Waves and quantum physics on fractals :

From continuous to discrete scaling symmetry

Eric Akkermans
Physics-Technion

Four lectures

- General introduction - Photons and Quantum Electrodynamics on fractals
- Interplay between topology and discrete scaling symmetry : Quasi-crystals
- Critical behaviour on fractals : BEC and superfluidity
- Efimov physics from geometric and spectral perspectives

Benefitted from discussions and collaborations with:

Technion:

Evgeni Gurevich (KLA-Tencor)
Dor Gittelman
Ariane Soret (ENS Cachan)
Or Raz
Omrie Ovdat
Ohad Shpielberg
Alex Leibenzon

Rafael:

Eli Levy
Assaf Barak
Amnon Fisher

Elsewhere:

Gerald Dunne (UConn.)
Alexander Teplyaev (UConn.)
Raphael Voituriez (LPTMC, Jussieu)
Olivier Benichou (LPTMC, Jussieu)
Jacqueline Bloch (LPN, Marcoussis)
Dimitri Tanese (LPN, Marcoussis)
Florent Baboux (LPN, Marcoussis)
Alberto Amo (LPN, Marcoussis)
Julien Gabelli (LPS, Orsay)

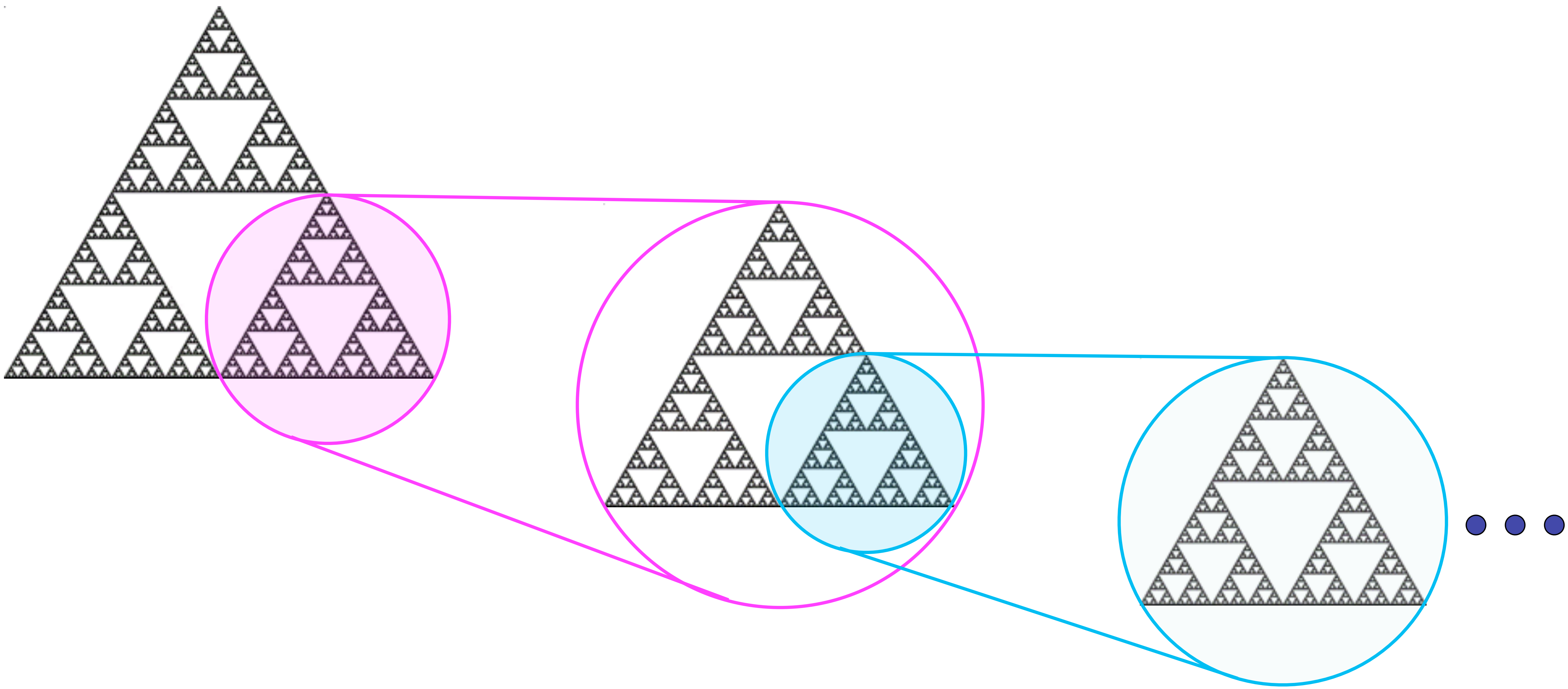
Part 1

A brief digest of some salient previous results

As opposed to Euclidean spaces characterised by translation symmetry, fractals possess a dilatation symmetry.

Fractals are self-similar objects

Fractal ↔ Self-similar



Discrete scaling symmetry

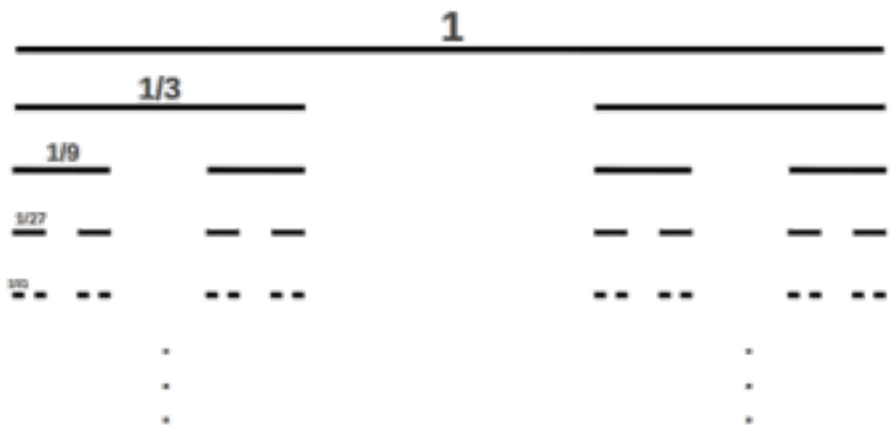
Discrete scale invariance (DSI)

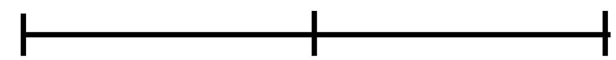
discrete scale invariance is expressed by a weaker version of scale invariance, *i.e.*,

$$f\left(ax\right)=b\,f\left(x\right),\quad \textit{with fixed}\left(a,b\right)$$

whereas this relation is satisfied $\forall\, b\left(a\right)\in\mathbb{R}$, for continuous scale invariance

