LECTURES – COLLEGE de FRANCE
MAY 2016

PCE STAMP (UBC)

For all online material:

http://www.phas.ubc.ca/~berciu/PHILIP/TEACHING/CF/index.html

http://www.college-de-france.fr/site/jean-dalibard/guestlecturer-2015-2016__1.htm

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PLAN of LECTURES

LECTURE 1: FORMAL RESULTS, and an EXAMPLE

The Essence of Quantum Mechanics
Path Integral formulation of QM; Reduced Density Matrix Dynamics
Effective Hamiltonians – remarks on derivation
Oscillator Baths and Spin Baths; Environmental Decoherence

REAL WORLD EXAMPLE: Dynamics of a Superfluid Vortex

LECTURE 2: FROM QUBITS to SPIN NETS - 1

Qubits in the real world
"Central Spin" & "Spin-Boson" models
Decoherence for a single Qubit

REAL WORLD EXAMPLE: Spin in a solid state insulator

LECTURE 3: FROM QUBITS to SPIN NETS - 2

Networks of qubits vs Quantum walks – the equivalence
Dynamics of the Quantum Ising system
Long-range decoherence ("correlated errors") & 3rd-Party decoherence

REAL WORLD EXAMPLE 1: Buckyballs & 2-slit diffraction
REAL WORLD EXAMPLE 2: Quantum Ising system

LECTURE 4: QUANTUM MECHANICS & GRAVITATION

The conflict between Quantum Mechanics & General Relativity
Path integrals in Quantum Gravity
Correlated WorldLine Theory

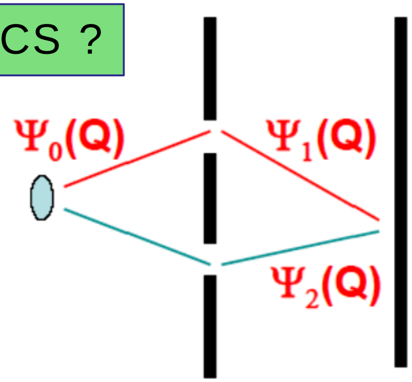
REAL WORLD EXAMPLE: An optomechanical experiment

WHAT is the ESSENCE/MYSTERY of QUANTUM MECHANICS ?

(1) **According to Feynman (1965), ' the fundamental mystery of QM' is encapsulated in the '2-slit' experiment:**

$\Psi_0(q)$ **evolves according to**

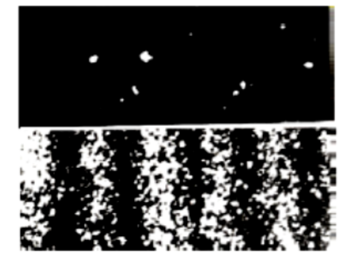
$$\Psi_o(q) \rightarrow [a_1 \Psi_1(q) + a_2 \Psi_2(q)]$$

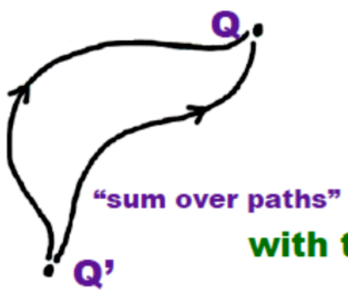


The probability of seeing particle at position Q on screen:

$$P(Q) = |a_1 \Psi_1(Q) + a_2 \Psi_2(Q)|^2 = P_1 + P_2 + 2P_{12}$$

with cross-term $P_{12}(Q) = |a_1 a_2 \Psi_1(Q) \Psi_2(Q)|$





Feynman gave a beautiful formulation of QM, perfectly encapsulating this 'superposition'. He writes

$$\psi(Q,t) = \int dQ' \, G(Q,Q';t,t') \, \psi(Q',t')$$

with the 'path integral' sum:

$$G(Q,Q';t,t') = \int_{q(t')=Q'}^{q(t)=Q} \mathcal{D}q(\tau) e^{\frac{i}{\hbar} S[q,\dot{q}]}$$

This formulation perfectly captures the non-local character of QM (cf Aharonov-Bohm/Berry phase effects)

(2) A second mysterious feature – also best formulated in path integral language – is that of INDISTINGUISHABILITY. It yields particle statistics (which can be fractional), & is inherently NON-LOCAL.

$$\mathbf{G}_o(2,1) = \int_1^2 \mathcal{D}\mathbf{r}(\tau) \, e^{\frac{i}{\hbar} S_{21}[\mathbf{r}(\tau)]}$$

$$= \sum_\alpha \chi(\alpha) G_o^\alpha(2,1)$$

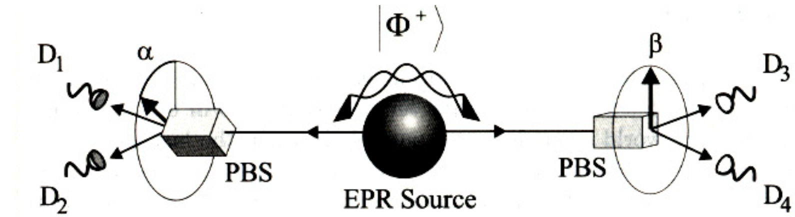
Sum over topological classes

Thus, Feynman's formalism gives directly an unambiguous answer to global problems. Other formalisms use ad hoc, extraneous conditions to deal with global problems, such as boundary conditions on wave functions, symmetry or antisymmetry property of the wave function, etc. ... and their answers are not necessarily identical with Feynman's.

C. Morette-DeWitt,
Comm. Math. Phys. 28, 47 (1972)

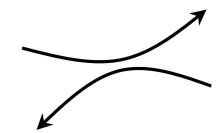
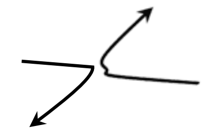
(3) Long before Feynman, Einstein & Schrodinger (1935) fingered “ENTANGLEMENT” as the real essence of QM – embodied in states like

$$\Psi = [\, \phi_+ (A)\, \phi_- (B) + \phi_- (A)\, \phi_+ (B)\,]$$

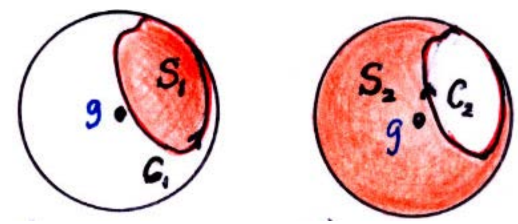


for which the quantum state of either individual system is literally meaningless!

In the path integral formulation, entanglement is a CONSEQUENCE of superposition, for multi-particle systems; in this sense it is NOT a new feature of QM (NB: Feynman already showed this in path integral language in 1965).

K(1,2;1’2’) =  + 

(4) Another essential feature of non-relativistic QM, The SPIN-STATISTICS THEOREM, can only be explained using relativistic Quantum Field Theory (that the spin be integer or half-integer comes from the form of the spin path integral).

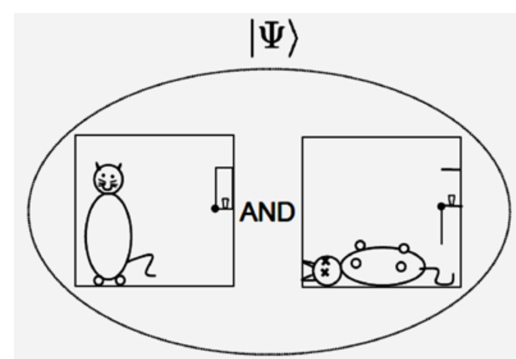


The enclosed area is defined modulo 4π

(5) The most mysterious essential feature of QM is the transition from probabilities to concrete results (often called the measurement problem”). Whether we deal with density matrices or wave-functions, we end up writing the “results of measurements”, in the form

$$\langle M_j \rangle = Tr\{\hat{M}_j \hat{\rho}\}$$

in terms of a density matrix & measurement operator.



REMARKS on PATH INTEGRALS

In traditional non-relativistic QM one defines $\hat{G}(t_2 - t_1) \;=\; e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}\,\theta(t_2 - t_1)$

which automatically satisfies $(\hat{H} - i\hbar\partial_t)\hat{G}(t-t') \;=\; -i\hbar\hat{\mathbf{1}}\delta(t-t')$

(ie., the Schrodinger eqtn), and which has matrix elements

$$G(x, x'; \omega) \;=\; \langle x|\hat{G}(\omega)|x'\rangle \;=\; i\hbar \sum_n \frac{\phi_n^*(x)\phi_n(x')}{\hbar\omega - \epsilon_n}$$

However (cf. Morette-DeWitt), the Feynman form is more general. For a single non-relativistic particle, one has

$$K(x, x') \;=\; \int_{x'}^{x} \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}$$

and for some field $\phi(x)$ we have $K(\Phi(x), \Phi'(x)) \;=\; \int_{\Phi'}^{\Phi} \mathcal{D}\phi(x)\, e^{\frac{i}{\hbar}S[\phi]}$ between different field configurations

Note that these propagators are NOT the same as the <u>correlators</u> of the theory. This is most obvious when we deal with fields

<u>ADVANTAGES of this FORMULATION</u>

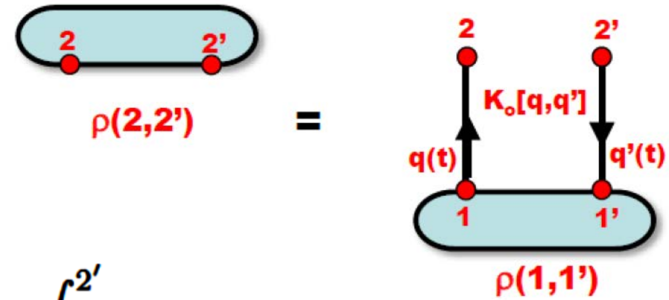
- Captures non-local, topological features of QM, & relation between phase & action along worldliness: clearer connection to classical theory
- No need for wave-functions, or operator algebra, or canonical field theory
- Fundamentally non-perturbative; clear definition of interacting fields

One needs to define the measure for the path integration – ie., decide how to assign weight to the different paths. We discuss this later

For a tutorial article:

P Storey, C. Cohen-Tannoudji, J de Physique II, <u>4</u>, 1999 (1999)

DYNAMICS of a BARE DENSITY MATRIX



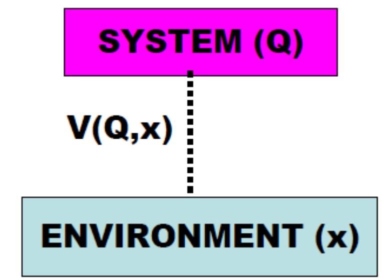
For a FREE system the density matrix has equation of motion

$$\rho_o(2,2') = \int d1 \int d1' K_o(2,2';1,1')\rho_o(1,1')$$

with density matrix propagator $K_o(2,2';1,1') \;=\; \int_1^2 \mathcal{D}q(s) \int_{1'}^{2'} \mathcal{D}q'(s')\; A_o[q,q']$

where $A_o[x(s),x'(s)] = e^{\frac{i}{\hbar}(S_o[x(s)] - S_o[x'(s)])}$

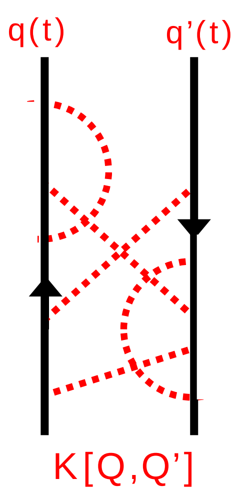
DYNAMICS of the REDUCED DENSITY MATRIX



Consider the action $S[Q,x] \;=\; S_o[Q] + S_{env}[x] + S_{int}[Q,x]$

We define $K(1,2) \;=\; \int_{Q_1}^{Q_2} dQ \int_{Q_1'}^{Q_2'} dQ'\; e^{-i/\hbar(S_o[Q]-S_o[Q'])}\mathcal{F}[Q,Q']$

where we introduce Feynman's influence functional $\mathcal{F}[Q,Q'] = \prod_k \langle \hat{U}_k(Q,t)\hat{U}_k^\dagger(Q',t)\rangle$



