Embeddings: Algorithms and Combinatorics

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Computational Geometry and Topology in the Sciences

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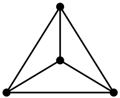
Starting Point: Graph Planarity

Question. Given a graph G, can we draw it in $\mathbb{R}^2$ without crossings?



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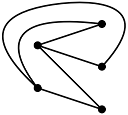

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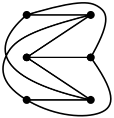

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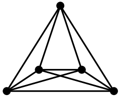

K_5 not planar

Classical and well-understood:

- Necessary and Sufficient Criteria for Planarity, e.g.,
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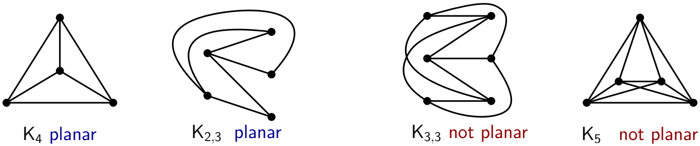

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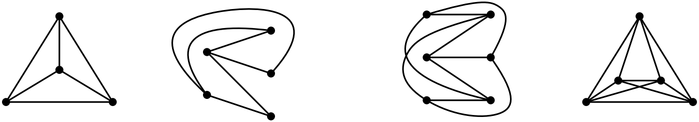

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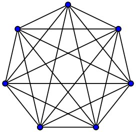
- Linear-time planarity testing algorithms (Hopcroft-Tarjan)

Graphs and Simplicial Complexes

- graphs



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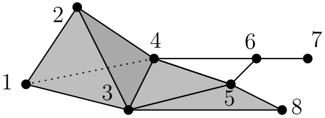
- model *pairwise interactions*
- 1-dimensional spaces

- simplicial complexes (hypergraphs)



- higher-dimensional spaces built from simple building blocks (simplices)
- combinatorial description, basic input model in computational topology

$$K = \{1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 14, 23, 24, 34, 35, 38, 45, 46, 56, 58, 123, 124, 134, 234, 345, 358, 1234\}$$

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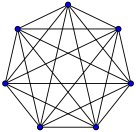

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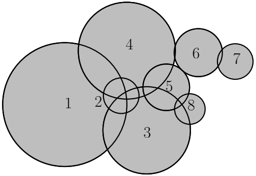

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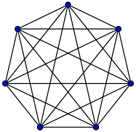

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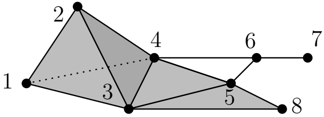

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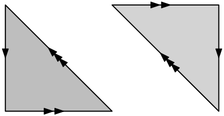
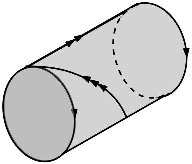
* Many Important Topics Omitted
   - Classificiation of embeddings (knot theory)
   - Graphs on surfaces
   - Quantitative nonembeddability (crossing numbers)
   - ...

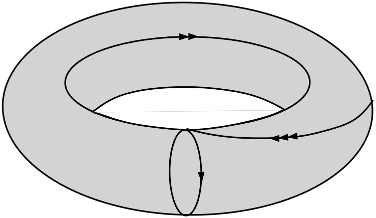
Embeddability in 3 Dimensions

Given a 2-dimensional simplicial complex K, does it embed into $\mathbb{R}^3$?

(Can we "assemble" K without self-intersections?







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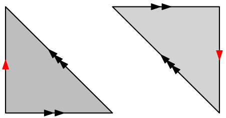

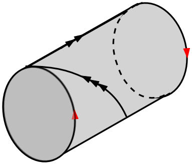

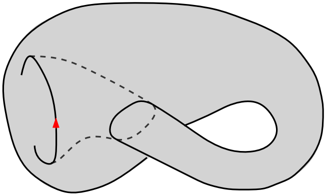

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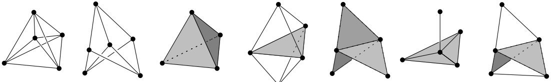

$K_{\mathrm{I}} \cong K_5$ $K_{\mathrm{II}} \cong K_{3,3}$ $K_{\mathrm{III}} \cong S^2$ K_{IV} K_{V} K_{VI} K_{VII}