Relation between the two cases : discrete *vs*. continuous

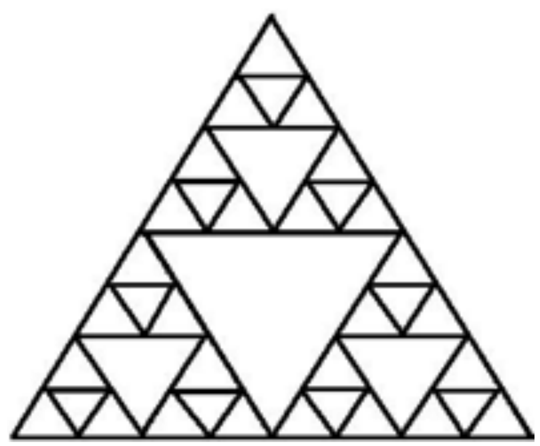


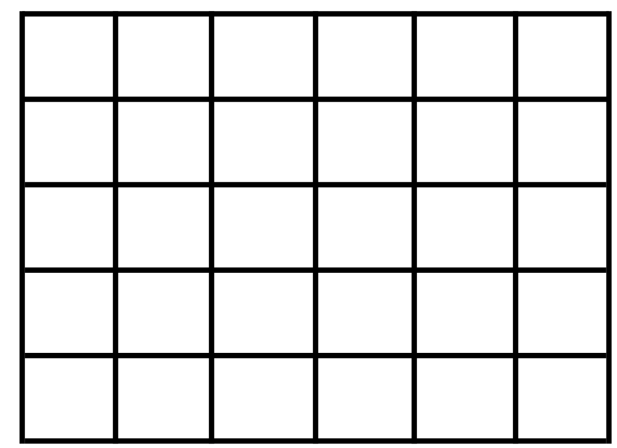


$d = 1$

$m(2L) = 2\ m(L)$ $\quad \forall b(a) \in \mathbb{R}$

$m(3L) = 2m(L)$ $\quad (a,b) = (3,2)$





$d = 2$

$m(2L) = 3m(L)$ $\quad (a,b) = (2,3)$

Both satisfy $f(ax) = b\,f(x)$ but with fixed (a,b) for the fractals.

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f(ax) = b\, f(x)$$

If satisfied $\forall\, b(a) \in \mathbb{R}$ (CSI),

General solution (by direct inspection)

$$f(x) = C\, x^{\alpha}$$

with $\alpha = \dfrac{\ln b}{\ln a}$

If satisfied *with fixed* (a,b) (DSI),

whose general solution is

$$f(x) = x^{\alpha}\, G\left(\frac{\ln x}{\ln a}\right)$$

where $G(u+1) = G(u)$ is a periodic function of period unity

Power laws are signature of scale invariance

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Break CSI into DSI ?

Complex fractal exponents and oscillations

For a discrete scale invariance, $f(x) = x^{\alpha} G\left(\frac{\ln x}{\ln a}\right)$

and $G(u+1) = G(u)$ is a periodic function of period unity

Fourier expansion: $f(x) = \sum_{n=-\infty}^{\infty} c_n x^{\alpha + i\frac{2\pi n}{\ln a}}$

The scaling quantity $f(x)$ is characterised by an infinite set of complex valued exponents,

$$d_n = \alpha + i\frac{2\pi n}{\ln a}$$

Power laws with complex valued exponents are signature of discrete scale invariance (DSI)

Today's program

- To investigate situations where continuous scale invariance is spontaneously broken into discrete scale invariance.
- Physical examples (generically, Efimov physics).
- Renormalisation group and limit cycles.

Part 2

A simple example of continuous scale invariance (a.k.a. conformal) in quantum physics

Illustration of scale invariance in quantum mechanics

In order to make the idea more explicit, consider the seemingly simple quantum problem :

Schrodinger equation for a particle of mass μ in d-dimensions with an attractive (enough) $V(r) = -\frac{\xi}{r^2}$ potential.

$$\boxed{\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}}$$

Redefining $k^2 = -2\mu E$

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{l(l+d-2)}{r^2}\right)\psi(r) + \frac{2\mu\xi}{r^2}\psi(r) = k^2\psi(r)$$

l is the orbital angular momentum and $\hbar = 1$

$$\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)$$

$$\zeta=2\mu\xi-l(l+d-2)$$

This equation displays an unexpected behaviour distinct from hermitian Hamiltonian eigenvalue problems.

The only parameter ζ in the problem is dimensionless : no characteristic length (energy) scale, *e.g.* Bohr radius $a_0 = \hbar^2/\mu e^2$ for the Coulomb potential.

Consequence: Schrodinger eq. displays continuous scale invariance, *i.e.*, it is invariant under the transformation:

$$\begin{cases} r \to \lambda r \\ k \to \frac{1}{\lambda} k \end{cases}$$

To every normalisable wave function $\psi(r,k)$ sol. of the Schr. eq. corresponds a family of wave functions $\psi(\lambda r,k)$ of energy $(\lambda k)^2$, $\forall \lambda \in \mathbb{R}$

solution of

$$\left(\frac{d^2}{dr^2}+\frac{d-1}{r}\frac{d}{dr}+\frac{\zeta}{r^2}\right)\psi(\lambda r,k)=(\lambda k)^2\psi(\lambda r,k)$$

$\zeta \;=\; 2\mu\xi - l(l+d-2)$

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The existence of one bound state implies those of a continuum of related bound states !

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Related to the fact that the Hamiltonian $\hat{H}$ is not self-adjoint over L_2, the space of square integrable functions.

This is a more general property also characteristic of potentials $V(r) \sim 1/r^n, \quad n \geq 3$

Adding further restrictions on the space on which $\hat{H}$ operates, it is always possible to define a family of new operators $\hat{H}_\theta$ associated to $\hat{H}$ and self-adjoint.

These restrictions show up as boundary conditions specific to each new operator $\hat{H}_\theta$

All operators $\hat{H}_\theta$ have the same formal expression but act in a new space (L_2 restricted by the corresponding boundary conditions). They are called self-adjoint extensions of $\hat{H}$.

Summary of the main results

For the case $\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$

$\hat{H}$ is scale invariant, $r \to \lambda r \Rightarrow \hat{H} \to \frac{1}{\lambda^2}\hat{H}$ and not self-adjoint

Leads to the remarkable result : for $\zeta > \zeta_{cr}$ with $\zeta_{cr} = \frac{(d-2)^2}{4}$ $\quad \zeta = 2\mu\xi - l(l+d-2)$

Boundary conditions needed to find self-adjoint extensions break CSI spontaneously into DSI.

As a result, the energy spectrum appears in the form of a geometric sequence:

$$k_n \propto e^{\frac{\pi n}{\Lambda}}, \quad n \in \mathbb{Z} \qquad \text{with} \quad \Lambda = \sqrt{\zeta - \zeta_{cr}}$$

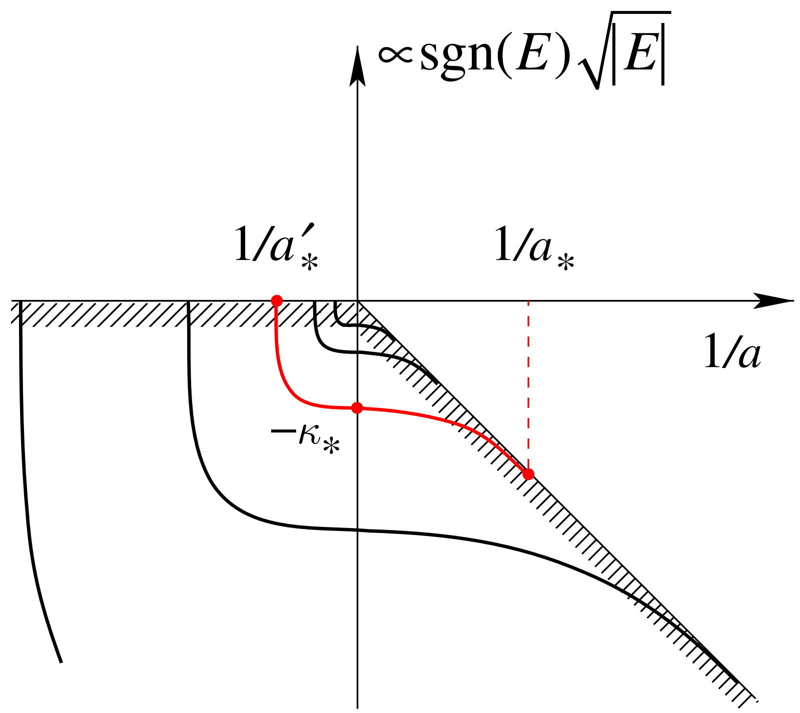
Note : no ground state for this spectrum. Breaking of continuous scale invariance in the quantum domain is known as a scale anomaly.

