Incidences, Distinct Distances, Other Erdős Problems in Geometry, And Their Applications: The New Algebraic Era

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Combinatorial and Computational Geometry

- Ask questions about the combinatorial complexity of geometric structures

- For pure intellectual curiosity (Combinatorial)

- And for analysis of algorithms (Computational)

- Strongly interwoven co-existence

Combinatorial Geometry owes its roots to (many, but especially to) **Paul Erdős** (1913–1996)

Paul Erdős

26 March 1913 (Budapest) – 20 September 1996 (Warsaw)

Paul Erdős

The most prolific mathematician ever

One of the most influential mathematicians of the 20th century

Wrote $>$ 1500 papers

With $>$ 500 collaborators

Worked in (and helped to revolutionize):
Combinatorics, Graph Theory, Number Theory, Classical Analysis, Approximation Theory, Set Theory, and Probability Theory

and Geometry

Combinatorial Geometry

Erdős asked, in 1946, two simple questions
which have kept many good people sleepless for many years

Distinct distances: Estimate the smallest possible number $D(n)$ of distinct distances determined by any set of n points in the plane

Repeated distances: Estimate the maximum possible number of pairs, among n points in the plane, at distance exactly 1

Distinct Distances, Incidences, and more

In this talk:

- Distinct distances and related incidence problems in geometry

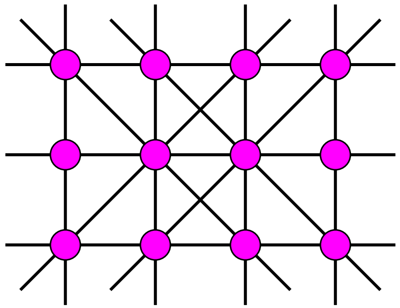
- The new algebraic revolution that has made it possible to solve all these problems

- Other applications of the algebraic machinery

What are incidences?
Incidences between points and lines in the plane

P: Set of m distinct points in the plane
L: Set of n distinct lines

$I(P, L) =$ Number of incidences between P and L
$= |\{(p, \ell) \in P \times L \mid p \in \ell\}|$



$I(m, n) = \max\{I(P, L) \mid |P| = m,\ |L| = n\}$

$I(m, n) = \Theta(m^{2/3}n^{2/3} + m + n)$ [Szemerédi–Trotter 83]

Why incidences?

- Because it's there—another Erdős-like cornerstone in geometry

- Simple question; Unexpected bounds; Nontrivial analysis

- Arising in / related to many topics:

Repeated and distinct distances and other configurations
Range searching in computational geometry
The Kakeya problem in harmonic analysis

- Triggered development of sophisticated tools

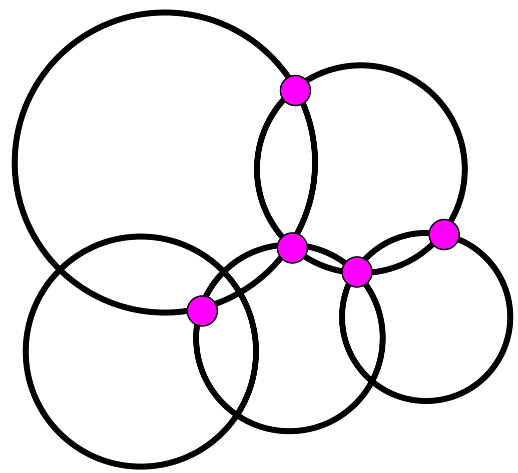
(space decomposition) with many other applications

Many extensions

• Incidences between points and curves in the plane

• Incidences with lines, curves, flats, surfaces, in higher dimensions

• In most cases, no known sharp bounds
Point-line incidences is the exception...



Distinct distances

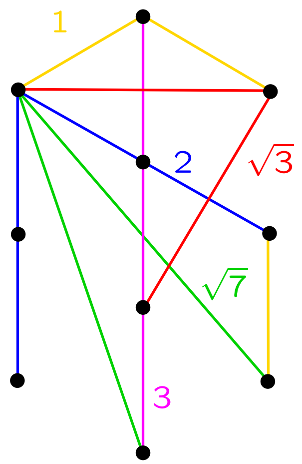
The present high profile of incidence geometry:
Due to the dramatic breakthrough on Erdős's Distinct Distances problem

Near-complete solution by Larry Guth and Nets Hawk Katz (2010)
Using techniques from Algebraic Geometry

Gave the whole area a huge push
Many new results, new deep techniques, and a lot of excitement

Erdős's distinct distances problem:

Estimate the smallest possible number $D(n)$ of distinct distances determined by any set of n points in the plane



$5 \ll \binom{10}{2} = 45$

[Erdős, 1946] conjectured:
$D(n) = \Omega(n/\sqrt{\log n})$
(Cannot be improved: Tight for the integer lattice)

Erdős's distinct distances problem:

A 1946 classic

A hard nut; Slow steady progress

Best bound before the revolution:

$\Omega(n^{0.8641})$ [Katz-Tardos 04]

The founding father of the revolution:

György Elekes (passed away in September 2008)

A brief history: Elekes's insights

Circa 2000, Elekes was studying
Erdős's distinct distances problem
Found an ingenious transformation of this problem to an
incidence problem between points and curves (lines) in 3D

For the transformation to work, Elekes needed a couple of deep
conjectures on the new setup
(If proven, they yield the almost tight lower bound $\Omega(n/\log n)$)

