The Complex Dynamics of Financial Prices

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Outline of this talk

- The nature of trading
- Options as experimental tools
- The Black-Scholes model
- The volatility surface
- Stochastic volatility
- Rough volatility

The trader's perspective

- There are many ways to approach an understanding of the world in which we live, in particular the world of finance:
  - As a physicist
  - As a mathematician
  - As an economist
- In this talk, I will try to approach the world of finance from the point of view of a trader.
  - More specifically, from the point of view of an options trader.

What is a trader?

- A trader buys and sells financial assets.
- Traditionally, traders are either hedgers, speculators or arbitrageurs.
- Today, trading is overwhelmingly automated and electronic.
  - Today's trader programs and monitors computer algorithms.

Old-style trading

Figure 1: The trading floor of the New York Stock Exchange.

Computerized trading

Figure 2: The trading floor of IMC in Chicago.

Trading algorithms

- Trading algorithms perform the functions of hedger, speculator and arbitrageur all at once.
  - Every algorithm needs to be able to hedge accumulated inventory.
  - Detecting and following trends in the price series is popular.
  - Statistical arbitrage where profits are not guaranteed but considered likely based on historical price patterns is a favorite activity.
    - True arbitrage, where profit is riskless, assumed not to exist in theory, almost never arises in practice either.
  - Market making, capturing the spread between bid and offer, is another typical objective of an algorithm.

Our picture of the market

- Our picture will be one of a game between competing algorithms.
  - Each time an algorithm trades, it impacts the market price.
    - A buy order tends to increase the price of the asset being traded.
    - A sell order tends to decrease the price.
  - That change in the price is detected by other algorithms that respond by trading.
  - And so on for ever ...

- As will see, such a game creates complex dynamics and wonderful patterns that can be modeled mathematically.
  - Mathematical beauty emerges from the complex interaction between thousands of trading algorithms, all trading on different time scales.
- Traders do not need to understand finance theory or mathematics to be able to trade profitably!
- To paraphrase Nassim Taleb, birds do not need ornithologists to teach them how to fly.
  - A typical trading algorithm requires imagination to come up with the idea, but then just simple regression to come up with parameters.
  - Much to the chagrin of physicists, mathematicians and economists, the only essential skill is computer programming.

The point of modeling

- If scientists are not necessary for trading, what is the point of modeling?
  - Mathematical models help us trade optimally, taking into account our portfolio and the history of prices (time series).
    - In other words, models help us maximize profits whilst minimizing risk.
  - Traders become more comfortable handling risk, and spreads tighten, reducing transaction costs.
  - This is a feedback effect: Modeling affects prices.

The point of markets

- If trading is just a game between algorithmic traders, what is the point of markets?
  - According to economic theory, one of the key functions of markets is to allocate resources efficiently.
- From our trader's perspective, there are end users, not just traders in the market. For example
  - Corporates raising capital for new initiatives.
  - Long term investors such as pension funds.
- Nevertheless, most trading activity appears to be endogenous, traders trading with other traders.
- The complex game played by traders makes the process of resource allocation (we hope) more efficient.

SPX from 1927 to present

Figure 3: Time series of SPX prices since 1927. This looks like exponential growth.

Log(SPX) from 1927 to present

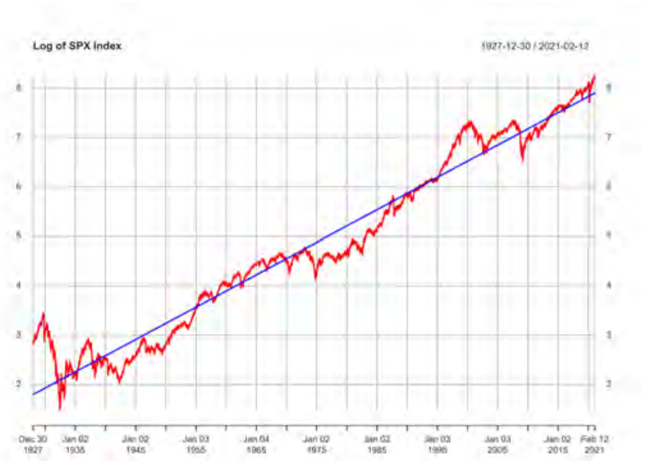
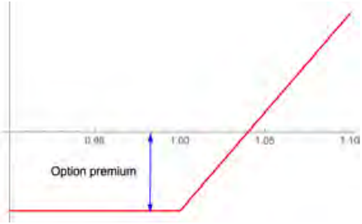
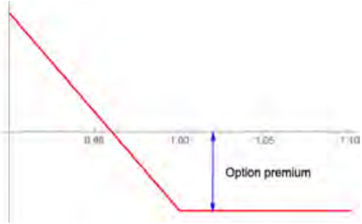


Figure 4: Time series of SPX prices since 1927. This looks approximately linear. Average continuous return is about 4.5% per annum.

Call and put options



(a) Call option payoff



(b) Put option payoff

- A call option gives the holder the right but not the obligation to buy stock at the strike price at expiration.
- A put option gives the holder the right but not the obligation to sell stock at the strike price at expiration.

An option strategy: The condor

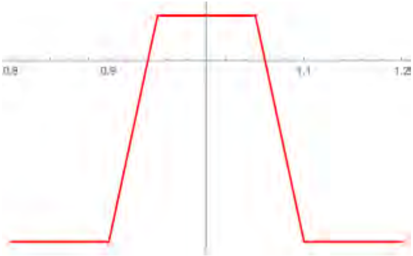


Figure 6: This strategy is long a 0.9 strike put, short a 0.95 strike put, short a 1.05 strike call, and long a 1.10 strike call.