An example of quantum anomaly is the Efimov effect which occurs in the non relativistic quantum 3-body problem.

Efimov (1970) analysed the 3-nucleon system interacting through zero-range interactions (r_0). He pointed out the existence of universal physics at low energies, $E \ll \hbar^2 / m r_0^2$

When the scattering length a of the 2-body interaction becomes $a \gg r_0$ there is a sequence of 3-body bound states whose binding energies are spaced geometrically in the interval between $\hbar^2 / m a^2$ and $\hbar^2 / m r_0^2$

As $|a|$ increases, new bound states appear at critical values a^* of a that differ by a multiplicative factor e^{π/s_0} where $s_0 \approx 1.00624$ is a universal number.



$$E_n = \frac{\hbar^2 \kappa_*^2}{2m} e^{\frac{2\pi n}{s_0}}$$

Efimov spectrum

Efimov showed that the corresponding 3-body pb. reduces to a simple Schr. eq. with an effective attractive potential :

$$V(r) = -\frac{s_0 + \frac{1}{4}}{r^2}$$

Efimov quantum states have been beautifully evidenced (Rudi Grimm (06), Randy Hulet (09),…) in ultracold gases using Feshbach resonances.

Back to our spectral problem…

How does it work ?
Solution of the spectral problem

$$\boxed{\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)}$$

$$\zeta = 2\mu\xi - l(l+d-2)$$

Look for bound states. General solution:

$$\psi(r)=r^{\frac{2-d}{2}}\left(C_1 K_{i\Lambda}(kr)+\cancelto{0}{C_2} I_{i\Lambda}(kr)\right) \quad \text{with} \quad \Lambda=\sqrt{\zeta-\frac{(d-2)^2}{4}}$$

Modified Bessel functions

Assume now, $\zeta > \zeta_{cr} \equiv \frac{(d-2)^2}{4} \Leftrightarrow \Lambda^2 > 0$

for $r \to 0$

It remains $\psi(r)=\frac{1}{\sqrt{Z}}r^{\frac{2-d}{2}}K_{i\Lambda}(kr) \sim \frac{1}{\sqrt{Z}}r^{\frac{2-d}{2}}(kr)^{i\Lambda}$

which does not provide a quantisation condition :

Hamiltonian is not self-adjoint.

Self-adjoint ?

$$\langle\psi_2|\hat{H}^\dagger|\psi_1\rangle=\langle\psi_2|\hat{H}|\psi_1\rangle \Leftrightarrow \left(u_2'^*(r)u_1(r)-u_2^*(r)u_1'(r)\right)\Big|_0^\infty=0$$

defining $\begin{cases} u_1(r)=\sqrt{r}\,K_{i\Lambda}(k_1 r) \\ u_2(r)=\sqrt{r}\,K_{i\Lambda}(k_2 r) \end{cases}$ $\quad \Lambda=\sqrt{\zeta-\frac{(d-2)^2}{4}} \quad \zeta=2\mu\xi-l(l+d-2)$

with $k_1 \neq k_2 \quad \Longrightarrow \quad \left(u_2'^*(r)u_1(r)-u_2^*(r)u_1'(r)\right)\Big|_0^\infty \propto \sin\left(\Lambda\log\left(\frac{k_1}{k_2}\right)\right)\neq 0$

so that $\hat{H}$ is not self-adjoint

To cure this, we add the additional boundary condition:

$$\left(u_2'^*(r)u_1(r)-u_2^*(r)u_1'(r)\right)\Big|_0^\infty \propto \sin\left(\Lambda\log\left(\frac{k_1}{k_2}\right)\right)=0 \qquad \forall\, k_1,k_2>0 \text{ and } k_1\neq k_2$$

Solving for k_1,k_2 $\quad \boxed{\frac{k_1}{k_2}=e^{\frac{\pi n}{\Lambda}}\equiv\lambda^n \Leftrightarrow k_n=k_0\,e^{\frac{\pi n}{\Lambda}}} \quad n\in\mathbb{Z}$

Self-adjoint ?

Efimov spectrum

$$\frac{k_1}{k_2} = e^{\frac{\pi n}{\Lambda}} \equiv \lambda^n \Leftrightarrow k_n = k_0\, e^{\frac{\pi n}{\Lambda}} \qquad n \in \mathbb{Z}$$

k_0 is an arbitrary energy parameter introduced for dimensional considerations : **Efimov parameter**

The Efimov parameter should be determined by an exact solution of the 3-body problem which properly takes into account the short distance physics (which cannot be of the form $1/r^2$ all the way to $r \to 0$).

The Efimov spectrum

$$\boxed{\frac{k_1}{k_2} = e^{\frac{\pi n}{\Lambda}} \equiv \lambda^n \Leftrightarrow k_n = k_0\, e^{\frac{\pi n}{\Lambda}}}$$

- The Efimov spectrum is invariant under a discrete scaling *w.r.t.* the parameter :

$$\lambda \equiv e^{\frac{\pi}{\Lambda}} \qquad \text{where} \qquad \Lambda = \sqrt{\zeta - \frac{(d-2)^2}{4}} \qquad \zeta = 2\mu\xi - l(l+d-2)$$

$$\boxed{\{k_n;\ n\in\mathbb{Z}\} \to \{\lambda k_n;\ n\in\mathbb{Z}\} = \{k_{n+1};\ n\in\mathbb{Z}\} = \{k_n;\ n\in\mathbb{Z}\}}$$

- The eigenfunctions : $\psi_n(r) = \sqrt{2\frac{\sinh(\pi\Lambda)}{\pi\Lambda}}\, k_n\, r^{\frac{2-d}{2}}\, K_{i\Lambda}(k_n r)$

are also invariant under a discrete scaling transformation :

$$\boxed{\psi_n(\lambda r) = \lambda^{\frac{2-d}{2}}\, \psi_n(r)}$$

$$\psi_n(r) = (\cdots)\, r^{\frac{2-d}{2}} \left(\cos\left(\Lambda \ln\left(k_n r\right) + \phi\right) + O\left(r^2\right)\right)$$

(Single harmonic approx.)

Density of states $\rho(E)=\sum_{n\in\mathbb{Z}}\delta(E-E_n)$

$$\rho(\lambda^2 E)=2\mu\sum_{n\in\mathbb{Z}}\delta(\lambda^2 k^2-k_n^2)=\cdots=\lambda^{-2}\rho(E)$$

so that $\boxed{\rho(E)=\frac{1}{E}G\left(\frac{\ln E}{\ln\lambda}\right)}$ where $G(u+1)=G(u)$

Conclusion : The original continuous scaling symmetry

$$\begin{cases} r\to\lambda r \\ k\to\frac{1}{\lambda}k \end{cases} \quad \forall\lambda\in\mathbb{R}$$

is broken into a discrete scaling symmetry

$$\boxed{\frac{k_1}{k_2}=e^{\frac{\pi n}{\Lambda}}\equiv\lambda^n\Leftrightarrow k_n=k_0\,e^{\frac{\pi n}{\Lambda}}}$$

Underlying effective fractal structure ?

Part 3

Rephrasing the same problem from another point of view

Renormalisation group (RG) and limit cycles

It is interesting to re-phrase the previous problem using the language of RG transformations.

Why ?

- It provides another (more physical ?) point of view on the $V(r) = -\frac{\xi}{r^2}$ problem.
- It allows to insert that problem in a broader perspective.
- to make a connexion with other physical problems.

The need of self-adjoint extensions results from the ill-defined behaviour of the potential $V(r) = -\frac{\xi}{r^2}$ for $r \to 0$ and from the absence of characteristic length.