Nobody managed to prove his conjectures; he passed away in
2008, three months before the revolution began

A brief history: The first breakthrough

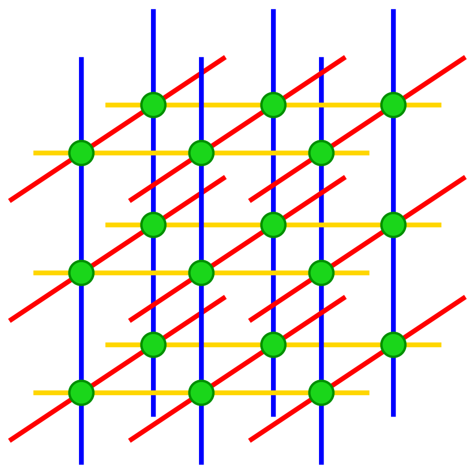
[Guth, Katz, 08]:

Algebraic Methods in Discrete Analogs of the Kakeya Problem

Showed: The number of joints in a set of n lines in 3D is $O(n^{3/2})$

A joint in a set L of n lines in $\mathbb{R}^3$:

Point incident to (at least) three non-coplanar lines of L



A brief history: Joints and distinct distances

Joints and Elekes's setup have a lot in common:
Incidences between points and lines in 3D

"Truly 3-dimensional":
If all points and lines lie in a common plane,
Cannot beat the planar Szemerédi-Trotter bound

The Guth-Katz proof uses simple tools from algebraic geometry:
A new twist in the plot...
(Follows similar ideas of Dvir for finite fields)

An even simpler proof coming up in a minute...

A brief history: From joints to distinct distances

I aired out Elekes's ideas in 2010, Guth and Katz picked them up, used more advanced algebraic methods, and obtained their second breakthrough:

• [Guth, Katz, 10]: The number of distinct distances in a set of n points in the plane is $\Omega(n/\log n)$

Settled Elekes's conjectures (in a more general setup)

And solved (almost) completely the distinct distances problem

A brief history:

- In both problems, new algebraic machinery applied to
Incidence problems between points and lines in 3D
(With better bounds than in the planar case)

- Hard problems, resisting decades of “conventional” geometric
and combinatorial attacks

- Algebraic machinery picked up, extended and adapted
Yielding solutions to many old and new difficult problems

Et ils vécurent heureux et eurent beaucoup d‘enfants

Attempting to present:

- Joints
- Polynomial partitioning
- Incidences in three and higher dimensions
- Distinct distances
- Polynomials vanishing on grids
- Quick review of recent progress

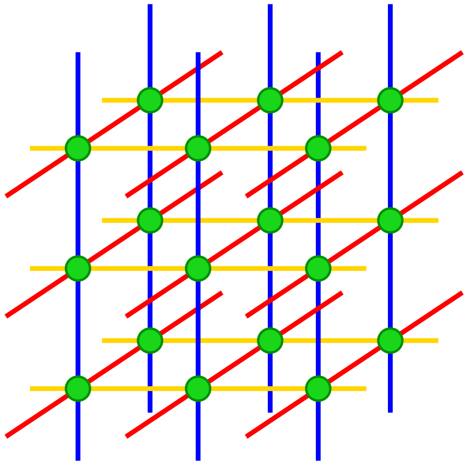
The joints problem

L – Set of n lines in $\mathbb{R}^3$

Joint: Point incident to (at least) three non-coplanar lines of L

[Chazelle et al., 1992] Conjectured:
The number of joints in L is $O(n^{3/2})$

Worst-case tight: $\sqrt{n} \times \sqrt{n} \times \sqrt{n}$ lattice; $3n$ lines



Joints

- Solved completely in [Guth, Katz, 08]

- Following a similar earlier technique of [Dvir, 09] for finite fields

In retrospect, a "trivial" problem

In general, in d dimensions
(Joint $=$ point incident to at least d lines, not all on a hyperplane)
Max number of joints is $\Theta(n^{d/(d-1)})$
[Kaplan, S., Shustin, 10],
[Quilodrán, 10]
(Similar, and very simple proofs)

One algebraic tool

S: set of m points in $\mathbb{R}^d$

Claim: There exists a nontrivial d-variate polynomial $p(x_1, \ldots, x_d)$ of degree b, vanishing at all the points of S, for

$$\binom{b+d}{d} \geq m+1 \qquad \text{or} \qquad b \approx (d!m)^{1/d} \approx (d/e)m^{1/d} = O(m^{1/d})$$

Proof: A d-variate polynomial p of degree b has $M = \binom{b+d}{d}$ monomials

Requiring p to vanish at m points $\Longrightarrow$
$m < M$ linear homogeneous equations in the M coefficients of the monomials

Always has a nontrivial solution $\square$

Proof of the bound on joints:

L – Set of n lines in $\mathbb{R}^d$

J – Set of their joints; put $m = |J|$

Assume to the contrary that $m > An^{d/(d-1)}$
($A \approx 2d/e$ a constant; to be set)

Step 1: Pruning

As long as L has a line ℓ incident to $< m/(2n)$ joints,

Remove ℓ from L and its incident joints from J

Left with subsets $L_0 \subseteq L$, $J_0 \subseteq J$, with

- $|J_0| > m/2$
- Each $\ell \in L_0$ is incident to $\geq m/(2n)$ points of J_0
- Each $a \in J_0$ is a joint of L_0