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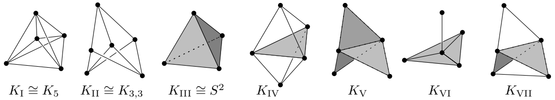

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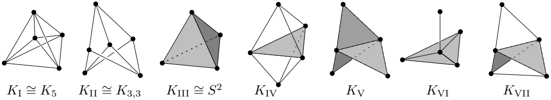

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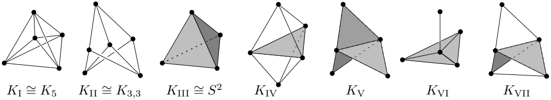

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  - $d = 2k$, $k \geq 3$: van Kampen–Shapiro–Wu obstruction $\rightsquigarrow$ polynomial-time algorithm

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  - $k \geq d - 1 \geq 4 \Rightarrow$ algorithmically undecidable
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  (Builds on work of Haken, Rubinstein–Thompson in knot theory / 3-manifold topology: unknot recognition, 3-sphere recognition)

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k \ d	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P												
2	P	Dec	NP										
3		Dec	NP	NP	P								
4			NP	Un	NP	NP	P						
5				Un	Un	NP	NP	P	P				
6					Un	Un	NP	NP	NP	P	P		
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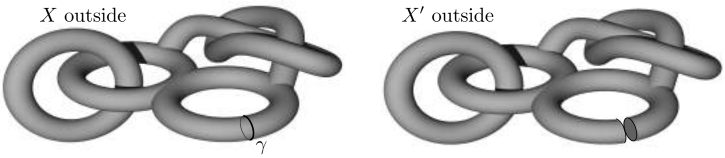

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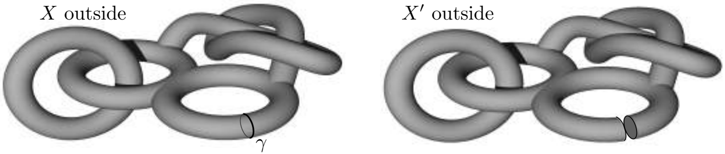


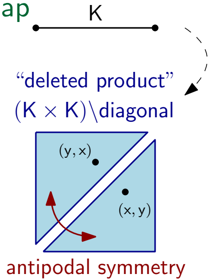
- **Key Theorem.** If X embeds in S^3, then there exists a short meridian γ (of length bounded by a computable function of the number of tetrahedra of X).

Embeddability $K \hookrightarrow \mathbb{R}^d$ and Deleted Products

Deleted Products and Embeddings

intersections of $f\colon K \to \mathbb{R}^d$ $\leftrightarrow$ zeros of auxiliary antipodal map $\widetilde{f}\colon (K \times K) \setminus \text{diagonal} \to \mathbb{R}^d$

$\widetilde{f}(x, y) := f(x) - f(y)$



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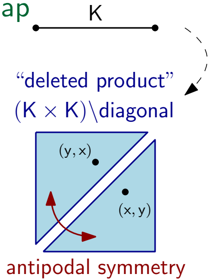

Nonembeddability via "Borsuk-Ulam Theorems"

No antipodal map $(K \times K) \setminus \text{diagonal} \to \mathbb{R}^d \setminus \{0\}$
$\Rightarrow$ no embedding $K \hookrightarrow \mathbb{R}^d$

- Classical example: Borsuk–Ulam Theorem
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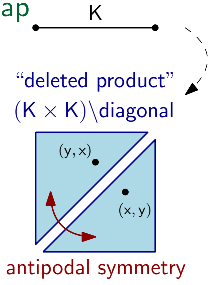

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van Kampen–Shapiro–Wu ($\Rightarrow$ embeddability)

$\dim K = \frac{d}{2} \geq 3$ and $\exists$ antipodal map $(K \times K) \setminus \text{diagonal} \to \mathbb{R}^d \setminus \{0\}$
$\Rightarrow \exists$ embedding $K \hookrightarrow \mathbb{R}^d$