- To cure these problems, introduce a short distance radial cutoff L_0 so that $\boxed{V_l(r) = \xi / r^s}$ for $r > L_0$ and some unknown short range structure $V_s(r)$ *for* $r < L_0$.

The cutoff L_0 is a physical parameter : characteristic scale at which the potential $V_l(r)$ is altered as a result of additional short range interactions.

The Schr. eq. becomes : $\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{l(l+d-2)}{r^2} + \frac{\xi}{r^s}\right)\psi(r) = -2\mu E \psi(r)$

$L_0 < r < \infty$

+ mixed boundary conditions $\boxed{L_0 \frac{\psi'(L_0)}{\psi(L_0)} = g_0}$ to encode the short distance contribution.

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structure $V_s(r)$ *for* $r < L_0$.

Obtain a complete solution, function of a set of parameters (L_0, ξ, g_0)

The cutoff L_0 is a physical ... aracteristic scale at which the potential $V_l(r)$ i ... esult of additional short range in

Th... Schr. eq. becomes :

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{l(l+d-2)}{r^2} + \frac{\xi}{r^s}\right)\psi(r) = -2\mu E \psi(r)$$

$$L_0 < r < \infty$$

+ mixed boundary conditions $\boxed{L_0 \frac{\psi'(L_0)}{\psi(L_0)} = g_0}$ to encode the short distance contribution.

Perform a RG transformation : change the cutoff distance $L_0 \to L$

Integrate out the Sch. eq. in the range $[L, L+dL]$ and obtain an equivalent effective description with a new cutoff

$$\boxed{L \to L + dL \equiv \lambda L} \quad \text{with} \quad 0 < \lambda - 1 \ll 1$$

and a new set of mixed boundary conditions

$$\lambda L \frac{\psi'(\lambda L)}{\psi(\lambda L)} = g(\lambda L)$$

New Schr. eq. defined in the range $\lambda L < r < \infty$

with a new coupling constant ξ and boundary cond. $g(L)$

$$\boxed{V_l(r) = \xi / r^s}$$

The rescaling $\begin{cases} r' = r/\lambda \\ E' = E/\lambda^2 \end{cases}$ $\quad \boxed{L \to L + dL \equiv \lambda L}$ with $0 < \lambda - 1 \ll 1$

leaves the Schr. eq. unchanged provided $\xi \to \xi \lambda^{2-s} \Leftrightarrow \boxed{L \dfrac{d\xi}{dL} = (2-s)\xi}$

We can also relate $g(\lambda L)$ to $g(L)$: $\boxed{L \dfrac{dg}{dL} = (2-d)g - g^2 - 2\mu\xi L^{2-s} + l(l+d-2) - 2L^2\mu E}$

The 2 previous eqs. are the renormalisation group (RG) eqs. and we define the corresponding β-functions :

$$\beta_{\xi,g} \equiv \frac{\partial(\xi,g)}{\partial \ln L}$$

$$\begin{cases} \beta_\xi = (2-s)\xi \\ \beta_g = (2-d)g - g^2 - 2\mu\xi L^{2-s} + l(l+d-2) - 2L^2\mu E \end{cases}$$

$$\begin{cases} \beta_\xi = (2-s)\xi \\ \beta_g = (2-d)g - g^2 - 2\mu\xi L^{2-s} + l(l+d-2) - 2L^2\mu E \end{cases}$$

Take $s = 2$ i.e., $V_l(r) = \xi / r^s \longrightarrow V(r) = \xi / r^2$

Assume low energy compared to potential and centrifugal barriers:

$$2L^2\mu|E| \ll |2\mu\xi - l(l+d-2)|$$

RG eqs. simplify to:

$\beta_\xi = 0$: the coupling is scale invariant

and $\boxed{\beta_g = (2-d)g - g^2 - \zeta = -(g-\zeta_+)(g-\zeta_-)}$ with $\zeta = 2\mu\xi - l(l+d-2)$

$$\zeta_\pm = \frac{2-d}{2} \pm \sqrt{\frac{(d-2)^2}{4} - \zeta}$$

Evolution of the coupling $g(L)$

$$\boxed{\beta_g = (2-d)g - g^2 - \zeta = -(g-\zeta_+)(g-\zeta_-)}$$

$$\zeta_\pm = \frac{2-d}{2} \pm \sqrt{\frac{(d-2)^2}{4} - \zeta}$$

Accounts for the change of boundary conditions $\lambda L \frac{\psi'(\lambda L)}{\psi(\lambda L)} = g(\lambda L)$

For $\zeta < \zeta_{cr} \equiv \frac{(d-2)^2}{4}$: two real fixed points (ζ_+, ζ_-)

$\beta_g = 0 \Leftrightarrow g(L) = const.$

stable (ζ_+), unstable (ζ_-)

Complete and well behaved solution of the Schr. eq.

Where the problem was ?

How does it work ?
Solution of the spectral problem

$$\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)$$

$$\zeta = 2\mu\xi - l(l+d-2)$$

Look for bound states. General solution:

$$\psi(r)=r^{\frac{2-d}{2}}\left(C_1 K_{i\Lambda}(kr)+C_2 I_{i\Lambda}(kr)\right) \quad \text{with} \quad \Lambda=\sqrt{\zeta-\frac{(d-2)^2}{4}}$$

0

It was here

Assume now, $\zeta > \zeta_{cr} \equiv \frac{(d-2)^2}{4} \Leftrightarrow \Lambda^2 > 0$

for $r \to 0$

It remains $\psi(r)=\frac{1}{\sqrt{Z}} r^{\frac{2-d}{2}} K_{i\Lambda}(kr) \sim \frac{1}{\sqrt{Z}} r^{\frac{2-d}{2}} (kr)^{i\Lambda}$

which does not provide a quantisation condition :

Hamiltonian is not self-adjoint.

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stable unstable

Complete and well behaved solution of the Schr. eq.

Where the problem was ?

namely, when the strength ξ of the potential $V(r) = \xi / r^2$ increases, then $\zeta = 2\mu\xi - l(l+d-2)$ increases.

and eventually, $\boxed{\zeta > \zeta_{cr} \equiv \frac{(d-2)^2}{4}}$

Then, the Hamiltonian : is not self-adjoint anymore

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

and CSI breaks down spontaneously into DSI leading to the (discrete) Efimov spectrum :

$$\frac{k_1}{k_2} = e^{\frac{\pi n}{\Lambda}} \equiv \lambda^n \Leftrightarrow k_n = k_0\, e^{\frac{\pi n}{\Lambda}}$$

Meaning in terms of RG eqs. ?

Evolution of the coupling $g(L)$

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Accounts for the change of boundary conditions

$$\lambda L \frac{\psi'(\lambda L)}{\psi(\lambda L)} = g(\lambda L)$$

For $\cancel{\zeta < \zeta_{cr} \equiv \frac{(d-2)^2}{4}}$ two real fixed points (ζ_+, ζ_-)

stable

unstable

Thus we need to consider :

$$\boxed{\zeta > \zeta_{cr} \equiv \frac{(d-2)^2}{4}}$$

Meaning in terms of RG eqs. ?