Step 2: Vanishing

Construct a polynomial p vanishing at all the points of J_0

Of minimal degree $0 < b \leq (d!m)^{1/d}$

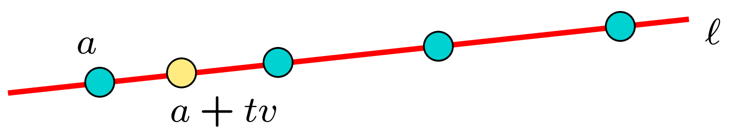
Crucial: Every line of L_0 contains more than b points of J_0:

$$\frac{m}{2n} > (d!m)^{1/d}, \qquad \text{or} \qquad m > \underbrace{(2^d d!)^{1/(d-1)}}_{A} n^{d/(d-1)}$$

$p = 0$ on more than b points on a line $\ell \Longrightarrow p \equiv 0$ on ℓ

So $p \equiv 0$ on every line of L_0

Step 3: Differentiating



Fix $a \in J_0$ and an incident line of L_0 $\quad \ell = \{a + tv \mid t \in \mathbb{R}\}$

$$p(a+tv) = p(a) + (\nabla p(a) \cdot v)t + O(t^2)$$

for t small

$p(a+tv) \equiv 0$ for all t and $p(a) = 0 \implies \nabla p(a) \cdot v = 0$

Step 3: Differentiating

$\nabla p(a) \cdot v = 0$
for all directions v of lines of L_0 incident to a

a is a joint $\Longrightarrow$ $\nabla p(a) = 0$

All first-order derivatives of p vanish at all the points of J_0

But these derivatives have degree $b - 1$
Contradicting the choice of b
(Minimal degree of a polynomial vanishing on J_0)

The End!

A new algebraic era in combinatorial geometry: Et ils vécurent heureux et eurent beaucoup d'enfants

- First, new proofs of old results (simpler, different)
  [Kaplan, Matoušek, S., 11]

- Unit distances in three dimensions
  [Zahl 13], [Kaplan, Matoušek, Safernová, S. 12]

- Point-circle incidences in three dimensions
  [S., Sheffer, Zahl 13], [S., Solomon 17]

- Complex Szemerédi-Trotter incidence bound and related bounds
  [Solymosi, Tao 12], [Zahl 15], [Sheffer, Szabó, Zahl 15]

- Range searching with semi-algebraic ranges
  An algorithmic application; [Agarwal, Matoušek, S., 13]

Et plus d'enfants:

- Incidences between points and lines in four dimensions
  [S., Solomon, 16]

- Incidences between points and curves in higher dimensions
  [S., Sheffer, Solomon, 15], [S., Solomon, 17]

- Incidences in general and semi-algebraic extensions
  [Fox, Pach, Sheffer, Suk, Zahl, 14]

- Algebraic curves, rich points, and doubly-ruled surfaces
  [Guth, Zahl, 15]

Et plus:

- Distinct distances between two lines
  [S., Sheffer, Solymosi, 13]

- Distinct distances: Other special configurations
  [S., Solymosi, 16], [Pach, de Zeeuw, 17],
  [Charalambides, 14], [Raz, 17], [S., Solomon, 17]

- Arithmetic combinatorics: Sums vs. products and related problems
  [Iosevich, Roche-Newton, Rudnev]

Et plus:

- Polynomials vanishing on grids:
The Elekes–Rónyai–Szabó problems revisited
[Raz, S., Solymosi, 16], [Raz, S., de Zeeuw, 16,17]

- Triple intersections of three families of unit circles
[Raz, S., Solymosi, 15]

- Unit-area triangles in the plane
[Raz, S., 15]

- Lines in space and rigidity of planar structures
[Raz, 16]

Et plus:

- Almost tight bounds for eliminating depth cycles for lines in three dimensions
  [Aronov, S. 16]

- Eliminating depth cycles among triangles in three dimensions
  [Aronov, Miller, S., 17]

- New bounds on curve tangencies and orthogonalities
  [Ellenberg, Solymosi, Zahl, 16]

- Cutting algebraic curves into pseudo-segments and applications
  [S., Zahl, 17]

Old-new Machinery from Algebraic Geometry and Co.

- Low-degree polynomial vanishing on a given set of points

- Polynomial ham sandwich cuts

- Polynomial partitioning

- Miscellany (Thom-Milnor, Bézout, Harnack, Warren, and co.)

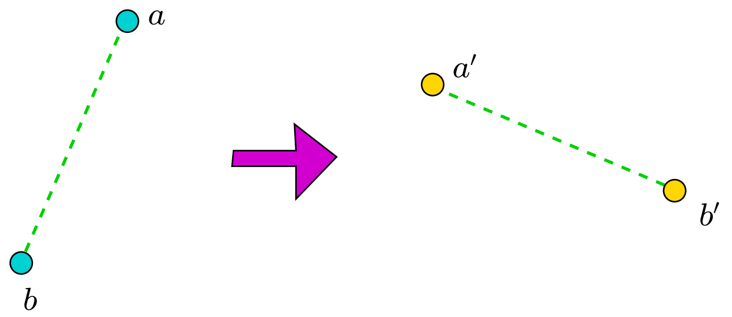
- Miscellany of newer results on the algebra of polynomials