- Analogous results for for $\dim K \leq \frac{2}{3}d - 1$ (Haefliger–Weber) , fails for $\dim K \geq \frac{2}{3}d$ (intuitively: presence of triple crossings)

Algebraic Intersection Numbers

Proposition

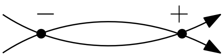
Suppose $\dim K = \frac{1}{2}d$. *There exists a symmetry-preserving map*

$$(K \times K) \setminus \text{diagonal} \to \mathbb{R}^d \setminus \{0\}$$

iff there exists a map $f\colon K \to \mathbb{R}^d$ *in general position such that*

$$\underbrace{f(\sigma) \cdot f(\tau)}_{\text{algebraic intersection number}} = 0$$

for any pair of vertex-disjoint simplices of K.



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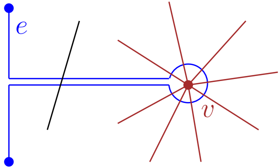
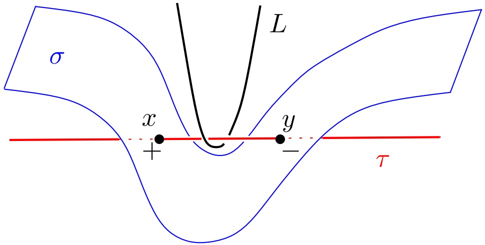


finger move

The Classical Whitney Trick

- Eliminate a pair of isolated double points of *opposite sign* by a local move (an ambient isotopy fixed outside a small ball), provided the codimension is at least 3.



- Idea: "push" σ upwards until the two intersections points x and y disappear, keeping τ and the boundary of σ and fixed.
- In low codimensions, doing this might require passing over some obstacles and/or introducing new double points, but in codimension 3 or more this can be avoided.

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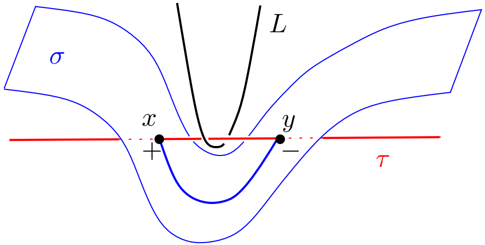

- Idea: "push" σ upwards until the two intersections points x and y disappear, keeping τ and the boundary of σ and fixed.
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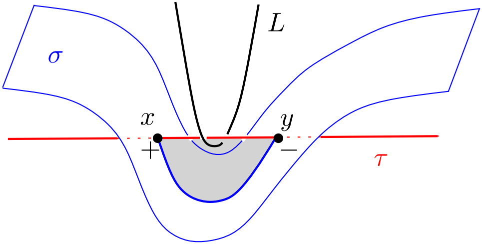

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Higher-Multiplicity intersections: Tverberg's Theorem

Theorem (Tverberg 1966)

Let $r \geq 2, d \geq 1$. Set $N := (d+1)(r-1)$.
Every $S \subseteq \mathbb{R}^d$ with $|S| \geq N+1$ has an "r-fold intersecting partition"

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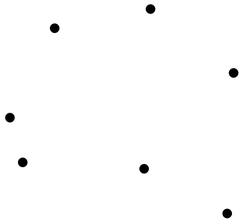

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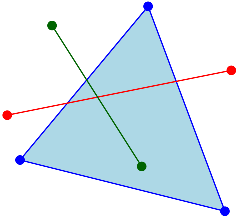

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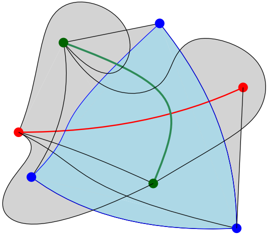

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Question

Is convexity/linearity necessary, or is continuity enough?

