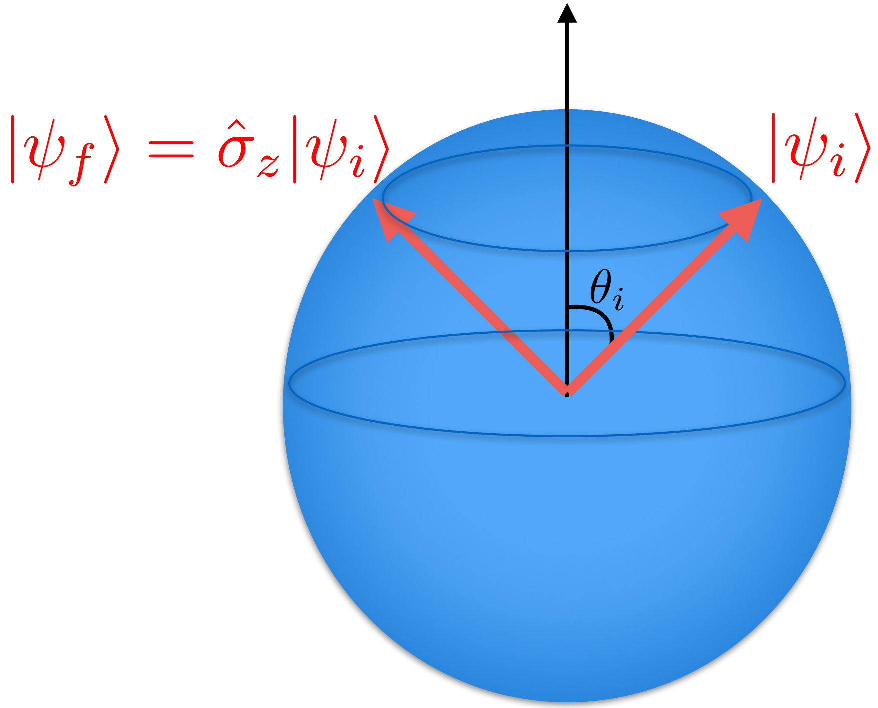
so there is indeed amplification.