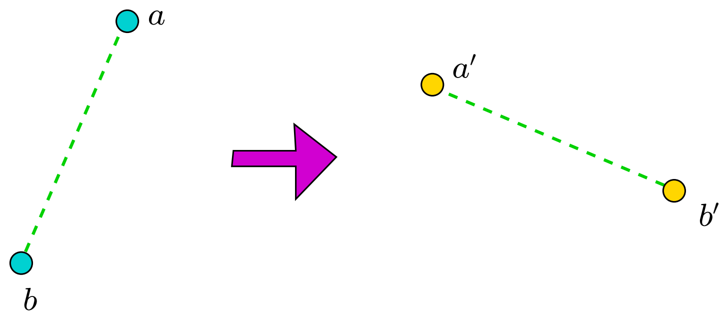
- And just plain good old stuff from the time when algebraic geometry was algebraic geometry
(Monge, Cayley–Salmon, Severi; 19th century)

Erdős's distinct distances problem
Elekes's transformation: Some hints

- Consider the 3D parametric space of rigid motions ("rotations") of $\mathbb{R}^2$

- There is a rotation mapping a to a' and b to b'
$\Leftrightarrow$ $dist(a, b) = dist(a', b')$



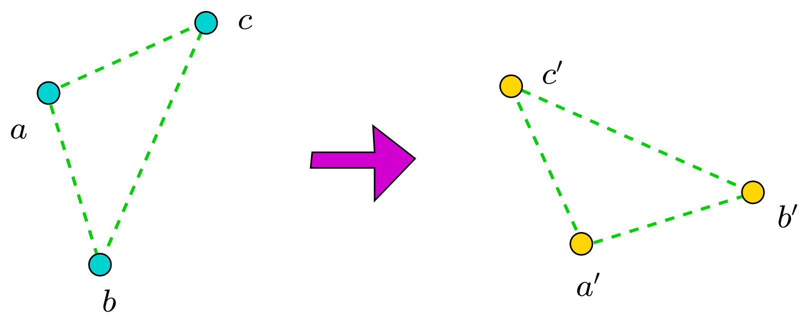


• Elekes assigns each pair $a, a' \in S$ to the locus $h_{a,a'}$ of all rotations that map a to a' (with suitable parameterization, $h_{a,a'}$ is a line)

• So if $dist(a,b) = dist(a',b')$ then $h_{a,a'}$ and $h_{b,b'}$ meet at a common point (rotation)

• After some simple (but ingenious) algebra,
Elekes's main conjecture was:

Number of rotations that map $\geq k$ points of S
to $\geq k$ other points of S
$=$ Number of points incident to $\geq k$ lines $h_{a,a'}$
$= O\left((\text{Num of lines})^{3/2}/k^2\right) = O\left(n^3/k^2\right)$



It is therefore all about incidences between points and lines in three dimensions

Opening up the door to questions about
Incidences between lines, or curves, or surfaces,
in three or higher dimensions

Unapproachable, “not-in-our-lifetime” problems before the revolution

Now falling down, one after the other, rather “easily”

Point-line Incidences in $\mathbb{R}^3$

Elekes's conjecture: Follows from the point-line incidence bound:

Theorem: ([Guth-Katz 10])
For a set P of m points
And a set L of n lines in $\mathbb{R}^3$, such that no plane contains
more than $O(n^{1/2})$ lines of L ("truly 3-dimensional")
(Holds in the Elekes setup)

$$\max I(P, L) = \Theta(m^{1/2}n^{3/4} + m + n)$$

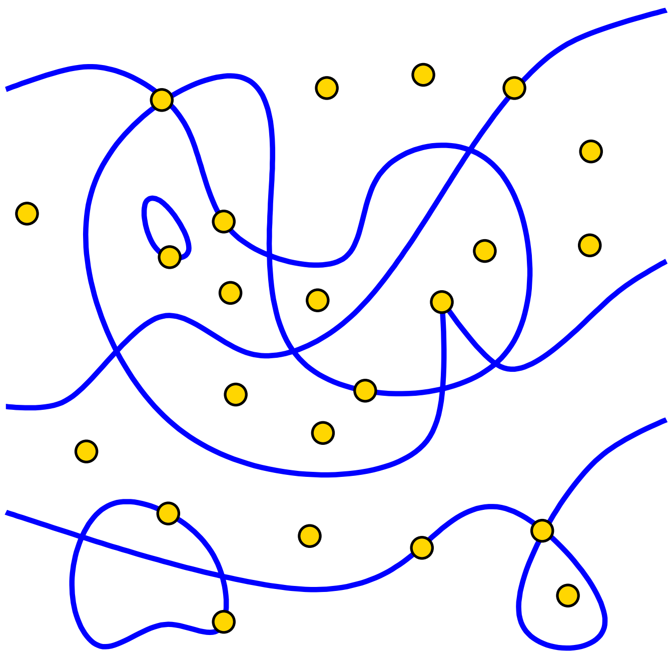
Proof uses **polynomial partitions**

Polynomial partitioning of a point set

[Guth-Katz 10]: A set S of n points in $\mathbb{R}^d$ can be partitioned into $O(t)$ subsets, each consisting of at most n/t points,
By a polynomial p of degree $D = O\left(t^{1/d}\right)$,
Each subset in a distinct connected component of $\mathbb{R}^d \setminus Z(f)$

Proof based on the polynomial Ham Sandwich theorem of [Stone, Tukey, 1942]