Evolution of the coupling $g(L)$

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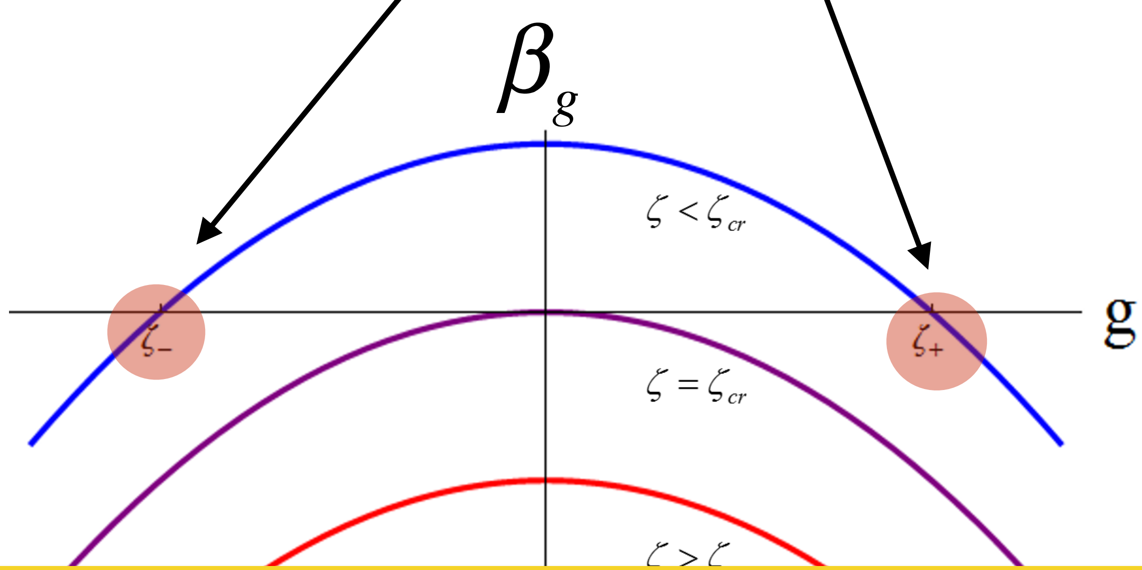
unstable



Complete and well behaved solution of the Schr. eq.

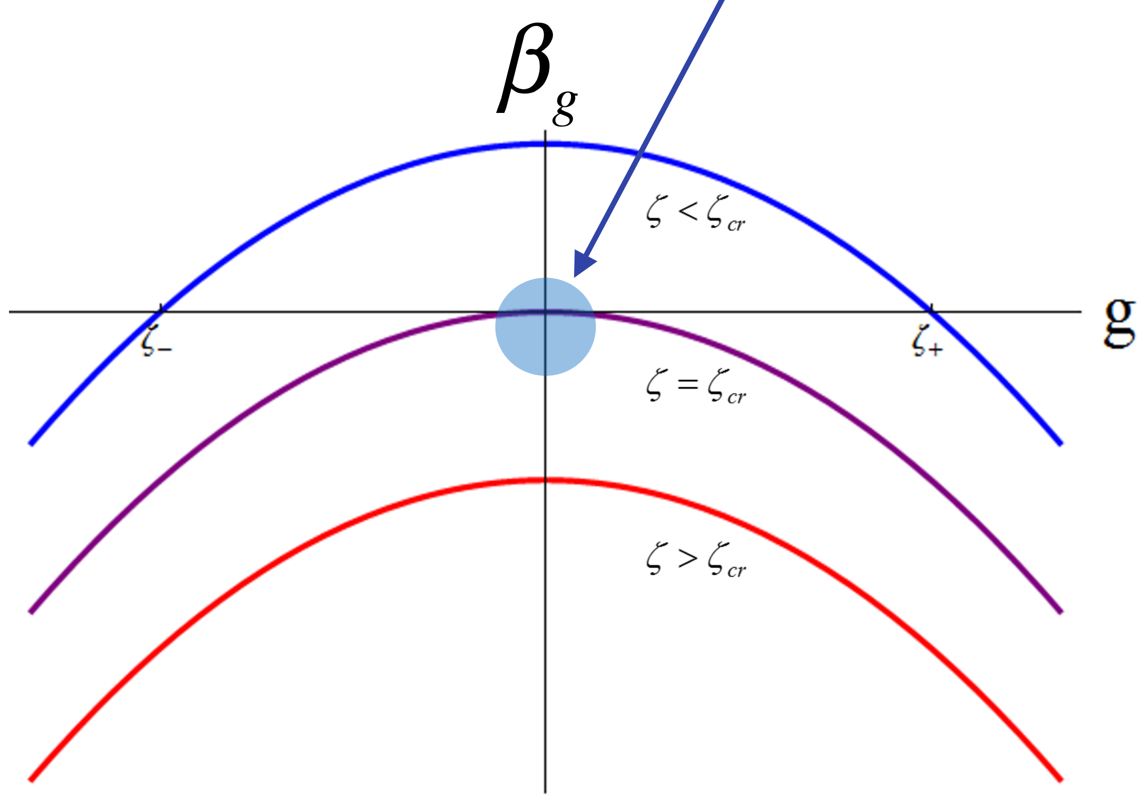
Weakly attractive $V(r) = \xi / r^2$ potential

Evolution of the coupling $g(L)$

$$\boxed{\beta_g = (2-d)g - g^2 - \zeta = -(g-\zeta_+)(g-\zeta_-)}$$

$$\zeta_\pm = \frac{2-d}{2} \pm \sqrt{\frac{(d-2)^2}{4} - \zeta}$$

For $\zeta = \zeta_{cr} = \dfrac{(d-2)^2}{4}$ two fixed points merge



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Evolution of the coupling $g(L)$

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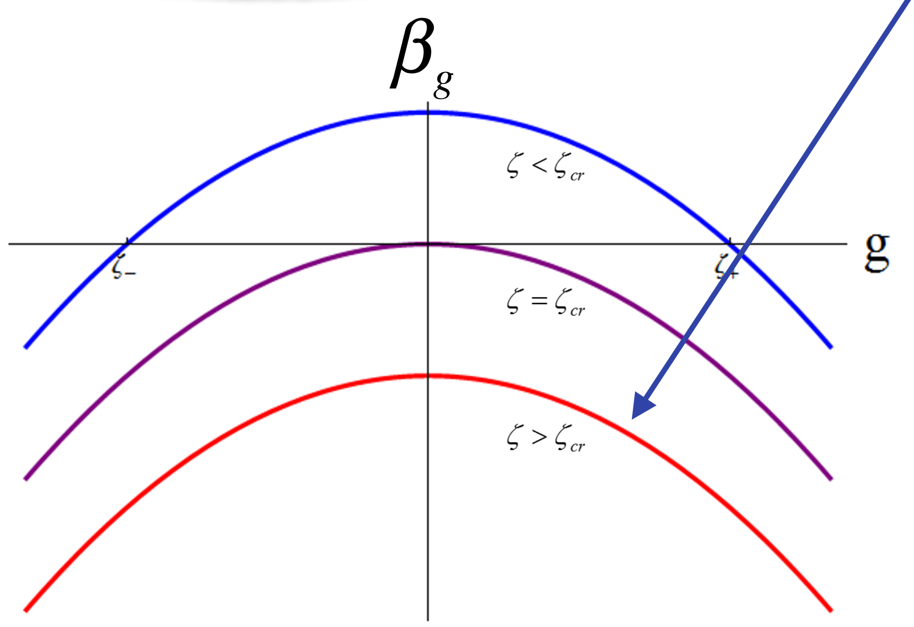
For $\zeta > \zeta_{cr} \equiv \frac{(d-2)^2}{4}$

No fixed point and (ζ_+, ζ_-) become complex.