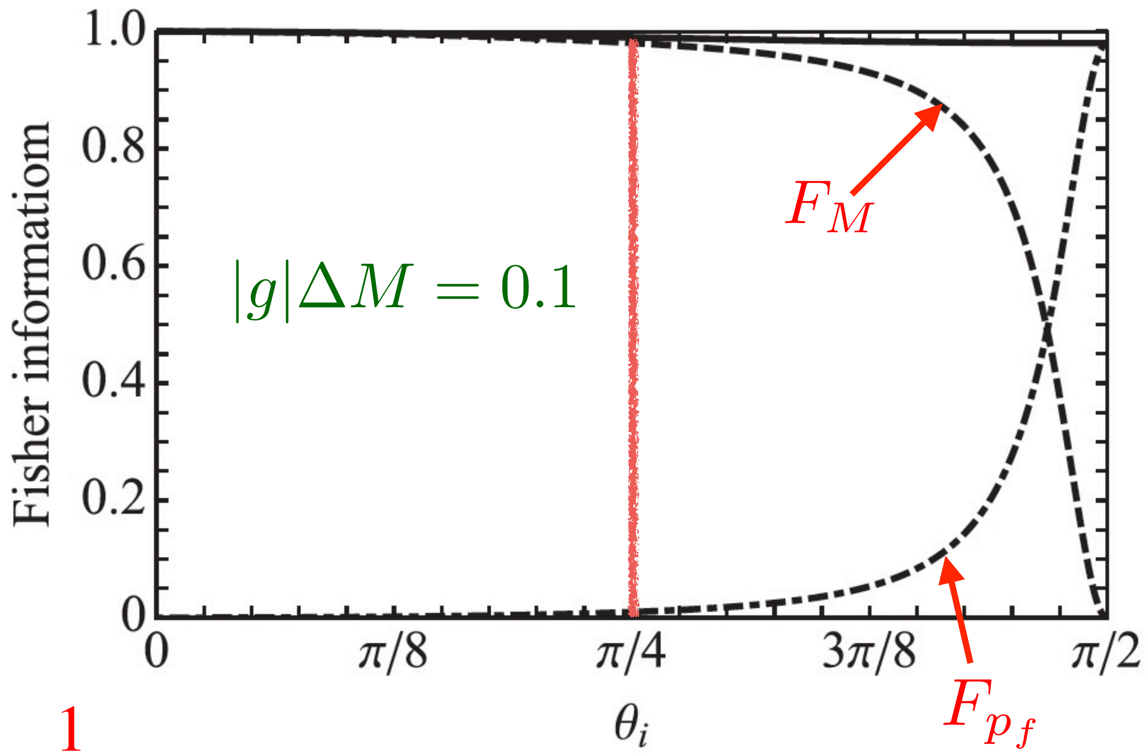
Example: spin measurement

$$\hat{A} = \hat{\sigma}_z \qquad \hat{A}^2 = \hat{1} \qquad \hat{U}(g) = \exp\left(-ig\hat{\sigma}_z \hat{M}\right) \qquad |\psi_f^{\rm opt}\rangle = \frac{\hat{A}|\psi_i\rangle}{\langle \hat{A}^2\rangle^{1/2}} = \hat{\sigma}_z|\psi_i\rangle$$

(Rotation of π around the z axis)

Initial state of the meter is a pure state with a Gaussian distribution of the eigenvalues of $\hat{M}$, with width $\Delta M = \langle \hat{M}^2\rangle^{1/2}$