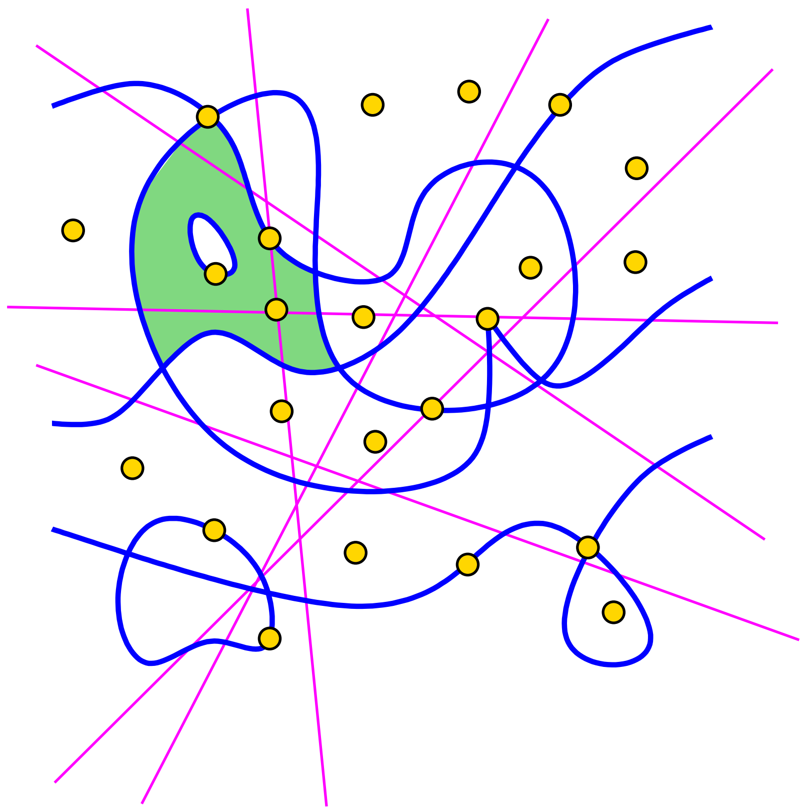
Polynomial partitioning



Polynomial partitioning: Restatement and extension

[Guth-Katz 10]: For a set S of n points in $\mathbb{R}^d$, and degree D
Can construct a polynomial p of degree D
Such that each of the $O(D^d)$ connected components of $\mathbb{R}^d \setminus Z(f)$
contains at most $O(n/D^d)$ points of S

[Guth 15]: For a set S of n k-dimensional constant-degree
algebraic varieties in $\mathbb{R}^d$, and degree D
Can construct a polynomial p of degree D
Such that each of the $O(D^d)$ connected components of $\mathbb{R}^d \setminus Z(f)$
is intersected by at most $O(n/D^{d-k})$ varieties of S



Polynomial partitioning

- A new kind of space decomposition
Excellent for Divide-and-Conquer

- Competes (very favorably) with cuttings, simplicial partitioning
(Conventional decomposition techniques from the 1990's)

- Many advantages (and some challenges)

- A major new tool to take home

Incidences via polynomial partitioning

In five easy steps (for Guth-Katz's m points / n lines in $\mathbb{R}^3$):

- Partition $\mathbb{R}^3$ by a polynomial f of degree D:
$O(D^3)$ cells, $O(m/D^3)$ points and $O(n/D^2)$ lines in each cell

- Use a trivial bound in each cell:
$O(\text{Points}^2 + \text{Lines}) = O((m/D^3)^2 + n/D^2)$

- Sum up: $O(D^3 \cdot (m^2/D^6 + n/D^2)) = O(m^2/D^3 + nD)$

- Choose the right value: $D = m^{1/2}/n^{1/4}$, substitute

- Et voilà: $O(m^{1/2}n^{3/4})$ incidences

But...

For here lies the point: [Hamlet]

What about the points that lie on the surface $Z(f)$?

Method has no control over their number

Here is where all the fun (and hard work) is:

Incidences between points and lines on a 2D variety in $\mathbb{R}^3$

Need advanced algebraic geometry tools:

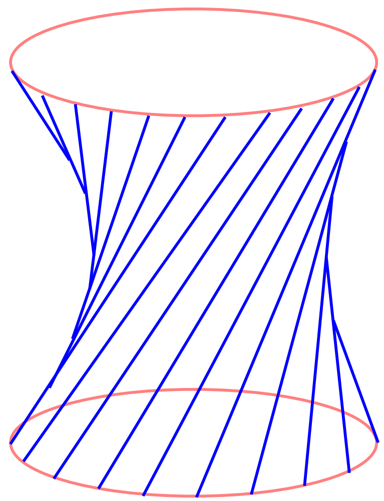
Can a surface of degree D contain many lines?!

Ruled surfaces

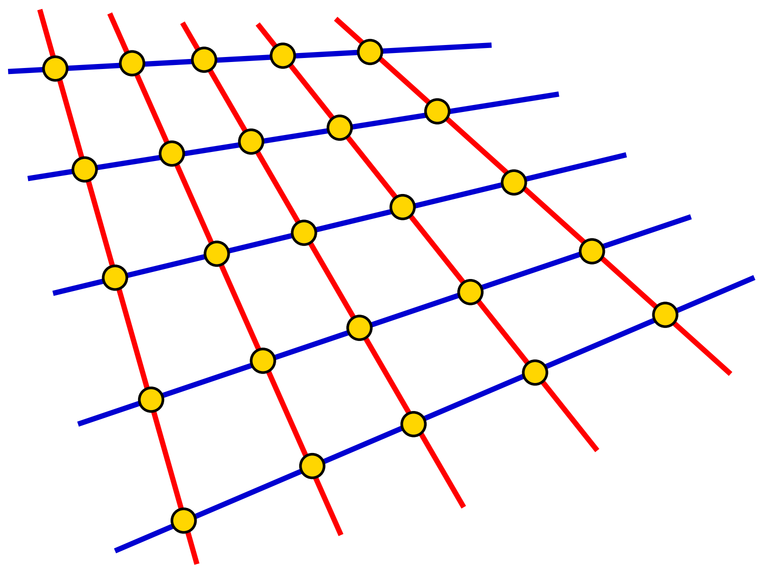
Can a surface of degree D contain many lines?!