Topological Tverberg Conjecture

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Let $r \geq 2$, $d \geq 1$, $N = (d+1)(r-1)$,
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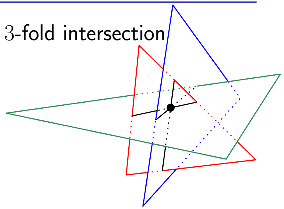

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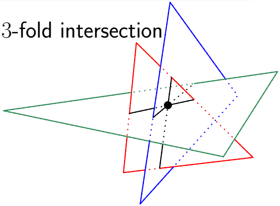

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- **True for**
  - $r = 2$ [Bajmoczy–Bárány 1979]
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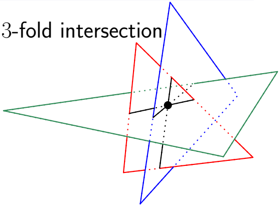

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Long-standing Open Problem

- What if r not a prime power?
- Conjecture commonly believed, existing methods insufficient!
  $\exists$ symmetry-preserving $(K \times \cdots \times K) \setminus$ diagonal $\to \mathbb{R}^{d(r-1)} \setminus \{0\}$ (Özaydin)

Eliminating Higher-Multiplicity Intersections

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Corollary

If $\dim K = \frac{r-1}{r}d$, codimension $d - \dim K \geq 3$, and r not a prime power then there exists a map $K \to \mathbb{R}^d$ without r-fold intersection.

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If $\dim K = \frac{r-1}{r} d$*, codimension* $d - \dim K \geq 3$*, and* r *not a prime power then there exists a map* $K \to \mathbb{R}^d$ *without r-fold intersection.*

Proof.

Symmetry-preserving map exists [Özaydin]
$\Rightarrow$ map $K \to \mathbb{R}^d$ without r-fold intersection exists. □

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- Counterexamples to "generalized Van Kampen–Flores conjecture" for non-prime powers (answers a question of Gromov)

Overcoming the Codimension Restriction

- *Solution 1 (Frick 2015)*: Reduction to lower-dimensional skeleton using a trick of Gromov and Blagojević–Frick–Ziegler

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Proposition (Gromov; Blagojević–Frick–Ziegler)

Let $r = 6$, $18 = 3 \cdot 6$, $15 = 3 \cdot (6-1)$, $100 = (18+2)(6-1)$.
If there is $g : : \mathrm{skel}_{15}(\sigma^{100}) \to \mathbb{R}^{18}$ *without* 6*-fold intersection, then there exists* $f : \sigma^{100} \to \mathbb{R}^{19}$ *without* 6*-fold intersection.*

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- *Solution 2 (Mabillard & W. 2015)*: *Prismatic maps*

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- Further improvement in codimension restriction [Avvakumov, Mabillard, Skopenkov, W.] $\rightsquigarrow$ counterexamples for $d \geq 2r$

  ($f: \sigma^{65} \to \mathbb{R}^{12}$ without 6-fold intersection)

r-Fold Algebraic Intersection Numbers

Proposition

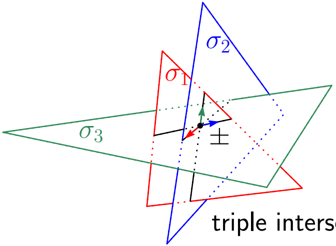
Suppose $\dim K = \frac{r-1}{r} d$. There exists a symmetry-preserving map

$$(K \times \cdots \times K) \setminus \text{diagonal} \to \mathbb{R}^{d(r-1)} \setminus \{0\}$$

iff there exists a map $f \colon K \to \mathbb{R}^d$ in general position such that

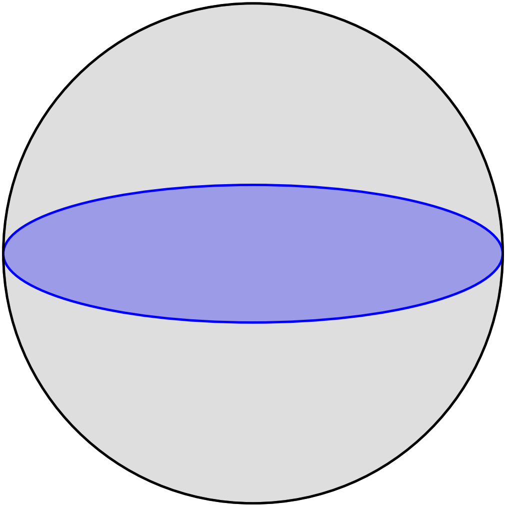
$$\underbrace{f(\sigma_1) \cdot \ldots \cdot f(\sigma_r)}_{r\text{-fold algebraic intersection number}} = 0$$

for any r-tuple of vertex-disjoint simplices of K.