Here the unitary operator $\hat{U}_k(Q,t)$ describes the evolution of the kth environmental mode, given that the central system follows the path $Q(t)$ on its 'outward' voyage, and $Q'(t)$ on its 'return' voyage

We can also write this as:

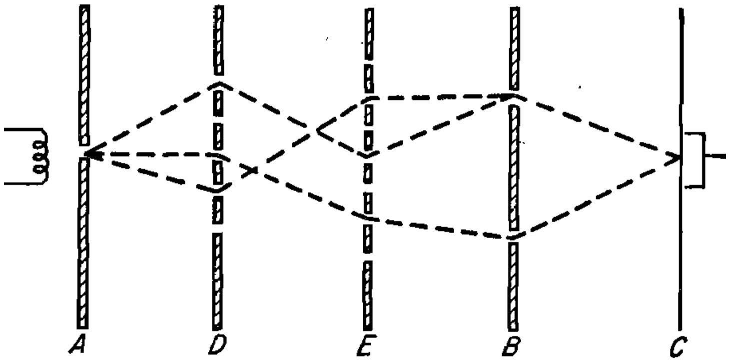
$$\mathcal{F}[Q,Q'] \;=\; \prod_k Tr_{x_k,x_k'} \int \mathcal{D}x_k \mathcal{D}x_k'\; e^{\frac{i}{\hbar}(S_{int}[x_k,Q] - S_{int}[x_k',Q'] + S_{env}[Q] - S_{env}[Q'])}$$

This defines the dynamics of the reduced density matrix:

$$\bar{\rho}(2,2') = \int d1 \int d1' K(2,2';1,1')\bar{\rho}(1,1')$$

RP Feynman, FL Vernon, Ann Phys 24, 118 (1963)
NV Prokofev, PCE Stamp, Rep Prog Phys 63, 669 (2000)

REMARKS on PATH INTEGRATION TECHNIQUE



PATH INTEGRATION MEASURE: On a 1st meeting With path integrals, the obvious question is – how to count paths? For simple problems, like a non-relativistic particle, one writes a phase space path integral of form

$$K(q_2,t_2;q_1,t_1) = \int \mathcal{D}q\mathcal{D}p \exp \frac{i}{\hbar}\left[\int_{t_1}^{t_2} dt[\dot{q} - H(p,q)]\right]$$
$$= lim_{N\to\infty} \prod_{j=1}^{N} \int dq_j \prod_{j=0}^{N} dp_j \exp \frac{i}{\hbar}\left[\sum_{n=0}^{N}[p_j(q_{j+1}-q_j) - \mathcal{H}(p_j,q_j)dt]\right]$$

which corresponds physically to a division of the path into infinitesimal segments (the composition rule for QM amplitudes). This allows a definition of path measures. But for more complicated theories (eg., non-Abelian gauge theories, or Q Gravity), we need to be more sophisticated; this will come up in the 4th lecture (see refs at left).

AM Polyakov, Phys Lett B103, 207, 211 (1981)
E Mottola, J Math Phys 36, 2470 (1995)

GAUSSIAN INTEGRALS: The functional integrations in these path integrals are actually for the most part very simple – they are just convolutions of Gaussians, of the form

$$\int_{-\infty}^{\infty} \exp\{i\lambda[(x_1-a)^2 + (x_2-x_1)^2 + \ldots + (b-x_n)^2]\}\, dx_1 \ldots dx_n = \left[\frac{i^n \pi^n}{(n+1)\lambda^n}\right]^{1/2} \exp\left[\frac{i\lambda}{n+1}(b-a)^2\right]$$

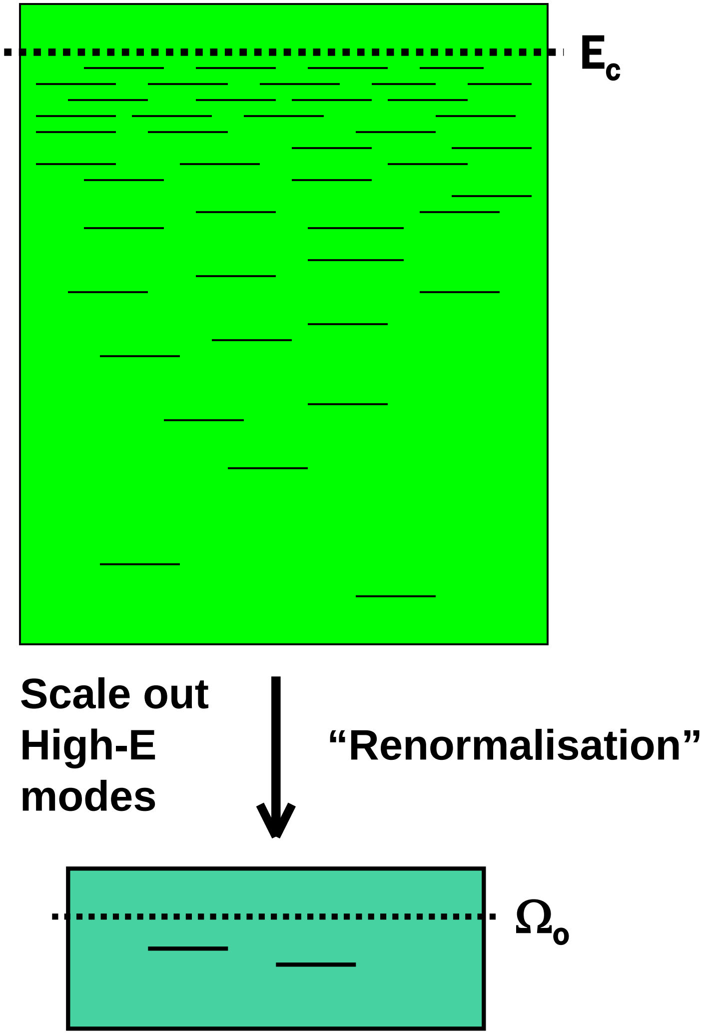
This leaves us free reign to appreciate the intuitive power of these path integrals

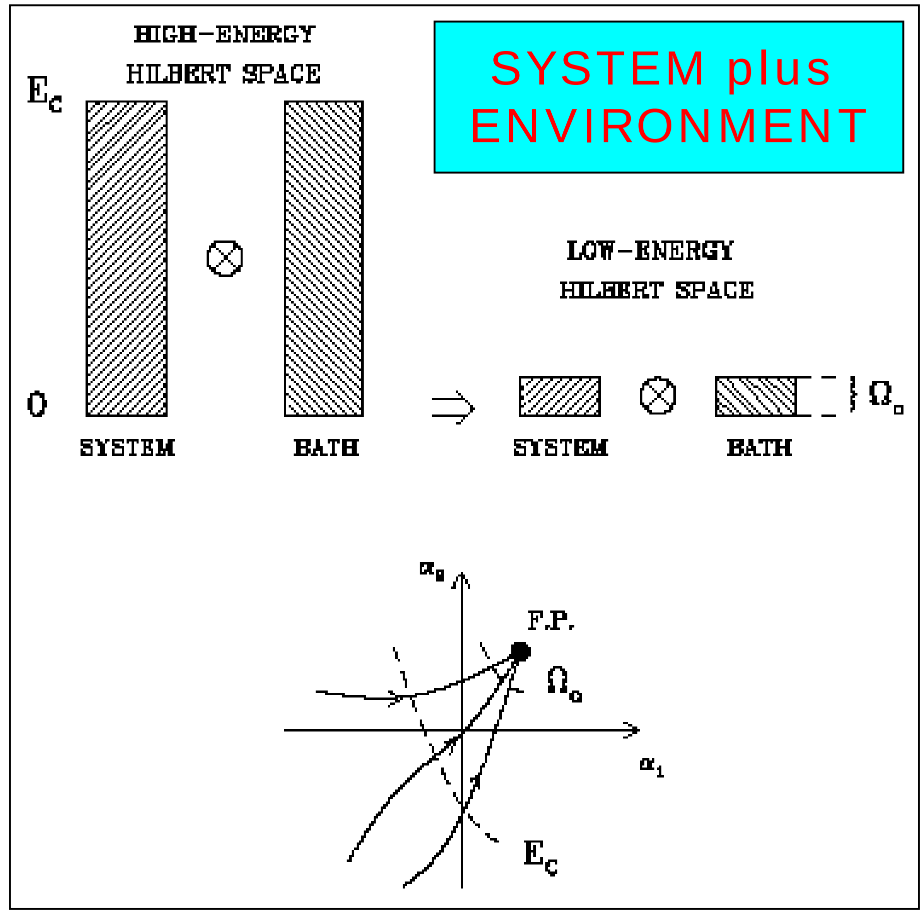
SPIN PATH INTEGRALS: This simplicity is partly lost in spin path integrals, of form

$$\mathcal{G}(\mathbf{n}_2,t_2;\mathbf{n}_1,t_1) = \int_{\mathbf{n}_1}^{\mathbf{n}_2} \mathcal{D}\mathbf{n}(t) \exp \frac{i}{\hbar}\left[\int_{t_1}^{t_2} dt\,[\hbar\mathcal{A}\cdot\dot{\mathbf{n}}(t) - \mathcal{H}(\mathbf{n})]\right]$$