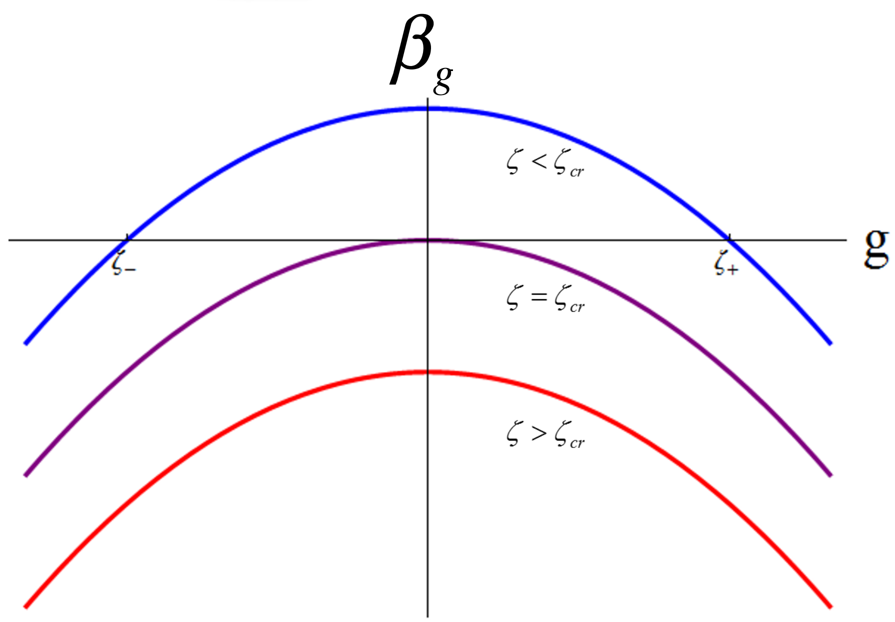
Evolution of the coupling $g(L)$

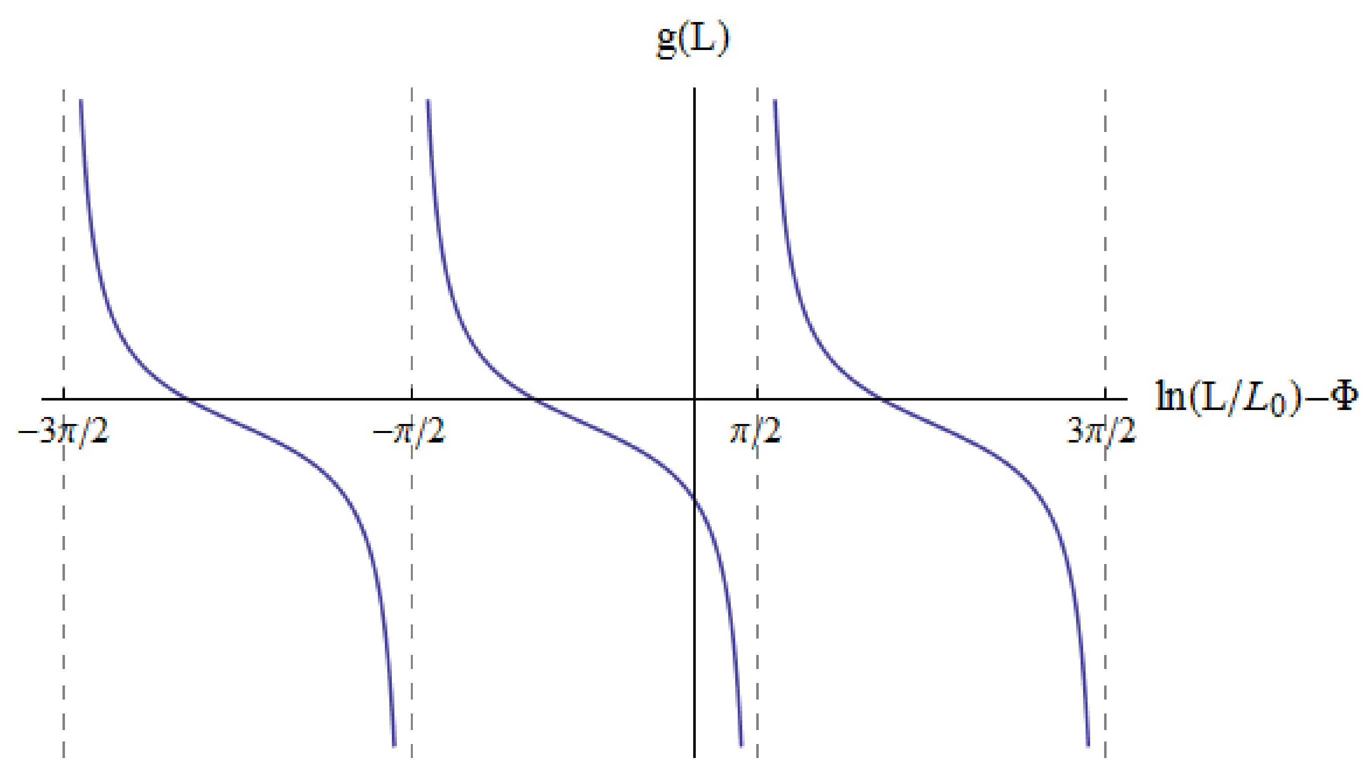
$$\boxed{\beta_g = (2-d)g - g^2 - \zeta = -(g-\zeta_+)(g-\zeta_-)}$$

$$\zeta_\pm = \frac{2-d}{2} \pm \sqrt{\frac{(d-2)^2}{4} - \zeta}$$

For $\zeta > \zeta_{cr} \equiv \dfrac{(d-2)^2}{4}$

The cycle completes a period for every $L \rightarrow e^{\frac{\pi}{\Lambda}} L$





The solution for $g(L)$ is a limit cycle.

$$g(L) = \frac{2-d}{2} + \Lambda \tan\left[Arc\tan\left(\frac{g_0 - \frac{2-d}{2}}{\Lambda} \right) - \Lambda \ln\left(\frac{L}{L_0} \right) \right]$$

with $\Lambda = \sqrt{\zeta - \dfrac{(d-2)^2}{4}}$

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f(ax) = b\, f(x)$$

If satisfied $\forall\, b(a) \in \mathbb{R}$ (CSI),

General solution (by direct inspection)

$$f(x) = C\, x^{\alpha}$$

with $\alpha = \dfrac{\ln b}{\ln a}$

Break CSI into DSI ?