Yes, but only if it is ruled by lines

Hyperboloid of one sheet



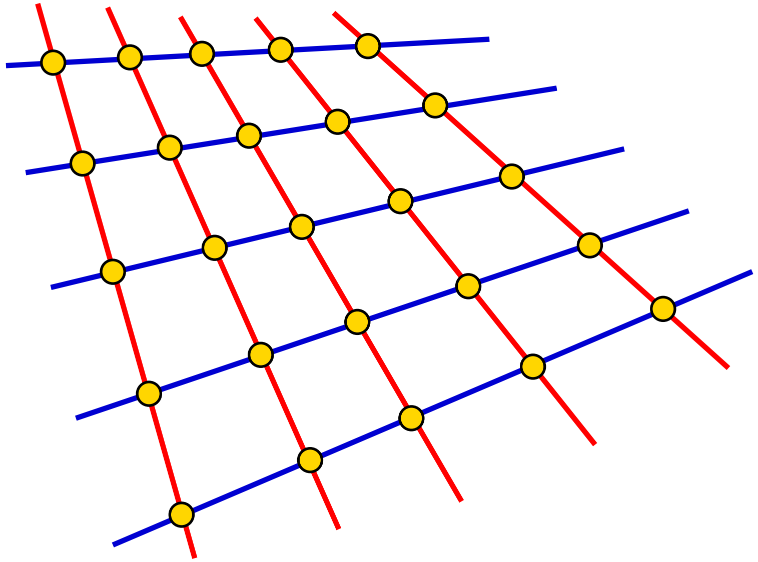
Ruled and non-ruled surfaces



Hyperbolic paraboloid

A non-ruled surface of degree D can contain at most $D(11D - 24)$ lines
[Monge, Cayley–Salmon, 19th century]

Ruled and non-ruled surfaces



If $Z(f)$ not ruled: Contains only "few" lines; "easy" to handle

If $Z(f)$ is (singly) ruled: "Generator" lines meet one another only at singular points; again "easy" to handle
(The only doubly ruled surfaces are these two quadrics)

Point-line incidences in $\mathbb{R}^3$

Finally, if $Z(f)$ contains planes
Apply the Szemerédi-Trotter bound in each plane
(No 3D tricks left...)

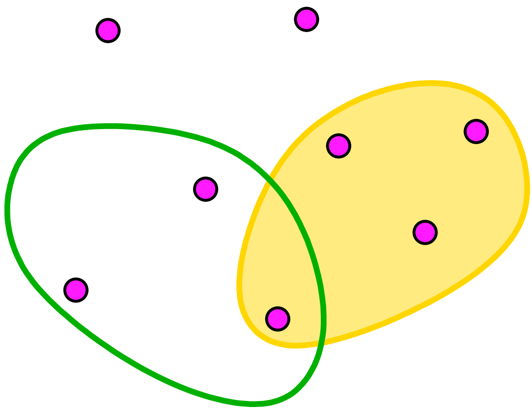
But we assume that no plane contains more than $n^{1/2}$ lines:
The incidence count on these planes is not too large

And we are done: $O\left(m^{1/2}n^{3/4} + m + n\right)$ incidences

Algorithmic applications

Not too many at the moment... Most ubiquitous:

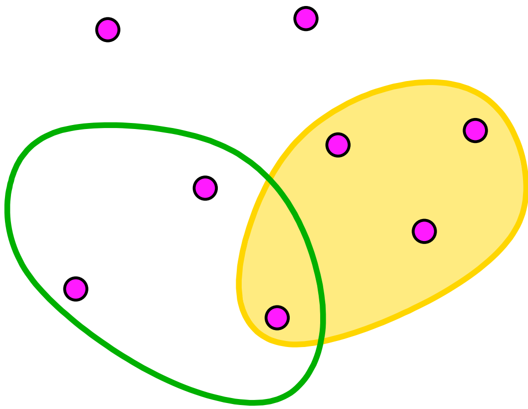
Range searching with semi-algebraic ranges
[Agarwal, Matoušek, S., 13]