triple intersection sign

Triple Whitney Trick



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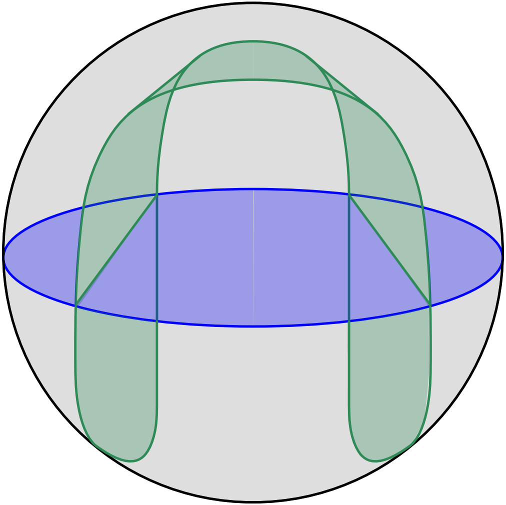

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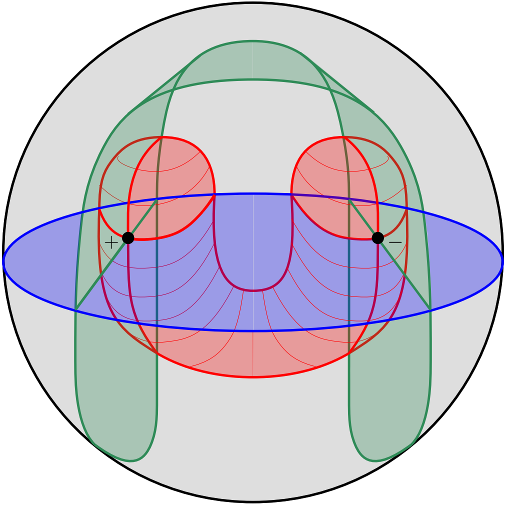

Three disks inside a ball intersecting in two points of opposite sign.

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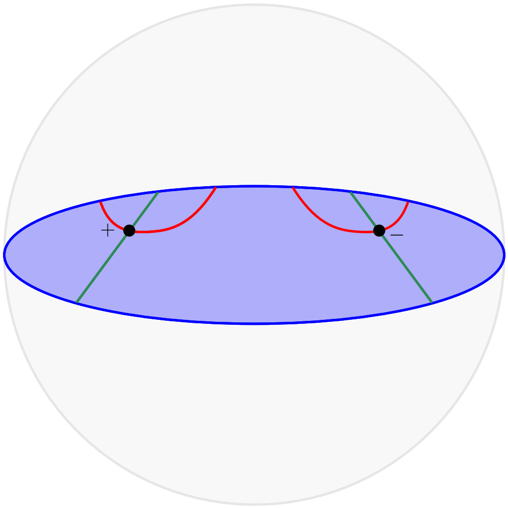

Restriction to blue disk: intersection points in different components of the intersections.

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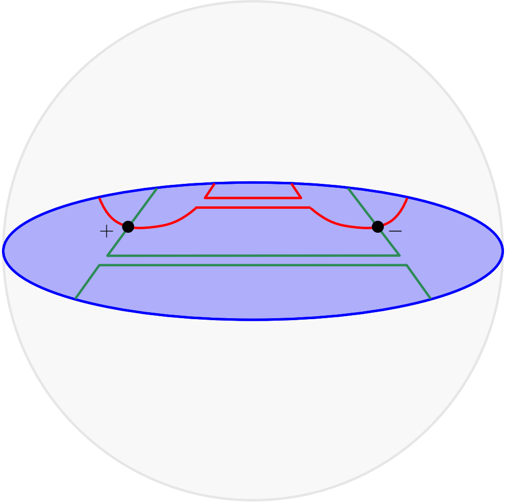

Change intersections with blue disk by "piping".