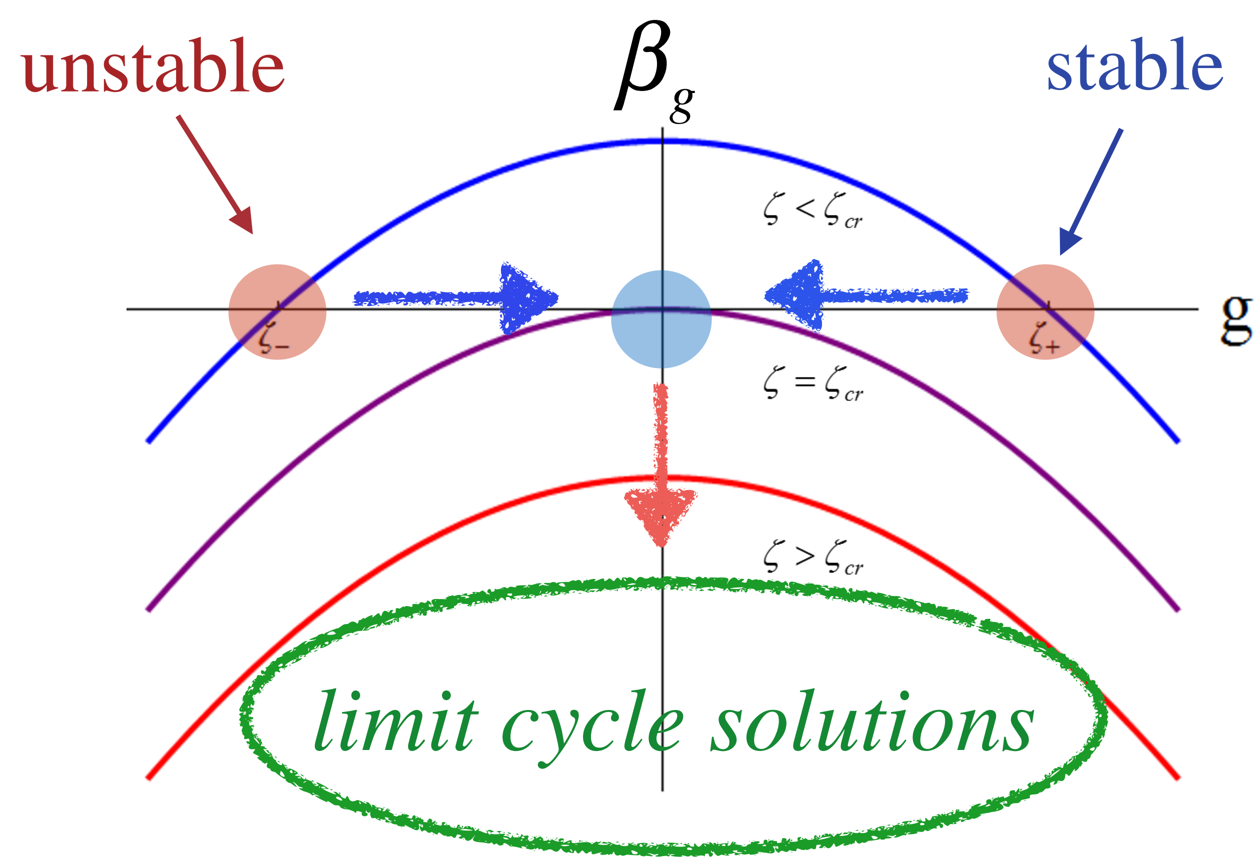
If satisfied *with fixed* (a,b) (DSI),

whose general solution is

$$f(x) = x^{\alpha}\, G\left(\frac{\ln x}{\ln a}\right)$$

where $G(u+1) = G(u)$ is a periodic function of period unity

Breaking of CSI into DSI previously understood as a requirement of self-adjointness, is now interpreted as a transition of the RG flow from an IR to UV fixed points into the emergence of limit cycle solutions.



The Efimov spectrum results immediately from $L\frac{\psi'(L)}{\psi(L)} = g(L)$

$$\frac{k_1}{k_2} = e^{\frac{\pi n}{\Lambda}} \equiv \lambda^n \Leftrightarrow k_n = k_0\, e^{\frac{\pi n}{\Lambda}}$$

Part 4

Another interesting case :

The Dirac equation + Coulomb potential

Dirac equation

The whole issue of Efimov physics is based on the CSI of the Hamiltonian :

$$\boxed{\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}}$$

A natural question : What about the Dirac eq. with a Coulomb potential ?

Since $\boxed{i\sum_{\mu=0}^{d}\gamma^{\mu}\left(\partial_{\mu}+ieA_{\mu}\right)\Psi\left(x^{\nu}\right)=0}$ is first order in the momentum and

the Coulomb potential $\boxed{V\left(r\right)=-\frac{\xi}{r},\ \xi\equiv Z\alpha}$

fine structure constant

These two problems share the same continuous scale invariance property (CSI).

Don't we know everything about the Dirac Hydrogen atom ?

Old problem (Pomeranchuk, 1945) of a relativistic electron in a super critical Coulomb potential.

Success of QED lies in the domain of small (weak) fields and perturbation theory in the small dimensionless parameter :

$$\boxed{\alpha = e^2/\hbar c \approx \frac{1}{137} \ll 1}$$

(fine structure constant)

Calculations involving bound states of a nucleus of charge Ze involve the dimensionless combination $Z\alpha$

Perturbation theory fails for $Z\alpha \geq 1$

In that case, we expect instability of the vacuum (ground state) against creation of electron-positron pairs.

Problem : to observe this instability, we need $Z \geq 1/\alpha \simeq 137$

No such stable nuclei have been created.

Idea: consider analogous condensed matter systems with a "much larger effective fine structure constant".

Graphene : Effective massless Dirac excitations with a Fermi velocity $v_F \simeq 10^6\, m/s$ so that $\boxed{\alpha_G = e^2/\hbar v_F \approx 2.5}$

and $Z_c \geq 1/\alpha_G \simeq 0.4$

($Z_c \simeq 1$ with screening effects)

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and $Z_c \geq 1/\alpha_G \simeq 0.4$ $(Z_c \simeq 1$ with screening effects)

- Charge impurities in graphene (Coulomb potential)

$\Longrightarrow$ scattering of quasi- bound states

$\Longrightarrow$ singular behaviour of the total phase shift

$\Longrightarrow$ change of spectral and transport properties

(Shytov, Katsnelson, Levitov,2007)

$\longrightarrow$ Measurement of local DOS (STM spectroscopy)

(Wang et al. (2013)

How to understand this instability ?

$$\boxed{\alpha = e^2 / \hbar c \approx \frac{1}{137} \ll 1}$$ (fine structure constant)

Calculations involving bound states of a nucleus of charge Ze involve the dimensionless combination $Z\alpha$

Perturbation theory fails for $Z\alpha \geq 1$

In that case, we expect instability of the vacuum (ground state) against creation of electron-positron pairs.

The Dirac-Kepler problem

Heuristic argument : classical expression for the energy of an electron of mass m , momentum p in the field of a charge Ze

$$\boxed{\varepsilon = c\sqrt{p^2 + m^2c^2} - \frac{Ze^2}{r}}$$

<u>Estimate of the ground state energy :</u>

electron position cannot be determined to better than $\hbar/p$

$$\varepsilon(p) \geq c\left(\sqrt{p^2 + m^2c^2} - Z\alpha\, p\right)$$

Minimising w.r.t p : $\varepsilon_0 = mc^2\sqrt{1-(Z\alpha)^2}$

which reproduces well known features of the Hydrogen ground state in the non relativistic $(Z\alpha \ll 1)$ and relativistic limits.

For $Z\alpha > 1$ the ground state energy becomes imaginary.

Important remark : this instability is independent of the electron mass m (dimensional analysis) so that the Dirac-Kepler instability remains in the (so-called) Weyl-Kepler problem $(m = 0)$.

Good news since in graphene, Dirac excitations are massless.

This instability in the Dirac/Weyl-Kepler problem is an example of the spontaneous breaking of CSI into DSI.

Important remark : this instability is independent of the electron mass m (dimensional analysis) so that the Dirac-Kepler instability remains in the (so-called) Weyl-Kepler problem $(m = 0)$.

Good news since in graphene, Dirac excitations are massless.

Is there an Efimov like structure for the massless Dirac problem ?

$$\boxed{i \sum_{\mu=0}^{d} \gamma^{\mu} \left(\partial_{\mu} + ieA_{\mu}\right) \Psi\left(x^{\nu}\right) = 0}$$

+ scalar electromagnetic potential

$$\begin{aligned} eA_0 &= -\xi/r \\ A_i &= 0 \quad i = 1, \ldots, d. \end{aligned}$$

Using self-adjoint extension is more cumbersome.
The RG picture is rather simpler here.

Obtain a spontaneous breaking of the original scale symmetry into a discrete one with a corresponding **Efimov spectrum** in the same way as for the non relativistic case.

Important remark : this instability is independent of the electron mass m (dimensional analysis) so that the Dirac-Kepler instability remains in the (so-called) Weyl-Kepler problem $(m=0)$.

Good news since in graphene Dirac excitations are massless.

Is there an Efimov like structure for the massless Dirac problem ?