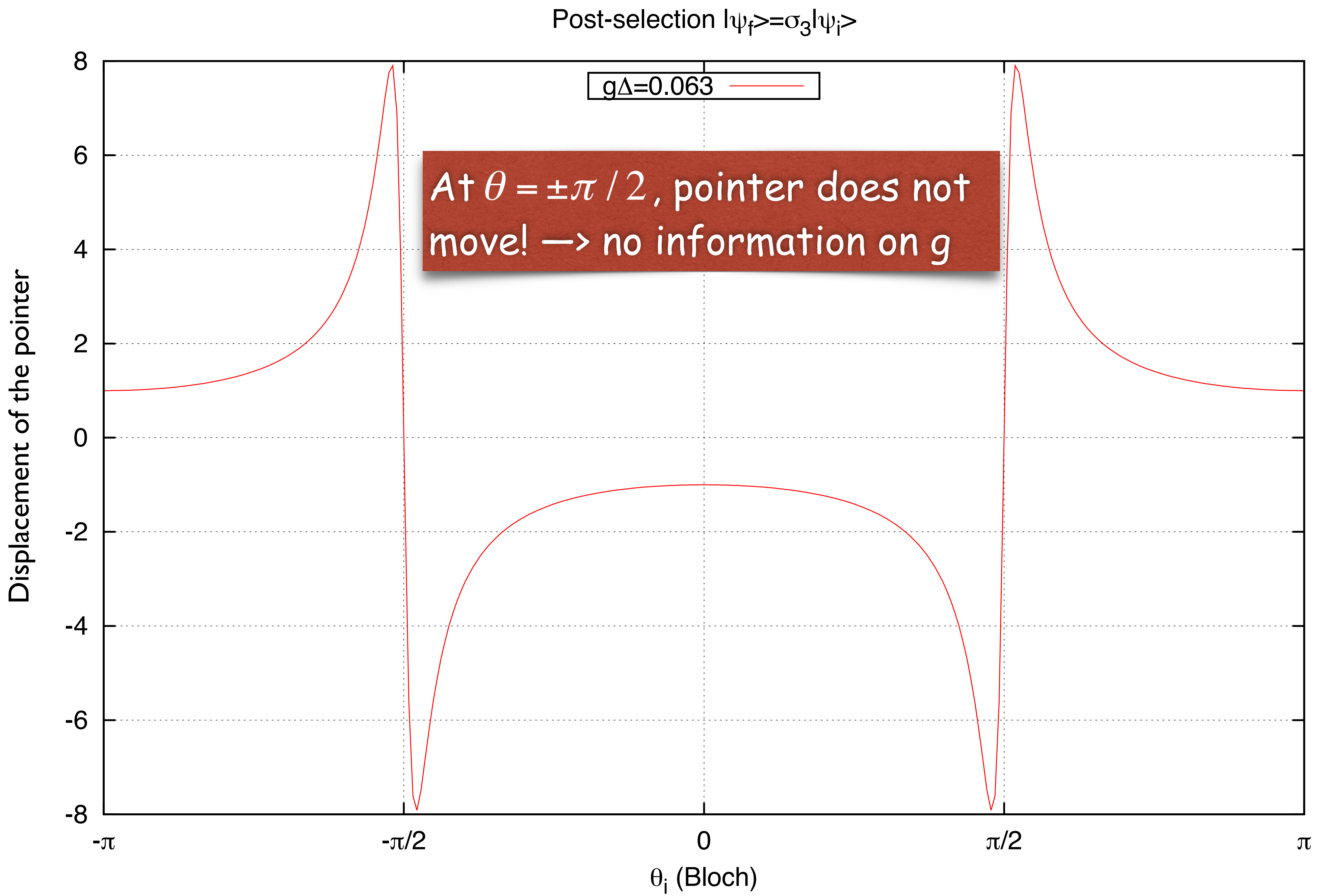




$$A_w = \frac{\langle\psi_i|\hat{A}^2|\psi_i\rangle}{\langle\psi_i|\hat{A}|\psi_i\rangle} = \frac{1}{\langle\psi_i|\hat{\sigma}_z|\psi_i\rangle} = \frac{1}{\underbrace{\langle\psi_f^{\rm opt}|\psi_i\rangle}_{\delta}}$$

Transition region: $|gA_w|\Delta M \approx 1$

Example: spin measurement



At $\theta = \pm\pi/2$, pointer does not move! —> no information on g

Value of weak-value amplification: a debate

PRL **112,** 040406 (2014) PHYSICAL REVIEW LETTERS week ending 31 JANUARY 2014

Weak Value Amplification is Suboptimal for Estimation and Detection

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(Received 25 July 2013; revised manuscript received 21 November 2013; published 31 January 2014)

We show by using statistically rigorous arguments that the technique of weak value amplification does not perform better than standard statistical techniques for the tasks of single parameter estimation and signal detection. Specifically, we prove that postselection, a necessary ingredient for weak value amplification, decreases estimation accuracy and, moreover, arranging for anomalously large weak values is a suboptimal strategy. In doing so, we explicitly provide the optimal estimator, which in turn allows us to identify the optimal experimental arrangement to be the one in which all outcomes have equal weak values (all as small as possible) and the initial state of the meter is the maximal eigenvalue of the square of the system observable. Finally, we give precise quantitative conditions for when weak measurement (measurements without postselection or anomalously large weak values) can mitigate the effect of uncharacterized technical noise in estimation.

PHYSICAL REVIEW X **4,** 011031 (2014)

Technical Advantages for Weak-Value Amplification: When Less Is More

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The technical merits of weak-value-amplification techniques are analyzed. We consider models of several different types of technical noise in an optical context and show that weak-value-amplification techniques (which only use a small fraction of the photons) compare favorably with standard techniques (which use all of them). Using the Fisher-information metric, we demonstrate that weak-value techniques can put all of the Fisher information about the detected parameter into a small portion of the events and show how this fact alone gives technical advantages. We go on to consider a time-correlated noise model and find that a Fisher-information analysis indicates that the standard method can have much larger information about the detected parameter than the postselected technique. However, the estimator needed to gather the information is technically difficult to implement, showing that the inefficient (but practical) signal-to-noise estimation of the parameter is usually superior. We also describe other technical advantages unique to imaginary weak-value-amplification techniques, focusing on beam-deflection measurements. In this case, we discuss combined noise types (such as detector transverse jitter, angular beam jitter before the interferometer, and turbulence) for which the interferometric weak-value technique gives higher Fisher information over conventional methods. We go on to calculate the Fisher information of the recently proposed photon-recycling scheme for beam-deflection measurements and show it further boosts the Fisher information by the inverse postselection probability relative to the standard measurement case.

Sommaire de la troisième leçon

Jeudi, 18 Février, 2016

Dans cette leçon, on a discuté l'extension pour la mécanique quantique de la théorie de Cramér-Rao-Fisher, qu'on a appliqué à des systèmes fermés, pour lesquels l'evolution de la sonde est décrite par une operation unitaire. La prochaine leçon introduira l'extension de cette théorie pour les systèmes ouverts, comme l'interféromètre optique qui subit des pertes de photons ou la diffusion de la phase. On considère aussi le problème d'estimation de forces faibles, qui agissent sur an oscillateur harmonique amorti.