Their meaning is discussed in the background notes

Integrating out to get $\mathcal{S}_{eff}$





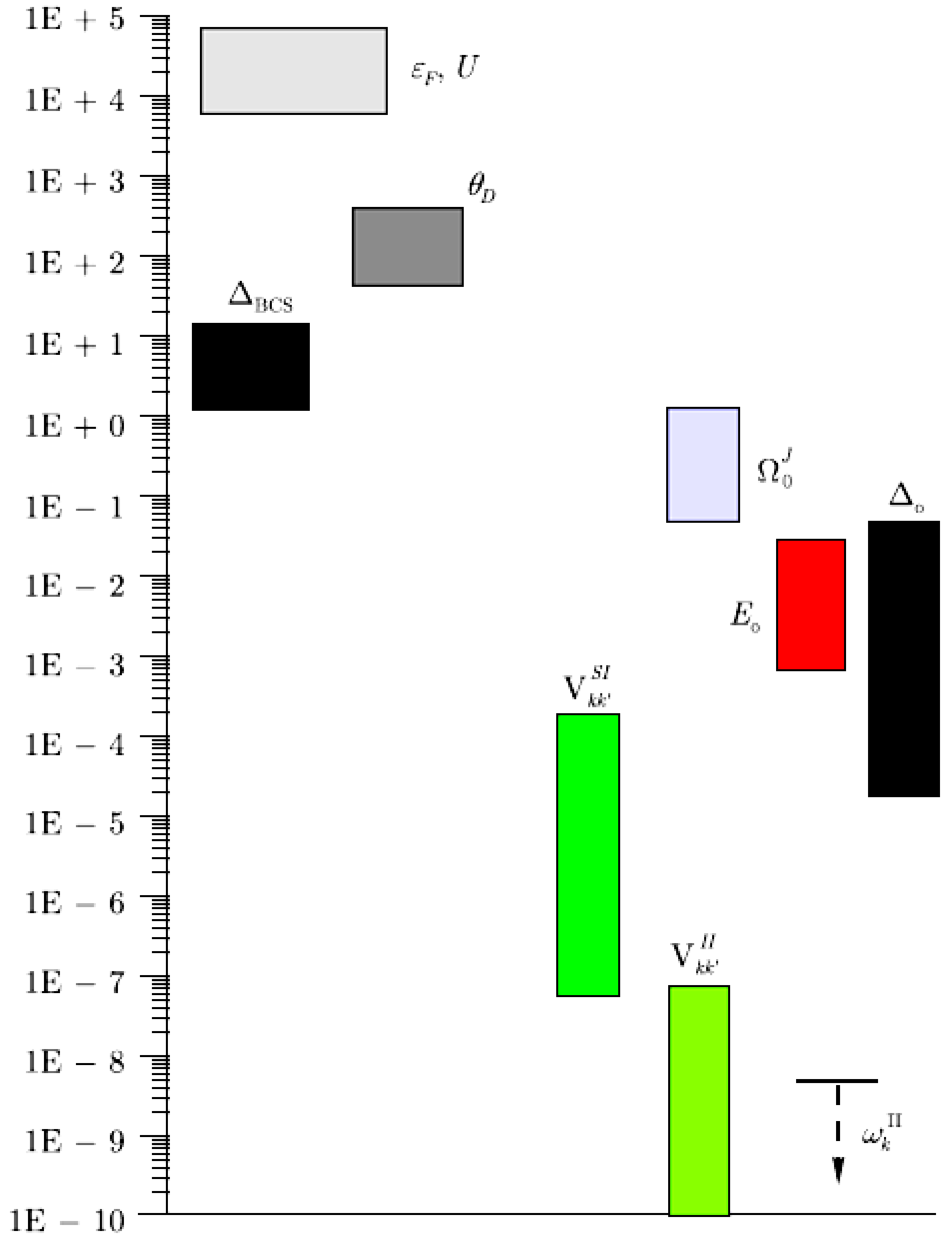
$\mathcal{H}_{eff}\,(E_c) \rightarrow \mathcal{H}_{eff}\,(\Omega_o)$

$|\psi_i\rangle\ \ H_{ij}(E_c)\ \ \langle\psi_j| \quad \rightarrow \quad |\phi_\alpha\rangle\ \ \mathcal{H}_{\alpha\beta}(\Omega_o)\ \langle\phi_\beta|$

Flow of Hamiltonian & Hilbert space with UV cutoff

The RG mantra is: **RG flow**
fixed points
low-energy $\mathcal{H}_{eff}$
universality classes

WHY DERIVATION of low-E H_{eff} is HARD: 1



Ex: ENERGY SCALES in SUPERCONDUCTORS

One has a broad hierarchy of energy scales (here shown for conventional s/c):

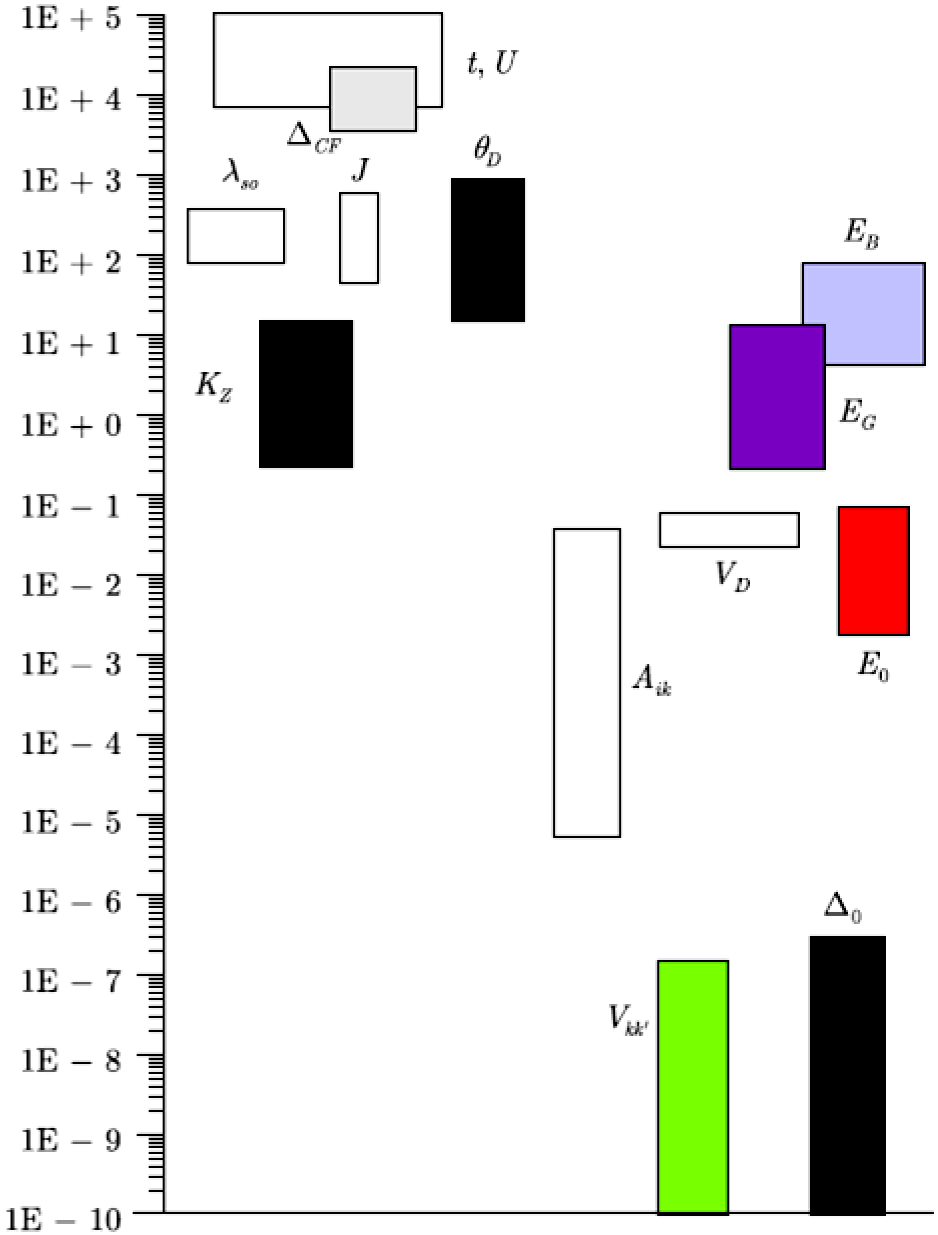
electronic energy scales:	U, ε_F, (or t)
phonon energies:	θ_D
Gap/condensation energy	Δ_{BCS}
p/m impurities (not shown)	J, T_K
Coupling of $\psi(r)$ to spins:	ω_k
Total coupling to spin bath:	$E_0 \sim N^{1/2}\omega_k$

A superconducting device has other energy scales – eg., in a SQUID:

Josephson plasma energy	Ω_0^J
Tunneling splitting	Δ_0

These are of course not all the energies that can be relevant in a superconductor. in particular, there are energy scales associated with the quasiparticle excitations that are not shown – nor do I show energy scales associated with impurities, which are crucial when considering decoherence phenomena.

WHY DERIVATION of low-E H_{eff} is HARD: 2



MICROSCOPIC ENERGY SCALES in MAGNETS

The standard electronic coupling energies are (shown here for Transition metals):

Band kinetic & interactions:	t, U
Crystal field:	D_{CF}
Exchange, superexchange	J
Spin-orbit:	l_{so}
Magnetic anisotropy	K_Z
inter-spin dipole coupling	V_D
p/m impurities (not shown)	J, T_K

which for large spin systems lead to

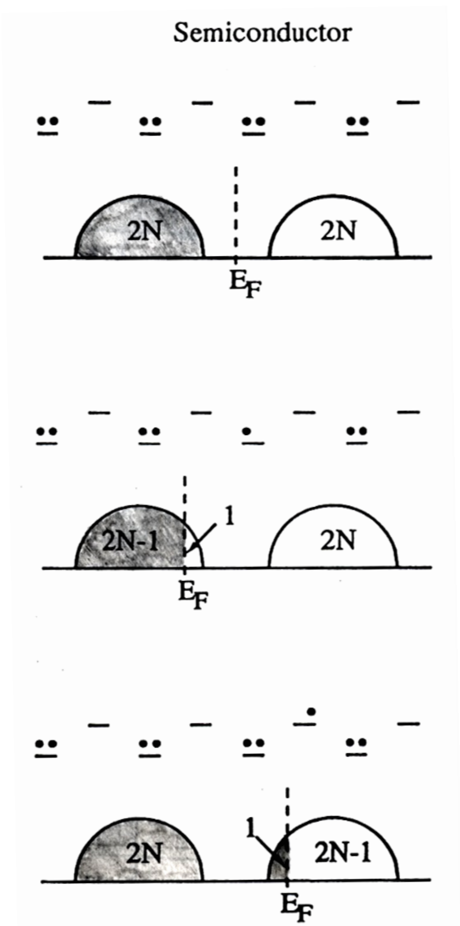
Anisotropy barriers:	$E_B \sim EK_Z$
small oscillation energies	$E_G \sim K_Z$
Spin tunneling amplitude	Δ_0

Also have couplings to various “thermal baths”, with energy scales:

Debye frequency:	θ_D
Hyperfine couplings	A_{ik}
Total spin bath energy	$E_0 \sim N^{1/2}\omega_k$
Inter-nuclear couplings	$V_{kk'}$

NOTE: all of these are parameters in effective Hamiltonians for magnets at low T.