Range searching

P: Set of n points in $\mathbb{R}^d$

Count / report / the subset of P inside a query range



"Off-line" incidence question

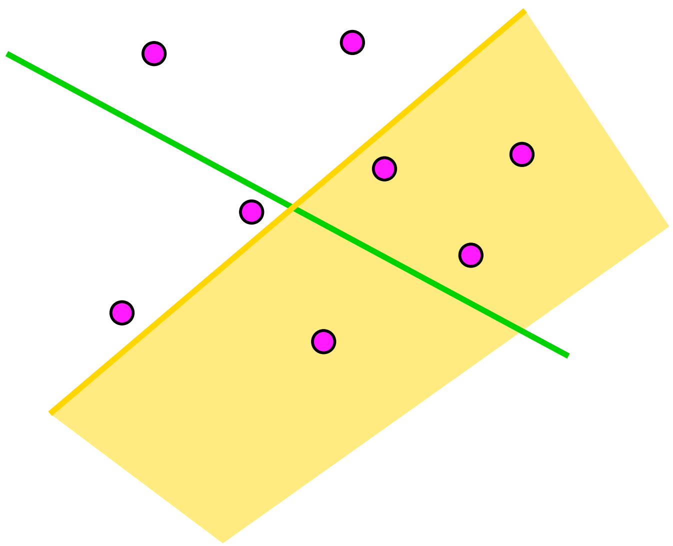
Range searching

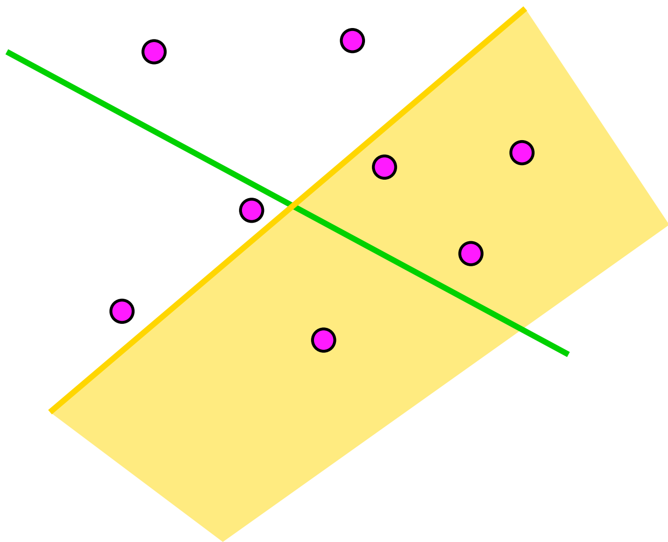
Fundamental problem in Computational Geometry

And in related areas (Data-bases)

Lots of applications

Traditionally (during the 1990's): Ranges = halfspaces





Most significant result:
With $\approx O(n)$ storage,
Can answer halfspace query in $\approx O\left(n^{1-1/d}\right)$ time

[Agarwal, Matoušek, S., 13]:
(Almost the) Same performance bounds for
semi-algebraic ranges of constant complexity

Range searching with semi-algebraic ranges

Using polynomial partitioning

Main difficulty:
Efficient construction of the partitioning polynomial
Existence based on non-constructive topological arguments
(Borsuk-Ulam, Polynomial Ham-Sandwich cuts)

Et comme dessert: Distinct distances between two lines

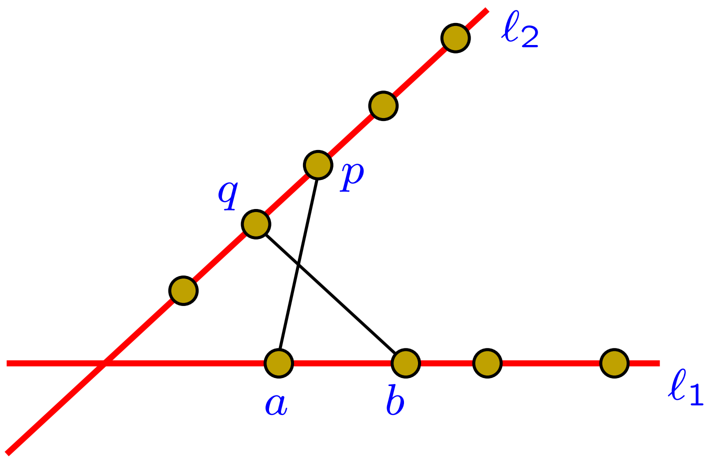
[S., Sheffer, Solymosi, 13]

ℓ_1, ℓ_2: Two lines in $\mathbb{R}^2$, non-parallel, non-orthogonal

P_1, P_2: Two n-point sets, $P_1 \subset \ell_1$, $P_2 \subset \ell_2$

$D(P_1, P_2)$: Set of distinct distances between P_1 and P_2

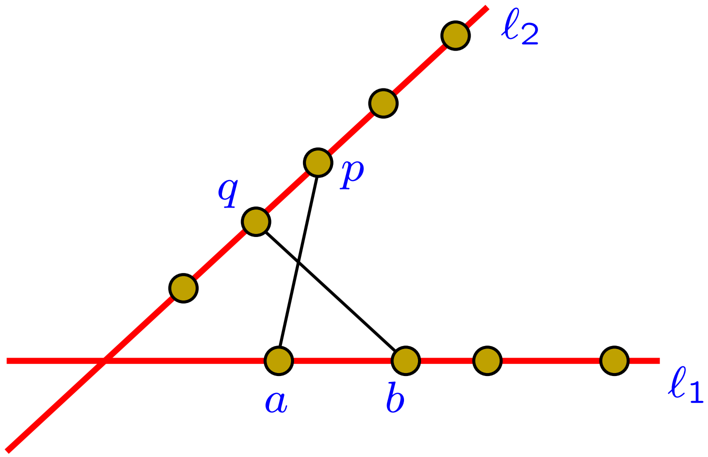
Theorem: $|D(P_1, P_2)| = \Omega\left(n^{4/3}\right)$