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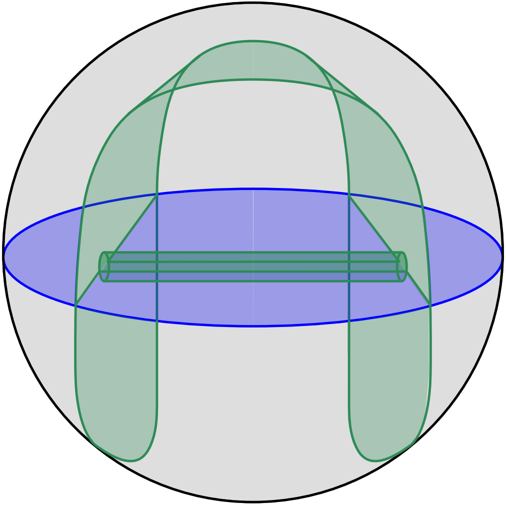

Change intersections with blue disk by “piping”.
This changes topology of the other disks.

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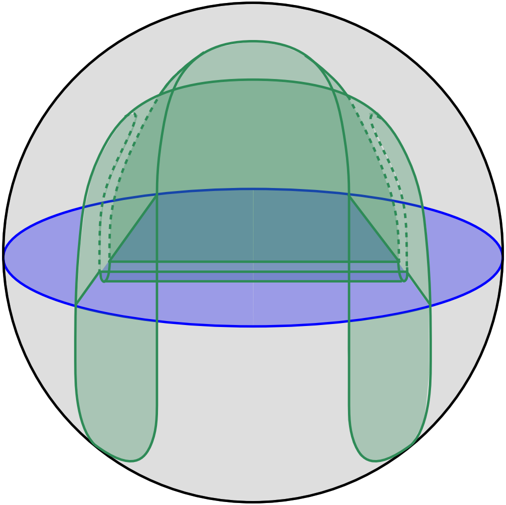

Repair the topology of the other disks by “unpiping”.

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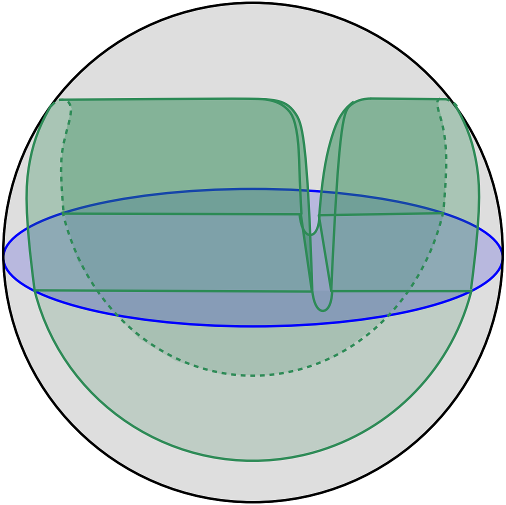

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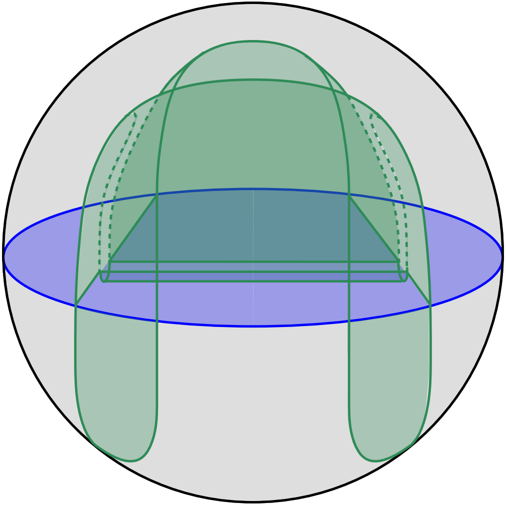

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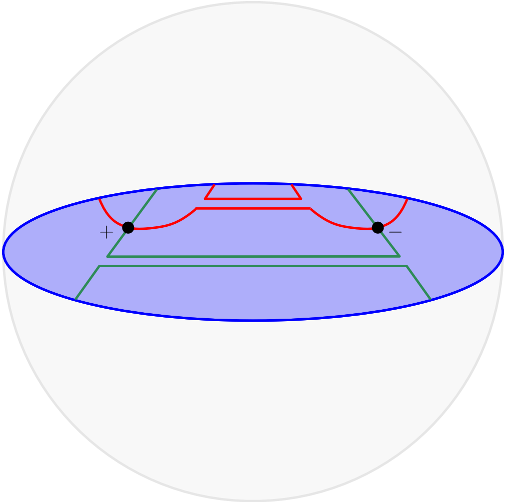