WHY DERIVATION of low-E H_{eff} is HARD: 3



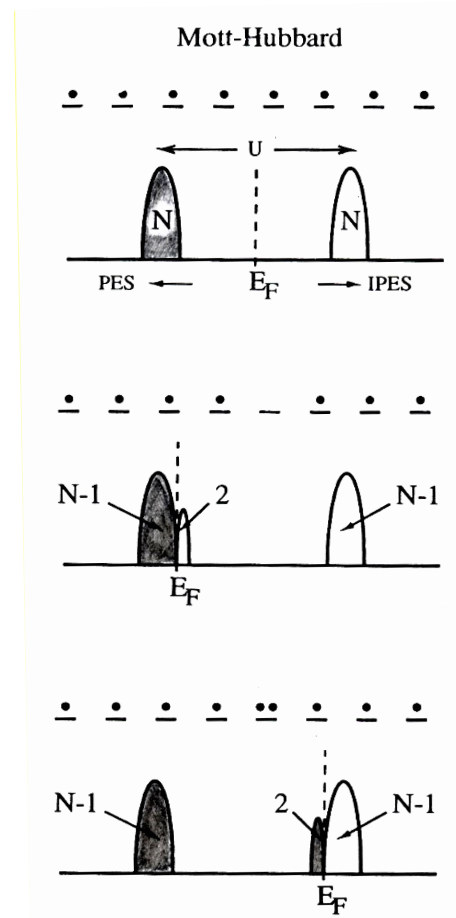
THE HUBBARD MODEL

The standard Hubbard effective Hamiltonian has the form

$$H = -t \sum_{\langle i,j \rangle} \left(c_{i\sigma}^{\dagger} c_{j\sigma} + h.c. \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

This is used to describe lattice solids. By including multiple on-site orbitals, & a local energy which changes from site to site, it can be used to describe molecules and more general solids.

Consider the case where U > t, with near half-filling of the system. Any attempt to find a low-E effective Hamiltonian runs into a problem – we have ‘spectral weight transfer’ between the upper and lower Hubbard bands. An effective Hamiltonian with UV cutoff << U

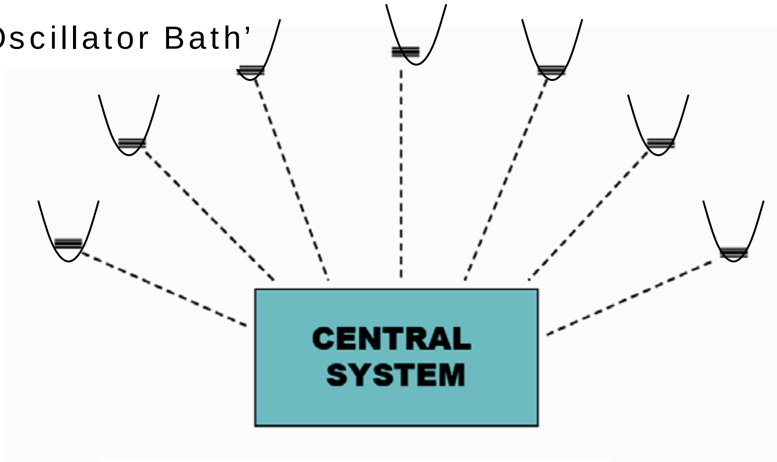


1) No well-defined low-E Hilbert space – no low-E effective **H**

2) Spectral weight transfer → analogue of "UV / IR mixing"

MODELS for SYSTEM/BATH INTERACTIONS

‘Oscillator Bath’



$$H_{\rm eff}^{\rm osc} = H_0 + H_{\rm int} + H_{\rm env}^{\rm osc}$$

Bath: $$H_{osc} = \sum_{q=1}^{N_o} \left(\frac{p_q^2}{m_q} + m_q \omega_q^2 x_q^2 \right)$$

Int: $$H_{\rm int}^{\rm osc} = \sum_{q=1}^{N} [F_q(Q) x_q + G_q(P) p_q]$$

Very SMALL (~ $O(1/N^{1/2})$)

Phonons, photons, magnons, spinons,
Holons, Electron-hole pairs, gravitons,..

Feynman & Vernon, Ann. Phys. 24, 118 (1963)

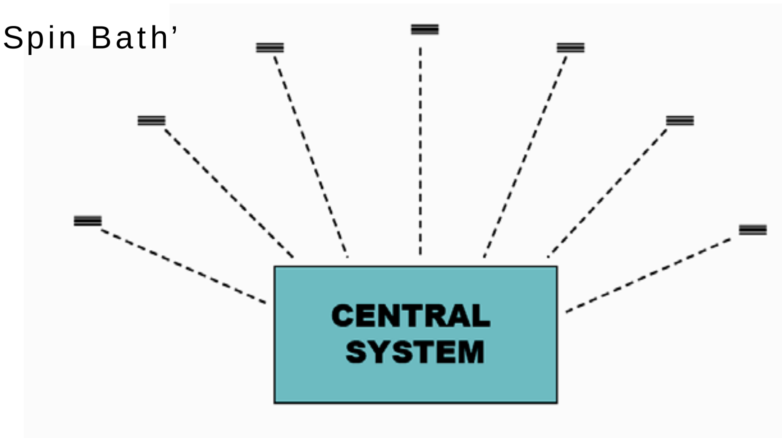
Caldeira & Leggett, Ann. Phys. 149, 374 (1983)
AJ Leggett et al, Rev Mod Phys 59, 1 (1987

DELOCALIZED BATH MODES

↓

OSCILLATOR BATH

‘Spin Bath’



$$H_{\rm eff}^{\rm sp}(\Omega_0) = H_0 + H_{\rm int}^{\rm sp} + H_{\rm env}^{\rm sp}$$

Bath: $$H_{\rm env}^{\rm sp} = \sum_{k}^{N_s} \mathbf{h}_k \cdot \boldsymbol{\sigma}_k + \sum_{k.k'}^{N_s} V_{kk'}^{\alpha\beta} \sigma_k^\alpha \sigma_{k'}^\beta$$

Interaction: $$H_{\rm int}^{\rm sp} = \sum_{k}^{N_s} \boldsymbol{F}_k(P,Q) \cdot \boldsymbol{\sigma}_k$$

NOT SMALL !

Defects, dislocation modes, vibrons,
Localized electrons, spin impurities,
nuclear spins, ...

LOCALIZED BATH MODES

↓

SPIN BATH

(1) P.C.E. Stamp, PRL 61, 2905 (1988)
(2) NV Prokof’ev, PCE Stamp, J Phys CM5, L663 (1993)
(3) NV Prokof’ev, PCE Stamp, Rep Prog Phys 63, 669 (2000)

CONDITIONS for DERIVATION of OSCILLATOR BATH MODELS

We get such models if one or other of 2 conditions is satisfied, viz.,

(1) PERTURBATION THEORY

Assume environmental states $\phi_\alpha(\mathbf{X})$ and energies ϵ_α

The system-environment coupling is $V(Q, \mathbf{X})$

Assume weak coupling: $|V_{\alpha\beta}| \ll |(\epsilon_\alpha - \epsilon_\beta)|$ where $V_{\alpha\beta} = \int d\mathbf{X}\, \phi^*_\alpha(\mathbf{X})\, V(Q,\mathbf{X})\, \phi_\beta(\mathbf{X})$

In this weak coupling limit we get oscillator bath with $\omega_q \equiv (\epsilon_\alpha - \epsilon_\beta)$

and couplings $F_q(Q) = V_q(Q)$

(2) BORN-OPPENHEIMER (Adiabatic) APPROXIMATION

Suppose now the couplings are not weak, but the system dynamics is SLOW, ie., Q changes with a characteristic low frequency scale E_o . We define slowly-varying environmental functions as follows:

Quasi-adiabatic eigenstates: $\tilde{\phi}_\alpha(\mathbf{X}, Q)$ Quasi-adiabatic energies: $\tilde{\epsilon}_\alpha(Q)$

'Slow' means: $E_o \ll \tilde{\epsilon}_\alpha$

Now define gauge potential: $iA_{\alpha\beta} = \int d\mathbf{X}\, \tilde{\phi}^*_\alpha(\mathbf{X})\, \partial/\partial Q\, \tilde{\phi}_\beta(\mathbf{X})$

We can now map to an oscillator bath if $|A_{\alpha\beta}| \ll |(\tilde{\epsilon}_\alpha - \tilde{\epsilon}_\beta)|$

Here the bath oscillators have energies $\omega_q \equiv (\tilde{\epsilon}_\alpha - \tilde{\epsilon}_\beta)$

and couplings $$F_q(P,Q) = \omega_q^2 \int_0^Q dQ'\, Re A_q(P,Q')$$

The oscillator bath models are good for describing delocalised modes; then usually $F_q(Q) \sim O(1/N^{1/2})$ (normalisation factor)

AO Caldeira, AJ Leggett, Ann Phys 149, 374 (1983)

All OK unless we have:
(i) degenerate bath modes
(ii) Localised bath modes

CONDITIONS for DERIVATION of SPIN BATH MODELS

We start again from a model of general form: $H_{\rm eff}^{\rm sp}(\Omega_0) = H_0 + H_{\rm int}^{\rm sp} + H_{\rm env}^{\rm sp}$

with interaction: $H_{\rm int}^{\rm sp} = \sum_k^{N_s} \boldsymbol{F}_k(P,Q)\cdot\boldsymbol{\sigma}_k$ and bath $H_{\rm env}^{\rm sp} = \sum_k^{N_s} \mathbf{h}_k\cdot\boldsymbol{\sigma}_k + \sum_{k,k'}^{N_s} V_{kk'}^{\alpha\beta}\sigma_k^\alpha\sigma_{k'}^\beta$

For this effective Hamiltonian to be valid we require that no other environmental levels couple significantly to the localised bath levels. We also require that the bath modes couple weakly to each other, satisfying the conditions:

(i) $|V_{kk'}| \ll |F_k(P,Q)| \quad \forall\, k,k'$ (intra-bath mode-mode coupling weak compared to the coupling to the central system); or:

(ii) $|V_{kk'}| \ll |h_k|$ The 'external fields' acting on the bath modes are much larger than the intra-bath couplings

There is no 'Born-Oppenheimer' requirement of 'slow' changes.
If the system changes on a timescale T,
with

$$\left|\frac{\partial_t F_k}{F_k}\right| T \sim O(1).$$

Then define:

$$u_k = \frac{1}{|F_k|T} \sim \left|\frac{\partial_t F_k}{F_k^2}\right|$$

The model is valid for all u_k:

$u_k \ll 1$ slow "adiabatic" limit

$u_k \gg 1$ fast "sudden" limit

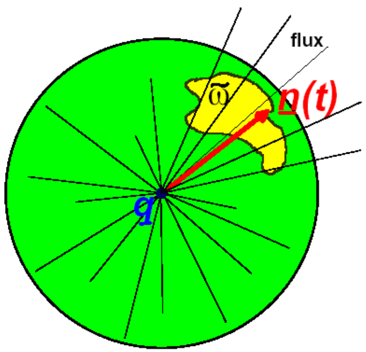
The bath action now becomes:

$$S_{env}^{sp} = \int d\tau \left[\sum_k^{N_s}\left(\mathcal{A}_k\cdot\frac{d\boldsymbol{\sigma}_k}{dt} - \mathbf{h}_k\cdot\boldsymbol{\sigma}_k\right) - \sum_{k,k'}^{N_s} V_{kk'}^{\alpha\beta}\sigma_k^\alpha\sigma_{k'}^\beta\right]$$

coupled to central system via: $S_{int}^{sp}(Q,\boldsymbol{\sigma}_k) = -\int d\tau \sum_k^{N_s} \boldsymbol{F}_k(P,Q)\cdot\boldsymbol{\sigma}_k$

Bath action contains topological term

$\phi_B = q/\hbar \int \mathcal{A}\cdot d\mathbf{n}$



NV Prokofev, PCE Stamp, Rep Prog 63, 669 (2000)
PCE Stamp, Stud Hist Phil Mod Phys 37, 467 (2006)