It would be very interesting to observe an Efimov spectrum in that case !

$$i\sum_{\mu=0}^{d}\gamma^{\mu}(\partial_{\mu}+ieA_{\mu})\Psi(x)=0$$

+ scalar electromagnetic potential

$$\begin{aligned} eA_0 &= -\xi/r \\ A_i &= 0 \quad i=1,\ldots,d. \end{aligned}$$

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Obtain a spontaneous breaking of the original scale symmetry into a discrete one with a corresponding **Efimov spectrum** in the same way as for the non relativistic case.

Summary-Further directions

- We have shown on two examples how continuous (conformal) scale invariance is spontaneously broken into a discrete scale invariance.

- For these examples, this breaking can be understood as the need of new boundary conditions to restore self-adjointness of the Hamiltonian (see also notions of deficiency indices and a theorem of Von Neumann).

- Breaking of the CSI can also be interpreted using the Renormalisation group picture : stable fixed points evolve into limit cycles described by complex valued exponents (characteristics of fractal structures, K.G. Wilson, RG and strong interactions, 1971).

- A large number of problems can be described similarly as "conformality lost" (Kaplan et al., 2009) and emergence of limit cycles:

  - Kosterlitz-Thouless transition (deconfinement of vortices in the XY-model at a critical temp. above which the theory is conformal): mapping between the XY-model and the T=0 sine-Gordon in 1+1 dim.

$$\boxed{L = \frac{T}{2}\left(\partial_\mu \phi\right)^2 - 2z\cos\phi}$$

  - Metal-insulator transition in d-dimensions (electron gas to Wigner crystal, Localisation transition).

  - Breitenlohner-Freedman bound for free massive scalar field on AdS_{d+1} space.

- Quantum gravity :

Basic tool : sum over histories

$$\int \mathcal{D}g \, e^{-S[g]}$$

Each path is a 4-dimensional, curved space time geometry "g" which can be thought of as a 3-dim., spatial geometry developing in time.
associated with each "g" is given by the corresponding Einstein-Hilbert action $S[g]$

$$S[g] = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \, (-R + 2\Lambda)$$

- Newton's constant: $G_N = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$
- cosmological constant: $\Lambda \approx 10^{-35} \ \mathrm{s}^{-2}$

A hard problem ! Several approaches on the market.

The other option : non perturbative renormalisation group flow analysis (M. Reuter, F. Saueressig, 2012)

Asymptotic Safety, Fractals, and Cosmology*

Martin Reuter and Frank Saueressig

Institute of Physics, University of Mainz
Staudingerweg 7, D-55099 Mainz, Germany

reuter@thep.physik.uni-mainz.de
saueressig@thep.physik.uni-mainz.de

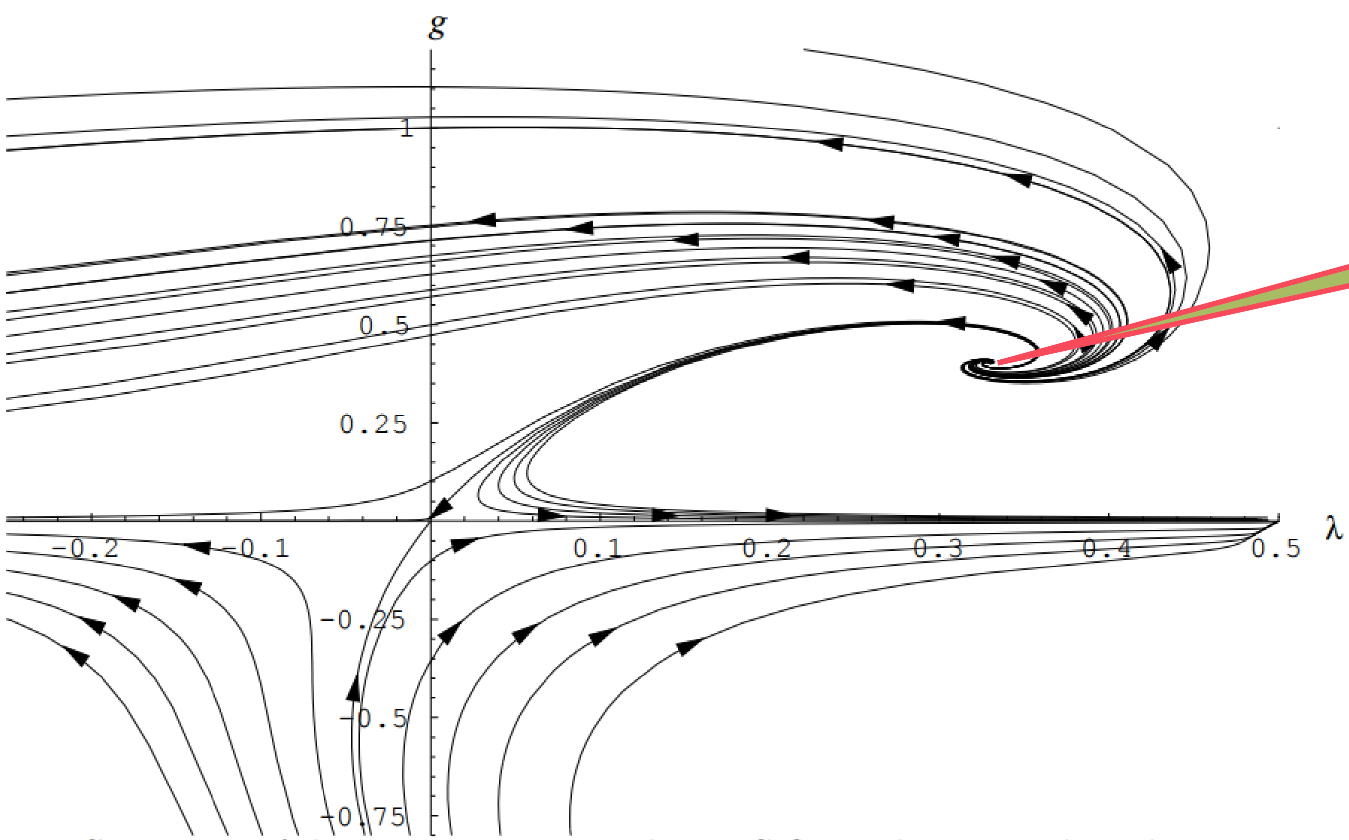
Abstract

These lecture notes introduce the basic ideas of the Asymptotic Safety approach to Quantum Einstein Gravity (QEG). In particular they provide the background for recent work on the possibly multifractal structure of the QEG space-times. Implications of Asymptotic Safety for the cosmology of the early Universe are also discussed.

Running coupling constants:

Newton constant G_k, dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k, dimensionless: $\lambda(k) = k^{-2} \Lambda_k$



close to the fixed point

$$\begin{pmatrix} g \\ \lambda \end{pmatrix} = \begin{pmatrix} g_* \\ \lambda_* \end{pmatrix} + A \left(\frac{k}{k_0} \right)^{-|\theta_R|} \begin{pmatrix} b \cos\left(\theta_I \ln \frac{k}{k_0} + \phi_b + \phi \right) \\ \cos\left(\theta_I \ln \frac{k}{k_0} + \phi \right) \end{pmatrix}$$

What I did not (yet) discuss

- Shot noise and quantum mesoscopic physics:
  universality of the Fano factor beyond 1D- SSEP on fractal structures-
  relation to electrical exponent - Dirichlet form - an entire field of research

- Off-diagonal propagator : relation to interesting probs. in statistical mechanics.

- Long-range correlations in disordered systems :
  generalisation of the Harris-Luck criterion for the relevance
  of discrete scaling symmetry (connexion to substitution matrices
  and quasi-periodic order).

Thank you for your attention.