After piping + unpiping, perform double Whitney trick inside the blue disk

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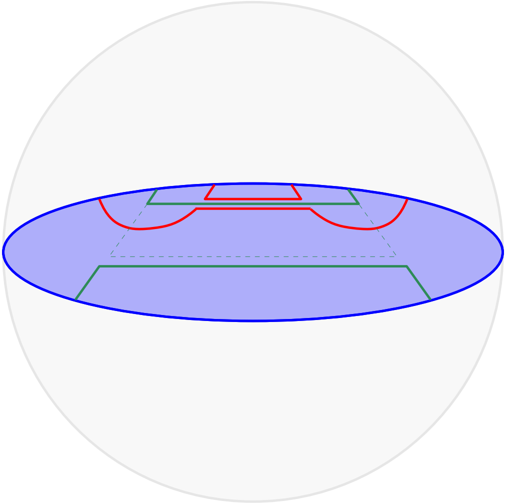

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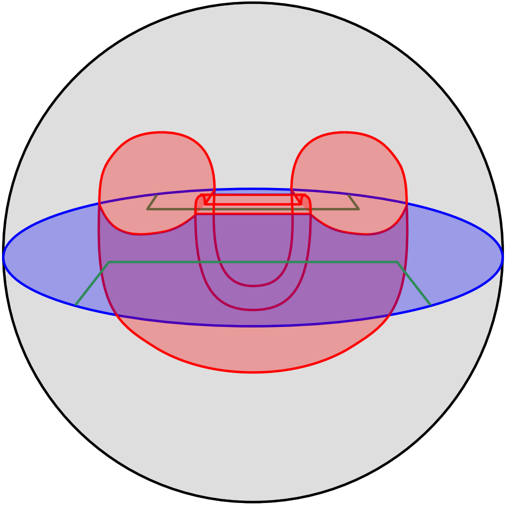

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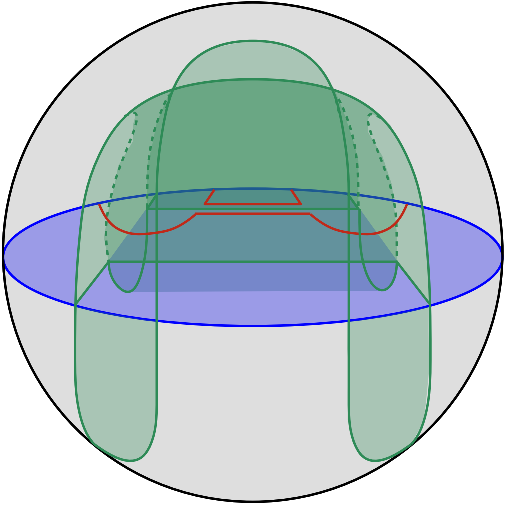

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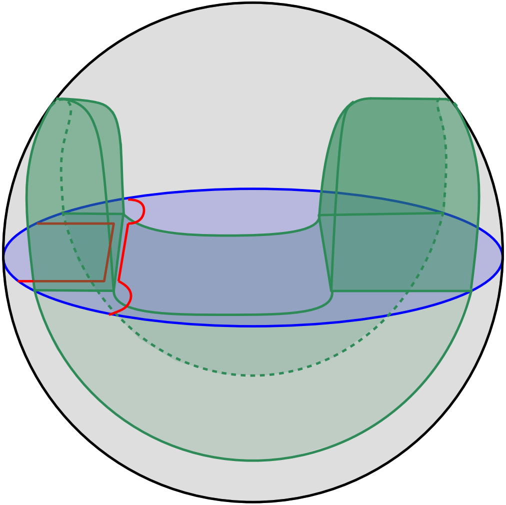

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- Other "*h*-principles" in topological combinatorics?

  (equivariant map from configuration space $\Rightarrow$ geometric solution?)

Merci de votre attention!