Summary – DYNAMICS of REDUCED DENSITY MATRIX

density matrix propagator: $$K(Q_2, Q_2'; Q_1, Q_1'; t, t') = \int_{Q_1}^{Q_2} \mathscr{D}q \int_{Q_1'}^{Q_2'} \mathscr{D}q' \, e^{-i/\hbar(S_0[q]-S_0[q'])} \mathscr{F}[q, q'],$$

with $$\mathcal{F}[Q, Q'] = \prod \langle \hat{U}_k(Q, t) \hat{U}_k^\dagger(Q', t) \rangle$$

OSCILLATOR BATHS

Each bath oscillator has Lagrangian: $$L_q(x_q, \dot{x}_q; t) = \frac{m_q \dot{x}_q^2}{2} - \Upsilon_q(t) x_q$$

where, including system-bath coupling, the force is $\Upsilon_q(t) = m_q \omega_q^2 x_q - F_q(Q(t))$

Then:

$$\mathcal{F}[Q, Q'] = \prod_q^{N_o} \int \mathcal{D}x_q(\tau) \int \mathcal{D}x_q(\tau') \; \exp\left[\frac{i}{\hbar} \int d\tau \frac{m_q}{2} [\dot{x}_q^2 - \dot{x}_q'^2 + \omega_q^2 (x_q^2 - x_q'^2)] \; + \; [F_q(Q) x_q - F_q(Q') x_q']\right]$$

$$= \exp\left[-\frac{1}{\hbar} \int_{t_o}^{t} d\tau_1 \int_{t_o}^{\tau_1} d\tau_2 [q(\tau_1) - q'(\tau_2)][\mathcal{D}(\tau_1 - \tau_2) q(\tau_2) - \mathcal{D}^*((\tau_1 - \tau_2) q'(\tau_2)]\right]$$

Bath propagator

NB: This last result is very easy to get – this is because the integral is of Gaussian form

SPIN BATHS

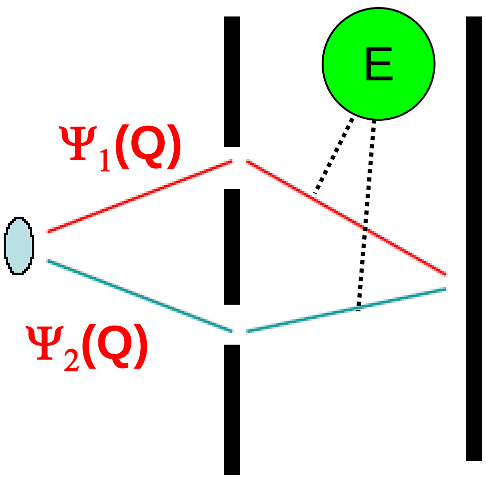
Each bath spin has Lagrangian: $$L(\boldsymbol{\sigma}_k, \dot{\boldsymbol{\sigma}}_k; t) = \mathscr{A}_k \cdot \frac{d\boldsymbol{\sigma}_k}{d\tau} - \boldsymbol{\Upsilon}_k(t) \cdot \boldsymbol{\sigma}_k$$

with force $\boldsymbol{\Upsilon}_k(t) = \mathbf{h}_k + \mathbf{F}_k(t) + \boldsymbol{\xi}_k(t)$ & 'noise' $\xi_k^\alpha(t) = \sum_{k'} V_{kk'}^{\alpha\beta} \sigma_{k'}^\beta(t) \sim \sum_{k'} V_{kk'}^{\alpha\beta} \langle \sigma_{k'}^\beta(t) \rangle$

Then: $$\mathcal{F}[Q, Q'] = \prod_k^{N_s} \int \mathcal{D}\boldsymbol{\sigma}_k(\tau) \int \mathcal{D}\boldsymbol{\sigma}_k(\tau') \; \exp\left[\frac{i}{\hbar} (S_{int}[Q, \boldsymbol{\sigma}_k] - S_{int}[Q', \boldsymbol{\sigma}_k'] \; + \; S_E[\boldsymbol{\sigma}_k] - S_E[\boldsymbol{\sigma}_k'])\right]$$

Typically there is no Gaussian integral here – the resulting form is more complex

The PROBLEM of ENVIRONMENTAL DECOHERENCE



Some quantum system with coordinate Q interacts with any other system (with coordinate x) ; typically they then form an entangled state

Example: In a 2-slit expt., the particle coordinate Q couples to photon coordinates, so that:

$$\Psi_o(Q) \;\; \Pi_q \phi_q^{in} \;\; \rightarrow \;\; [\, \Psi_1(Q)\, \Pi_q\, \phi_q^{(1)} \;\; + \;\; \Psi_2(Q)\, \Pi_q\, \phi_q^{(2)} \,]$$

Then, goes the story, if we have no control over, or knowledge of the photon states, we must then average over them. The result of this toy analysis is a reduced density matrix of form

$$\rho_{\alpha\beta} \;\; = \;\; \begin{pmatrix} |\Psi_1(Q)|^2 & \Psi_1^*(Q)\, \Psi_2(Q)\, D_{12} \\ \Psi_2^*(Q)\, \Psi_1(Q)\, D_{12}^* & |\Psi_2(Q)|^2 \end{pmatrix}$$

where we define $D_{12} \; = \; \Pi_q \;\; \Pi_{q'} < \phi_q^{(1)} \,|\, \phi_{q'}^{(2)} >$

This model tells us very little – it doesn't even have time dependence.
However it does suggest that:

Remark 1: The "Environment" E (which is in effect performing a measurement), there is no need for energy to be exchanged - only a communication of phase information.

Remark 2: What is crucial here is that the state of the environment be CORRELATED WITH (ie., CONDITIONAL ON) the state of the system.

We now want to see how environmental decoherence actually works

REAL WORLD PROBLEM #1

DYNAMICS of QUANTUM VORTICES

(application to Bose superfluids)

Only wimps specialize in the general case. Real scientists pursue examples.

MV Berry: Ann NY Acad Sci 755, 303 (1995)

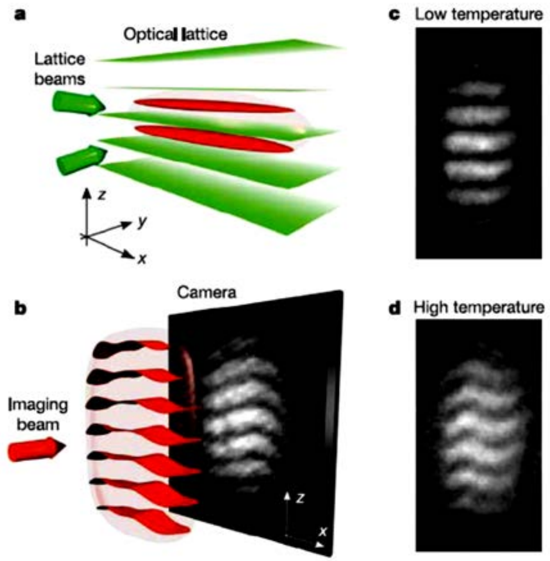
VORTEX DYNAMICS in BOSE SUPERFLUIDS

Since the original theory of superfluid vortices (Onsager, 1950; Feynman, 1951-53) & their discovery (Vinen, 1956) enormous experimental work has been done. Current interesting experiments:

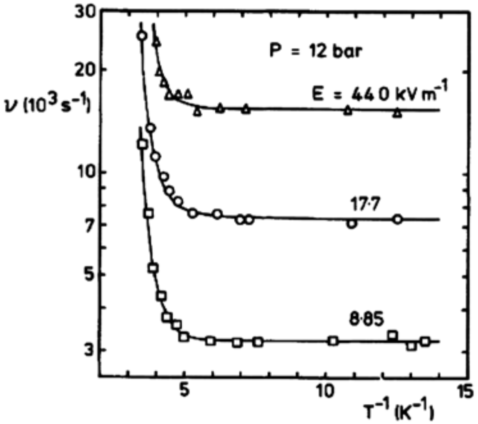
Quasi- 2-dimensional cold BEC systems

One can insert or nucleate a single vortex in this system, & watch its time dynamics as it spirals out from the centre.

Z Hadzibabic et al., Nature 441, 1118 (2006)



Vortex nucleation (He-4 & He-3)



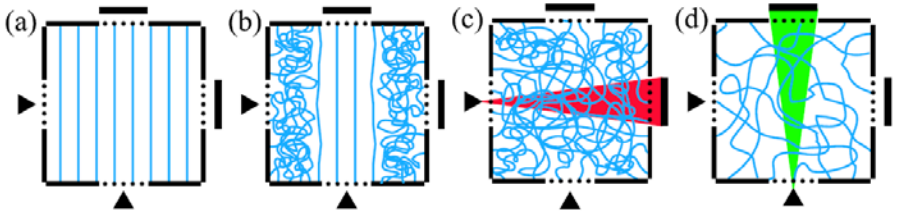
The tunneling rate & dynamics of vortex rings & other vortex configurations will be influenced by the quasiparticles in interesting ways.

PC Hendry et al., PRL 60, 604 (1988)

Turbulence

Intrinsically 3-dimensional and multi-vortex in nature.

P Walmsley et al., PRL 99, 265302 (2007)
P Walmsley, A Golov, PRL 100, 245301 (2008)



HISTORY of the PROBLEM: FORCES on a QUANTUM VORTEX

The fundamental question of quantum vortex dynamics has been highly controversial. Typically discussed in terms of FORCES:

"Berry" force

Superfluid Magnus force $\mathbf{F}_M = \rho_s \boldsymbol{\kappa}_V \times (\dot{\mathbf{r}}_V(t) - \mathbf{v}_s)$

Magnetic gyrotropic force $\mathbf{F}_g = G M_s q_V p_V \hat{\mathbf{z}} \times \dot{\mathbf{r}}_V$

"Iordanski" force

Superfluid Iordanski force $\mathbf{F}_I = \rho_n \boldsymbol{\kappa}_V \times (\dot{\mathbf{r}}_V(t) - \mathbf{v}_n)$

Drag force

Vortex-phonon drag - superfluid $\mathbf{F}_\parallel = -\gamma(\dot{\mathbf{r}}_V(t) - \mathbf{v}_n)$

Vortex-magnon drag - ferromagnet $\mathbf{F}_\parallel = \eta_m \dot{\mathbf{r}}_V(t)$

Inertial term

Inertial mass term: $M_V \ddot{\mathbf{r}}_V(t)$

Hall H E and Vinen W F 1956 *Proc. R. Soc.* A **238** 204,215

S.V. Iordanskii, Ann. Phys (NY) **29**, 335-49 (1964); and JETP **22**, 160-167 (1965)

E.B. Sonin, Phys. Rev. **B55**, 485 (1997)

D.J. Thouless, P. Ao, Q. Niu, Phys. Rev. Lett. **76**, 3758 (1996)

G.E. Volovik, Phys. Rev. Lett. **77**, 4687 (1997)

D. J. Thouless and J. R. Anglin, PRL **99,** 105301 (2007)

PROBLEM 1: Disagreement on most of the terms except Magnus term

PROBLEM 2: All these theories treat the vortex as a classical object!