$D(P_1, P_2)$ can be $\Theta(n)$ when

ℓ_1, ℓ_2 are parallel:
(Take $P_1 = P_2 = \{1, 2, \ldots, n\}$)

or orthogonal:
(Take $P_1 = P_2 = \{1, \sqrt{2}, \ldots, \sqrt{n}\}$)



- A superlinear bound conjectured by [Purdy]
- And proved by [Elekes, Rónyai, 00]
- And improved to $\Omega(n^{5/4})$ by [Elekes, 1999]

Distinct distances between two lines

In [S., Sheffer, Solymosi]: Ad-hoc proof; reduces to
Incidences between points and hyperbolas in the plane

But also a special case of old-new algebraic theory of
[Elekes, Rónyai, Szabó 00, 12]
Enhanced by [Raz, S., Solymosi 16], [Raz, S., de Zeeuw 16]

The Elekes–Rónyai–Szabó Theory

A, B, C: Three sets, each of n real numbers

$F(x, y, z)$: A real trivariate polynomial (constant degree)

How many zeroes does F have on $A \times B \times C$?

Focus only on "bivariate case": $F(x, y, z) = z - f(x, y)$

The bivariate case $F = z - f(x, y)$

$Z(F) = \{(a, b, c) \in A \times B \times C \mid c = f(a, b)\}$

$|Z(F)| = O(n^2)$

And the bound is worst-case tight:

$A = B = C = \{1, 2, \ldots, n\}$ and $z = x + y$

$A = B = C = \{1, 2, 4, \ldots, 2^n\}$ and $z = xy$

The bivariate case

The amazing thing ([Elekes-Rónyai, 2000]):
For a quadratic number of zeros,

$z = x + y$ (and $A = B = C = \{1, 2, \ldots, n\}$
$z = xy$ (and $A = B = C = \{1, 2, 4, \ldots, 2^n\}$

Are essentially the only two possibities!

The bivariate case

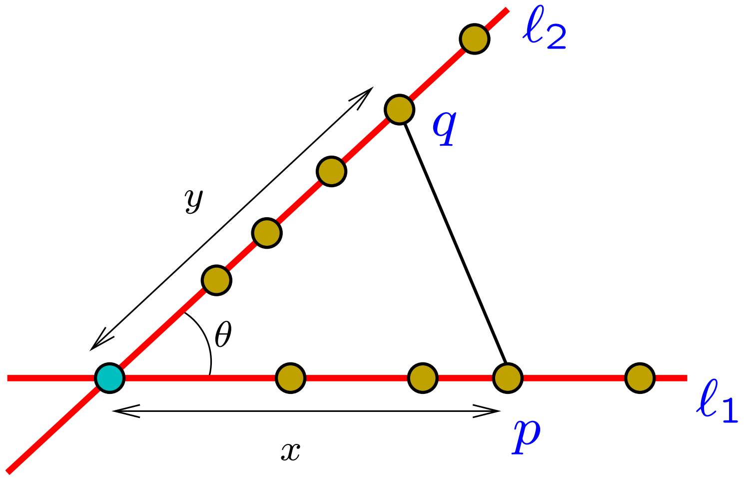
Theorem ([Elekes-Rónyai],
Strengthened by [Raz, S., Solymosi 16]):
If $z - f(x,y)$ vanishes on $\Omega(n^2)$ points of some $A \times B \times C$, with $|A| = |B| = |C| = n$, then f must have the special form

$$f(x,y) = p(q(x) + r(y)) \quad \text{or} \quad f(x,y) = p(q(x) \cdot r(y))$$

for suitable polynomials p, q, r

If f does not have the special form, then the number of zeros is always $O(n^{11/6})$ [Raz, S., Solymosi 16]

Distinct distances on two lines: What's the connection?



$$z = f(x,y) = \|p(x) - q(y)\|^2 = x^2 + y^2 - 2xy\cos\theta$$

$A = P_1$, $B = P_2$

$C =$ Set of (squared) distinct distances between P_1 and P_2

$z = f(x, y) = \|p(x) - q(y)\|^2 = x^2 + y^2 - 2xy\cos\theta$

$A = P_1$, $B = P_2$
$C =$ Set of (squared) distinct distances between P_1 and P_2

How many zeros does $z - f(x, y)$ have on $A \times B \times C$?
Answer: $|P_1| \cdot |P_2| = n^2$

Does f have the special form?
No (when $\theta \neq 0, \pi/2$)
Yes (when $\theta = 0, \pi/2$: parallel / orthogonal lines)

Here A, B, C have different sizes

Using unbalanced version of [Elekes, Rónyai, Raz, S., Solymosi]:

$$
\begin{aligned}
n^2 &= \text{Num. of zeros} = O\left(|P_1|^{2/3}|P_2|^{2/3}|C|^{1/2}\right) \\
&= O\left(n^{4/3}|C|^{1/2}\right)
\end{aligned}
$$

Hence $|C| =$ number of distinct distances $= \Omega(n^{4/3})$

The last slide (that nobody reads)

- A mix of

Algebra, Algebraic Geometry, Differential Geometry, Topology

In the service of Combinatorial (and Computational) Geometry

- Dramatic push of the area

Many hard problems solved

- And still many deep challenges ahead

Most ubiquitous: Distinct distances in three dimensions

Elekes's transformation leads to difficult incidence questions

Involving points and surfaces in higher dimensions (5 to 7)

And the really last slide (that everybody is so happy to read)

Thank You