In what follows we discuss a fully quantum treatment of this problem:

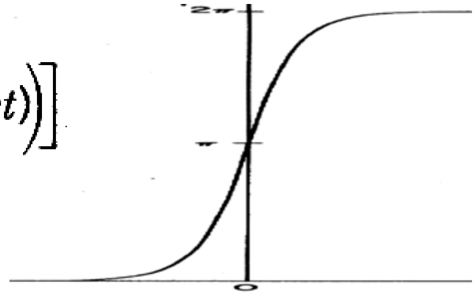
L Thompson. PCE Stamp, Phys Rev Lett 108, 184501 (2012)

L Thompson, PCE Stamp, J Low Temp Phys 171, 526 (2012)

T Cox, PCE Stamp, J Low Temp Phys 171, 459 (2012)

PRELIMINARY: QUANTUM SOLITON + QUASIPARTICLES

1-d Sine-Gordon model: $$\frac{\partial^2\psi}{\partial t^2} - c_0^2\frac{\partial^2\psi}{\partial x^2} + \omega_0^2 \sin\psi = 0$$



Single 'kink' soliton: $$\psi_\pm^v(x,t) = 4\tan^{-1}\left[\exp\left(\pm\frac{\omega_0}{c_0}\gamma(x-vt)\right)\right]$$

where: $$\gamma \equiv (1 - v^2/c_0^2)^{-1/2}, \quad |v| < c_0$$

Now add small oscillations: $$\psi_\pm(x,t) = \psi_\pm^v(x,t) + \phi(x,t)$$

Quasiparticle eqtn of motion: $$\frac{\partial^2\phi}{\partial t^2} - c_0^2\frac{\partial^2\phi}{\partial x^2} + \omega_0^2\left(1 - 2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right)\phi = 0$$

Corresponds to waves, but with 'kink potential': $$V(x) = \omega_0^2\left(1 - 2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right)$$

These 'quasiparticles' have Lagrangian: $$\mathcal{L} = \phi^*\partial_\mu\partial^\mu\phi - \phi^* V(x-Q)\phi$$
(for wall at position Q)

Assume: $$\phi(x,t) = f(x)\,e^{-i\omega t}$$

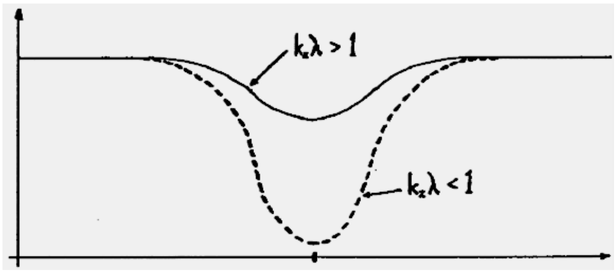
Bound QP mode:

$$f_b(x) = \frac{2\omega_0}{c_0}\,\mathrm{sech}\,\frac{\omega_0}{c_0}x$$

Extended QP modes: $$\omega_\kappa^2 = c_0^2\kappa^2 + \omega_0^2$$

$$f_\kappa(x) = \frac{1}{(2\pi)^{1/2}}\frac{c_0}{\omega_\kappa}e^{i\kappa x}\left(\kappa + i\frac{\omega_0}{c_0}\tanh\frac{\omega_0}{c_0}x\right)$$

This is typical – extended QP modes avoid the soliton and the bound states.



DESCRIPTION of a BOSE SUPERFLUID

Assume Actio $S = -\int d^2r dt \left[\frac{\rho}{m_0} \left(\hbar\dot{\Phi} + \frac{(\hbar\nabla\Phi)^2}{2m_0} \right) + \epsilon[\eta, \nabla\eta] \right]$

$$\boxed{\rho_s(\mathbf{r}, t) = \rho_s^o + \eta(\mathbf{r}, t)}$$

This is a 'long wavelength' action – valid for energy $\ll \quad \Lambda_o = \hbar c_o/a_o = m_o/\chi\rho_s$

and for lengths $\gg a_o$ where $a_o^2 \sim \hbar^2\chi\rho_s/m_o^2$

Sound velocity: $c_o = 1/\sqrt{\chi\rho_s}$

Superfluid velocity: $\mathbf{v} = (\hbar/m_o)\nabla\Phi$

Compressibility: χ

$$\epsilon(\eta, \nabla\eta) = \frac{\hbar^2}{8m_o^2\rho}(\boldsymbol{\nabla}\eta)^2 + \frac{\eta^2}{2\chi\rho_s^2} + V(\eta, \nabla\eta)$$

$$\dot{\Phi} + \frac{|\nabla\Phi|^2}{2} + \frac{\delta\epsilon}{\delta\eta} = 0$$

$$\dot{\eta} + \nabla\cdot((1+\eta)\nabla\Phi) = 0$$

Q Bernoulli Force

Mass conservation

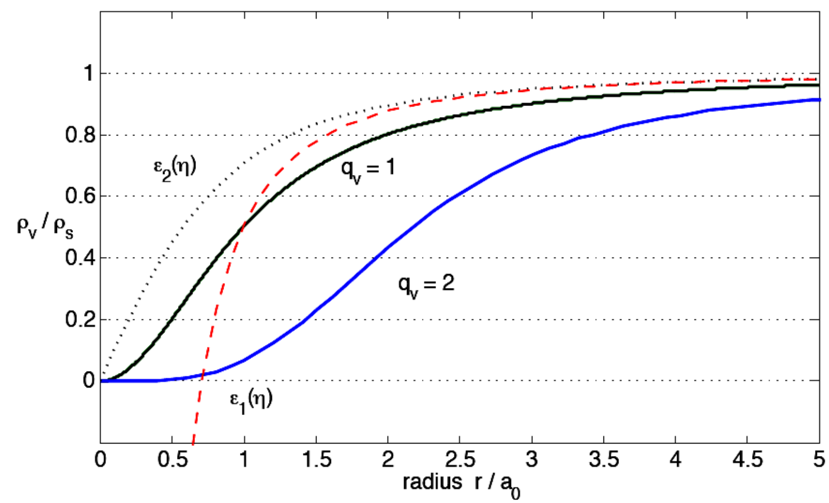
QUANTUM VORTEX

Separate out the vortex phase:

$$\Phi(\mathbf{r}, t) = \Phi_v(\mathbf{r}, t) + \phi(\mathbf{r}, t)$$

Quantized circulation: $\kappa = qh/m_o$

$$\rho_R^V = \rho_s \begin{cases} \frac{|\mathbf{r}-\boldsymbol{R}|^2}{\sqrt{2}a_0^2} & \text{for } |\mathbf{r}-\boldsymbol{R}| \ll a_0, \\ 1 - \frac{q_v^2 a_0^2}{\sqrt{2}|\mathbf{r}-\boldsymbol{R}|^2} & \text{for } |\mathbf{r}-\boldsymbol{R}| \gg a_0 \end{cases}$$



Vortex inertial mass: $M_V = -\frac{\partial^2 E_v}{\partial v^2}$

eg., for cylinder: $M_V = \frac{1}{4}\rho_s^2\chi\kappa^2 \log R_s/a_0$

VORTEX-PHONON SCATTERING: FORMAL

THE KEY POINTS HERE:

wrong to use free phonons

True phonons are altered by vortex, & couple quadratically to the vortex

$$\Delta S_{int}^{(2)} = \frac{\hbar}{2m_0}\int d^2r dt\, (\eta\nabla\phi - \phi\nabla\eta)\cdot(\dot{\boldsymbol{R}} - \boldsymbol{v}_n)$$

cylindrical coords:

$$\tilde{\phi}(r,\theta,t) = \tilde{\phi}_{lk}(r)\sin(\omega_k t + l\theta)$$
$$\tilde{\eta}(r,\theta,t) = -\tilde{\eta}_{lk}(r)\cos(\omega_k t + l\theta)$$

quasiparticle eqtn of motion:

$$\frac{\hbar}{m_0}\begin{pmatrix} \hat{\mathcal{D}}_1 & \frac{lq_V}{r} - \frac{m_0}{\hbar}\omega_k r \\ \frac{lq_V}{r} - \frac{m_0}{\hbar}\omega_k r & \hat{\mathcal{D}}_2 \end{pmatrix}\begin{pmatrix} \tilde{\phi}_{lk} \\ \tilde{\eta}_{lk} \end{pmatrix} = 0$$

where:

$$\hat{\mathcal{D}}_1 = \partial_r(r\rho_V(r)\partial_r) - \frac{l^2}{r}\rho_V(r)$$

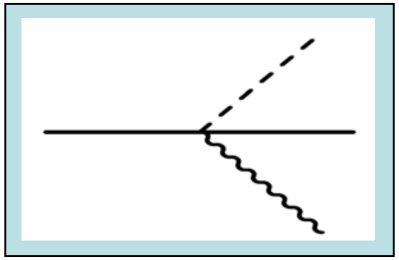
$$\hat{\mathcal{D}}_2 = \frac{1}{4\rho_V(r)}\partial_r\left(r\frac{\partial_r}{\rho_V(r)}\right) - \left[\frac{l^2}{4r\rho_V(r)} + \frac{r}{\rho_s a_0^2}\partial_r\left(r\frac{\partial_r\rho_V(r)}{\rho_V(r)}\right)\right]$$

This yields a vortex-quasiparticle interaction matrix:

$$\Lambda_{kq}^{\sigma l} = \int\frac{dr}{2m_0}\left[r(\tilde{\phi}_{lk}\partial_r\tilde{\eta}_{l+\sigma,q} + \tilde{\eta}_{lk}\partial_r\tilde{\phi}_{l+\sigma,q}) + \sigma(l+\sigma)(\tilde{\phi}_{lk}\tilde{\eta}_{l+\sigma,q} + \tilde{\eta}_{lk}\tilde{\phi}_{l+\sigma,q})\right]$$

so that

$$\Lambda_{kq}^{\sigma 0} = \frac{k+q}{4\sqrt{kq}}\delta(k-q) + \frac{a_0}{4}\begin{cases} \frac{\sigma}{2}\left(\frac{k}{q}\right)^{\frac{3}{2}} + \sigma\sqrt{kq}\frac{k}{q(k-q)} & \text{if } k < q \\ \frac{k+q}{16\sqrt{kq}}a_0 q + \frac{\sigma}{2}\left(\frac{q}{k}\right)^{\frac{1}{2}} + \sigma\sqrt{kq}\frac{q}{k(k-q)} & \text{if } q \le k \end{cases}$$



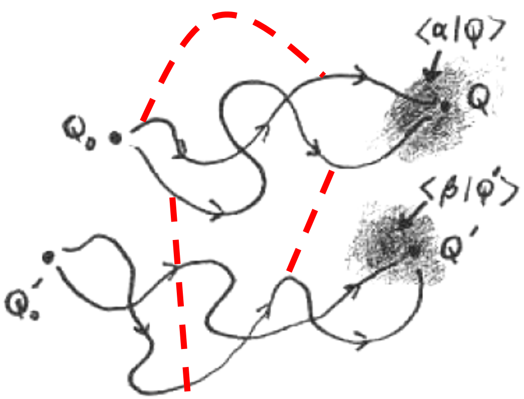
PATH INTEGRAL FORMULATION for N-Particle BOSE SUPERFLUID

For the N-particle system, we write the full N-particle density operator $\hat{\rho}_N = |\Psi\rangle\langle\Psi|$

where $\Psi(\{\mathbf{r}_j\}) = \langle\{\mathbf{r}_j\}|\Psi\rangle$, and the density matrix is $\rho_N(\{\mathbf{r}_j\}, \{\mathbf{r}'_j\}) = \langle\{\mathbf{r}_j\}|\hat{\rho}_N|\{\mathbf{r}'_j\}\rangle$.

We then introduce collective coordinates in which the vortex node position is singled out for special attention; this allows us to write

$$\rho_N(\{\boldsymbol{R}, \mathbf{q}_k\}; \{\boldsymbol{R}', \mathbf{q}'_k\}) = \langle\{\boldsymbol{R}; \mathbf{q}_j\}|\hat{\rho}_N|\{\boldsymbol{R}', \mathbf{q}'_j\}\rangle$$



with a reduced density matrix, for the vortex coordinate, given by

$$\begin{aligned}\bar{\rho}(\boldsymbol{R}, \boldsymbol{R}', t) &= \mathrm{Tr}_{\mathbf{q}_k}\, \rho_N(\{\boldsymbol{R}, \mathbf{q}_k\}; \{\boldsymbol{R}', \mathbf{q}_k\}) \\ &= \prod_k \int d\mathbf{q}_k\, \rho_N(\{\boldsymbol{R}, \mathbf{q}_k\}; \{\boldsymbol{R}', \mathbf{q}_k\})\end{aligned}$$

The propagator for the vortex density matrix is then:

$$K(2,1) = \int_{\boldsymbol{R}_1}^{\boldsymbol{R}_2} \mathcal{D}\boldsymbol{R}(t) \int_{\boldsymbol{R}'_1}^{\boldsymbol{R}'_2} \mathcal{D}\boldsymbol{R}'(t)\, e^{\frac{i}{\hbar}(\tilde{S}_V^0[\boldsymbol{R}] - \tilde{S}_V^0[\boldsymbol{R}'])} \mathcal{F}[\boldsymbol{R}(t), \boldsymbol{R}'(t)]$$

with bare vortex action: $\mathcal{S}_V = \frac{1}{2}\int dt \left[\rho_s(\boldsymbol{\kappa} \times (\dot{\boldsymbol{R}} - \boldsymbol{v}_s)) \cdot \boldsymbol{R} + M_v^0 \dot{R}^2\right]$

REMARK 1: All the complexity of this problem comes from the singular form of the matrix element coupling to the vortex to the phonons, and the fact that we couple to PAIRS of phonons – otherwise it is just a simple oscillator bath system

REMARK 2: There is no coupling to single phonons because the vortex is a soliton

VORTEX DYNAMICS

Define: $\mathbf{R}_v(t) = (\boldsymbol{R}+\boldsymbol{R}')/2$ Vortex 'centre of mass'

$\boldsymbol{\xi}(t) = \boldsymbol{R}-\boldsymbol{R}'$ Vortex 'quantum fluctuation'

Messy calculations give: $\mathcal{F}[\mathbf{R}_v, \boldsymbol{\xi}] = \exp\left[-i\sum_{l\sigma,kq}(\Lambda_{kq}^{\sigma l})^2\int_{t_1}^{t_2}dt\int_{t_1}^{t}ds\,\mathcal{D}_{kq}^{\sigma}(\dot{\mathbf{R}}_v, \dot{\boldsymbol{\xi}}; t, s)\right]$

where if we write $\mathcal{D} = \Phi + i\Gamma$

the reactive part is

$$\Phi_{kq}^{\sigma}(\dot{\mathbf{R}}_v, \dot{\boldsymbol{\xi}}; t, s) = (n_k - n_q)\,[\dot{\boldsymbol{\xi}}(t)\cdot\dot{\mathbf{R}}_v(s)\sin\Omega_{kq}(t-s) + \sigma\hat{\mathbf{z}}\cdot(\dot{\boldsymbol{\xi}}(t)\times\dot{\mathbf{R}}_v(s))\cos\Omega_{kq}(t-s)]$$

& the

Decoherence Functional is

$$\Gamma_{kq}^{\sigma}(\dot{\boldsymbol{\xi}}; t, s) = -n_k(1+n_q)\,[\dot{\boldsymbol{\xi}}(t)\cdot\dot{\boldsymbol{\xi}}(s)\sin\Omega_{kq}(t-s) - \sigma\hat{\mathbf{z}}\cdot(\dot{\boldsymbol{\xi}}(t)\times\dot{\boldsymbol{\xi}}(s))\cos\Omega_{kq}(t-s)]$$

This result is equivalent to two integro-differential eqtns of motion for the 2 vortex coordinates, taking the form of non-local Langevin eqns:

$$M_v^0\ddot{\boldsymbol{\xi}}(t) - f_M(\dot{\boldsymbol{\xi}}) - \mathbf{F}_{QP}^{\xi}[\dot{\boldsymbol{\xi}}(t)] = 0$$

$$M_v^o\ddot{\mathbf{R}}_v(t) - f_M(\dot{\mathbf{R}}_v) - \mathbf{F}_{QP}^{\mathbf{R}}[\dot{\mathbf{R}}_v - \mathbf{v}_n] = \mathbf{F}_{fluc}(t)$$

Magnus force Quantum noise force

with a non-local (in spacetime) force from the quasiparticles:

$$\mathbf{F}_{QP}^{\mathbf{R}}[\dot{\mathbf{R}}_v - \mathbf{v}_n] = \mathbf{F}_{\parallel}^{R}[\dot{\mathbf{R}}_v - \mathbf{v}_n] + \mathbf{F}_{\perp}^{R}[\dot{\mathbf{R}}_v - \mathbf{v}_n]$$

where, eg., $\mathbf{F}_{\parallel}^{R}[\dot{\mathbf{R}}_v - \mathbf{v}_n] = \frac{\hbar}{L_z}\sum_{m\sigma kq}(\Lambda_{kq}^{\sigma m})^2\Omega_{kq}(n_k - n_q)\int_{t_1}^{t}ds\,(\dot{\mathbf{R}}_v(s) - \boldsymbol{v}_n)\cos[\Omega_{kq}(t-s)]$

Both the quasiparticle & fluctuation forces, acting on the vortex coordinates, have long-time (highly non-Markovian) and long-range spatial components

COMPARISON – CLASSICAL vs QUANTUM DYNAMICS

Recall the classical Hall-Vinen-Iordanski equation: $M_v \ddot{\boldsymbol{r}}_v - \boldsymbol{f}_M - \boldsymbol{f}_{qp} - \boldsymbol{F}_{ac}(t) = 0$

with Magnus force: $\boldsymbol{f}_M = \rho_s \boldsymbol{\kappa} \times (\dot{\boldsymbol{r}}_v - \boldsymbol{v}_s)$

& LOCAL quasiparticle force: $\boldsymbol{f}_{qp} = D_0(\boldsymbol{v}_n - \dot{\boldsymbol{r}}_v) + D_0' \hat{\boldsymbol{z}} \times (\boldsymbol{v}_n - \dot{\boldsymbol{r}}_v)$

where $D_0'(T) = -\kappa \rho_n(T)$ (Iordanski coefficient)

Now let's FOURIER TRANFORM to the frequency domain:

$$R_v^i(\Omega) = A^{ij}(\Omega, n_{\boldsymbol{q}}) F^j(\Omega)$$

with "admittance" matrix $$\boldsymbol{A}_o(\Omega) = \frac{1}{\mathbb{D}_o(\Omega)} \begin{pmatrix} -\Omega^2 M_v + i\Omega D_o & -i\kappa\rho\Omega \\ i\kappa\rho\Omega & -\Omega^2 M_v + i\Omega D_o \end{pmatrix}$$

with determinant $\mathbb{D}_o(\Omega) = [\Omega^2 M_v - i\Omega D_o]^2 - [\kappa\rho\Omega]^2$

and with driving force: $\mathbf{F} = \mathbf{F}_{ac}(\Omega) - q_v \boldsymbol{\kappa} \times \mathbf{J}(\Omega)$ where $\mathbf{J} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$

CORRECT QUANTUM EQUATION of MOTION

Same form as above, except that now:

$$\boldsymbol{A}^R(\Omega, [n_{\boldsymbol{q}}]) = \frac{1}{\mathbb{D}_R(\Omega)} \begin{pmatrix} -\Omega^2 M_v(\Omega) + i\Omega D_{\|}(\Omega) & -i\kappa\rho\Omega - |\Omega| d_{\perp}(\Omega) \\ i\kappa\rho\Omega - |\Omega| d_{\perp}(\Omega) & -\Omega^2 M_v(\Omega) + i\Omega D_{\|}(\Omega) \end{pmatrix}$$

and we have a new driving force: $\mathbf{F} = [\mathbf{F}_{ac}(\Omega) + \mathbf{f}_{\perp} + \boldsymbol{F}_{fl}(\Omega, n_{\boldsymbol{q}})]$

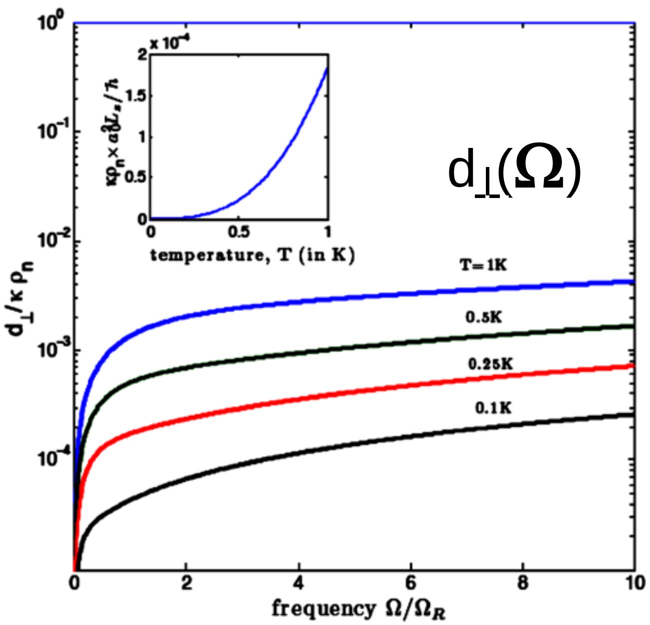
The long-time memory effects yield infrared strong terms in these eqtns.

VISCOUS DAMPING TERMS

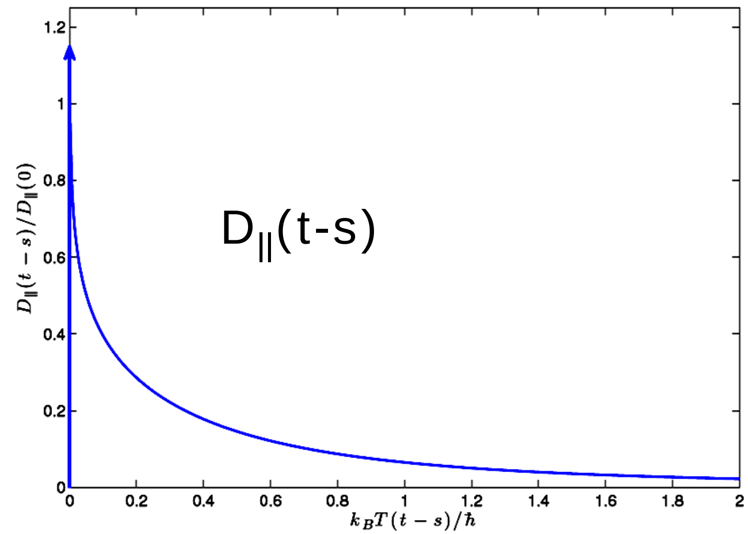
The longitudinal term looks like:

$$D_{\parallel}(\Omega) = \frac{3\hbar m_0 q_V^2 \pi}{16 L_z \tau_0} \left(\frac{k_B T}{\hbar \tau_0^{-1}} \right)^4$$

$$\times \begin{cases} 16\zeta(4) & \Omega \to 0 \\ \zeta(4) + \frac{6\zeta(5) k_B T}{\hbar\Omega} & \Omega \to \infty \end{cases}$$



Here we see the long-time tail behaviour



The transverse correction to the Iordanskii force is actually very small – it also has a long-time tail.

Note the totally different limits in quantum & classical regimes.

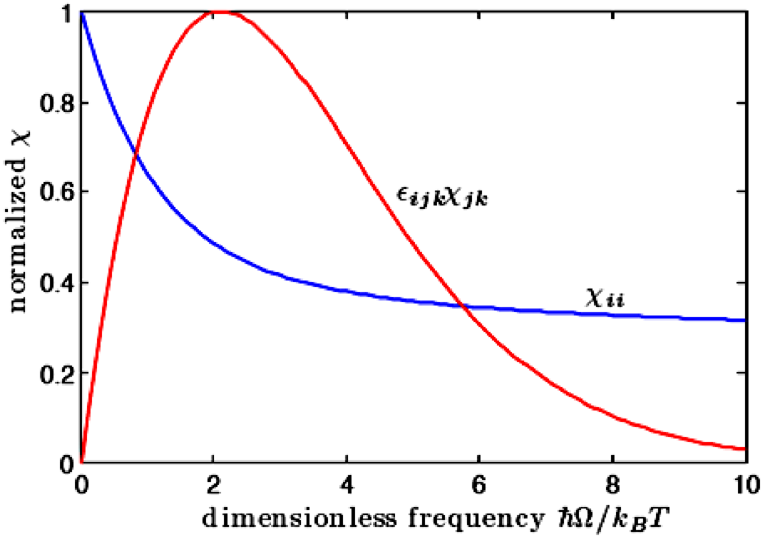
EFFECTIVE MASS

This is frequency-dependent, but this dependence depends on the sample geometry. All previous calculations have missed an essential feature – the 'self-acceleration' term. This problem is still unsolved.

FLUCTUATION FORCE

Define: $\chi_{ij}(t,s;[n_q]) = \langle F^i_{fluc}(t) F^j_{fluc}(s) \rangle$

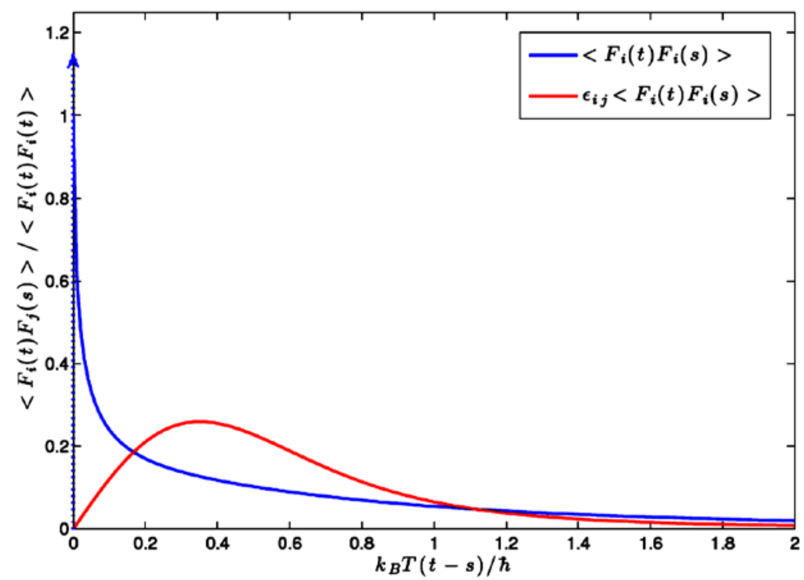
Then we find:
$$\chi_{ij}(t,s;[n_q]) = \frac{\hbar^2}{L_z^2} \sum_{m\sigma kq} \Omega_{kq}^2 n_k (1+n_q)(\Lambda_{kq}^{\sigma m})^2$$
$$\times \left(\delta_{ij}\cos[\Omega_{kq}(t-s)] - \epsilon_{ijz}\sigma \sin[\Omega_{kq}(t-s)]\right)$$



which gives:
$$\chi_{ii}(t-s;T) = \frac{3\hbar m_0 q_V^2 \pi}{2L_z^2 \tau_0} \left(\frac{k_B T}{\hbar \tau_o^{-1}}\right)^4 \zeta(5)\,\delta(t-s)$$
$$+\ \delta\chi_{ii}(t-s;T)$$

Long-time tail: $\delta\chi_{ii}(t-s) \propto (t-s)^{-2}$

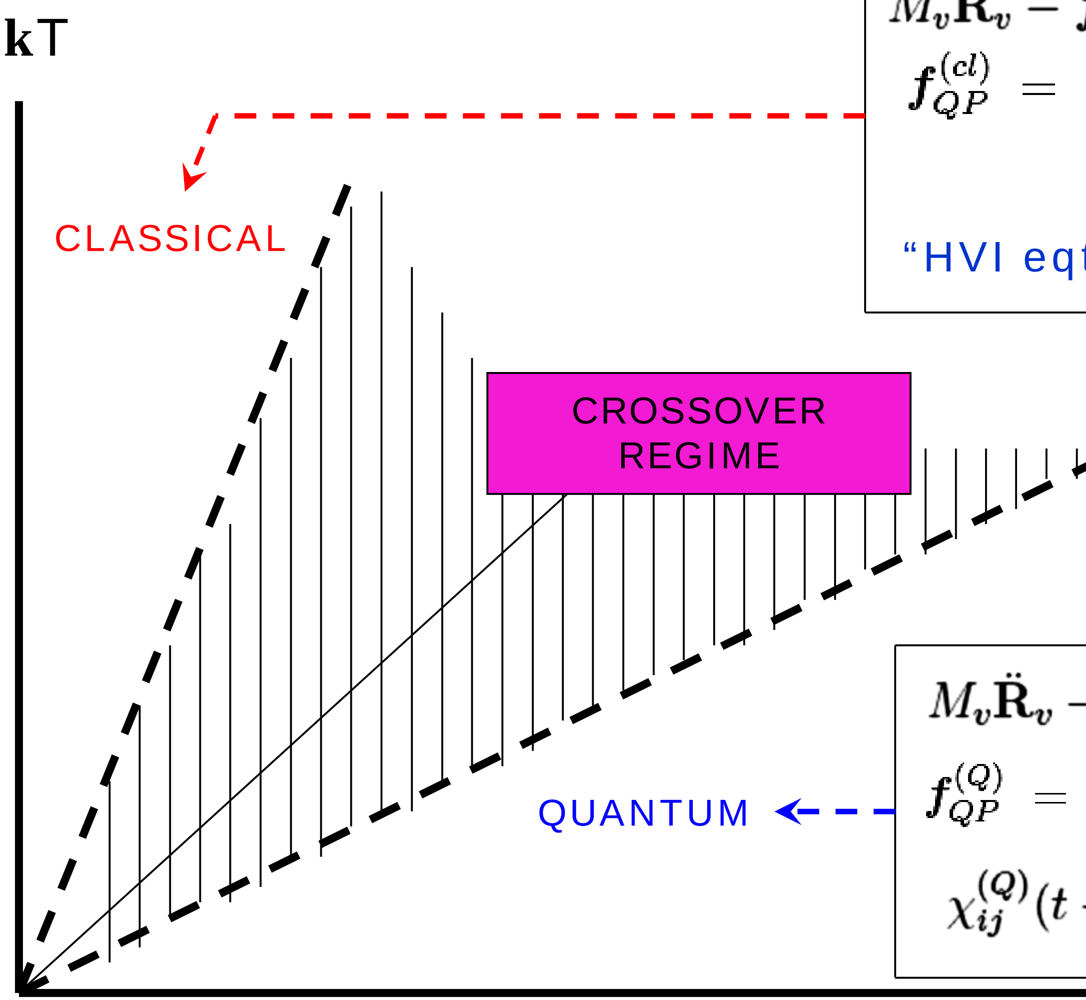
$$\chi_{ii}(\Omega;T) = \frac{3\hbar m_0 q_V^2 \pi}{2L_z^2 \tau_0} \left(\frac{k_B T}{\hbar \tau_0^{-1}}\right)^4$$
$$\text{x} \begin{cases} 4\zeta(5) & \Omega \to 0 \\ \zeta(5) + \frac{5\zeta(6) k_B T}{\hbar\Omega} & \Omega \to \infty \end{cases}$$



UPSHOT: This ‘noise’ term is very non-Markovian indeed → long-time memory effects.

OVERVIEW of RESULTS

All results depend simply on the ratio of thermal to quantum fluctuations



CLASSICAL REGIME

$$M_v \ddot{\mathbf{R}}_v - \boldsymbol{f}_M - \boldsymbol{f}_{QP}^{(cl)} - \mathbf{F}_{ac}(t) = \mathbf{F}_{fluc}^{(cl)}(t,T)$$

$$\boldsymbol{f}_{QP}^{(cl)} = D_o(\boldsymbol{v}_n - \dot{\mathbf{R}}_v) + D_o' \hat{\mathbf{z}} \times (\boldsymbol{v}_n - \dot{\mathbf{R}}_v)$$

$$\chi_{ij}^{(cl)}(T) = \delta_{ij} \chi_o^{\|}(T)$$

"HVI eqtns + Inertial and noise terms"

Local eqtns in 2 extremes

Highly non-local in crossover

QUANTUM REGIME

$$M_v \ddot{\mathbf{R}}_v - \mathbf{f}_M - \boldsymbol{f}_{qp}^{(Q)} - \mathbf{F}_{ac}(t) = \mathbf{F}_{fluc}^{(Q)}(t,T)$$

$$\boldsymbol{f}_{QP}^{(Q)} = \frac{1}{16} D_o(\boldsymbol{v}_n - \dot{\mathbf{R}}_v) + D_o' \hat{\mathbf{z}} \times (\boldsymbol{v}_n - \dot{\mathbf{R}}_v)$$

$$\chi_{ij}^{(Q)}(t-s,T) = \frac{\zeta(5)}{4\zeta(4)} \chi_o^{\|}(T) \delta_{ij} \delta(t-s)$$

There is a whole new Quantum regime to explore!!