- The condor makes a small amount of money if the stock price stays in the central band and loses a greater but limited amount of money if the stock price is very low or very high at expiration.

Trading options

- Options have been traded at least since ancient Greek times – see [Del] for great stories about options trading in Amsterdam in the 17th century.
- Expressing your market view using options is a fun and maybe even profitable game.
- Analyzing the market pricing of options is a great way to discover properties of the underlying time series.
  - To see why, consider the butterfly.

Butterfly

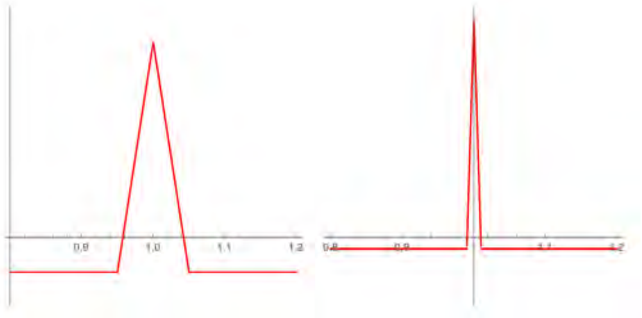


Figure 7: Two butterfly spreads, both with maximum payoff of 1 unit.

- A butterfly spread is long a put at $K - \epsilon$, short a put at K, short a call at K, and long a call at $K + \epsilon$.
    - Option prices convey information about the distribution of the underlying.

Options as experimental tools

- The profitability of option traders depends on their assessing correctly the relative probabilities of different events.
  - We can discover many properties of the underlying time series by looking at option prices.
- To be able to understand what option prices tell us about the behavior of the underlying price, we need a model.

Louis Bachelier

- Option traders had ingenious ways of pricing options long before the advent of option pricing theory.
- But how much are options really worth?
  - What is the fair premium?
- One of the first to consider this problem mathematically was Louis Bachelier in his 1900 Ph.D. thesis supervised by Henri Poincaré.
- Significantly, Louis Bachelier probably (in my view almost surely) traded options!

Put-call parity

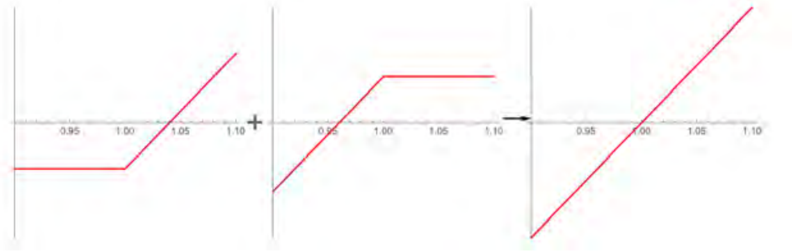


Figure 8: Long one call and short one put is equivalent to long the stock.

- It follows that short one share of stock is a perfect hedge for a portfolio that is long one call and short one put.
  - You can convince yourself that this means options should be priced as if the expected return were zero (ignoring financing and dividends).

Quote from Louis Bachelier

On page 38 of Théorie de la spéculation [Bac1900], Bachelier writes[1]:

> En résumé, la considération des cours vrais permet d'énoncer un principe fondamental:
>
> *L'espérance mathématique du spéculateur est nulle.*

[1] As translated by Mark Davis and Alison Etheridge: In summary, consideration of true prices allows us to state the fundamental principle: *The mathematical expectation of the speculator is zero.*

- Bachelier combined this assumption with the assumption that price moves are independent to effectively invent Brownian motion.
    - Five years ahead of Einstein!
- In modern notation, his price process reads

$$dS_t = S_0 \, \sigma \, dW_t.$$

- This process is known as arithmetic Brownian motion.
    - Brownian motion is a continuous process which is nowhere differentiable.
    - The constant σ is known as *volatility*.
    - $\mathrm{var}[S_t] = S_0^2 \, \sigma^2 \, t$ is directly proportional to the time interval t.
- Bachelier also wrote down an option pricing formula.

Comparison with real data

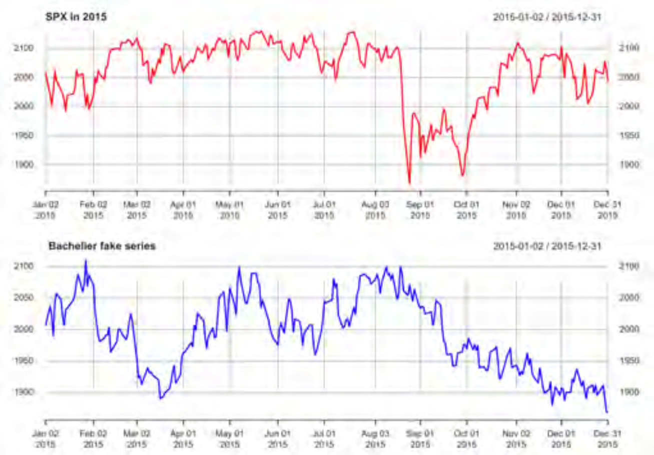


Figure 9: The fake time series looks pretty much like the real one.

Black and Scholes

- In 1973, Black and Scholes published a paper with their famous formula.
  - From a trader's perspective, this formula did not really add anything to Bachelier's work.
- Black and Scholes used geometric rather than arithmetic Brownian motion.

$$dS_t = S_t \, \sigma \, dW_t.$$

  - Log-returns over the interval τ are normally distributed with standard deviation $\sigma \sqrt{\tau}$.
- One key practical contribution however was the realization that in the limit of continuous trading with no transactions costs, risk could be eliminated.

Black and Scholes

The following quote is from Black and Scholes' 1973 paper "The pricing of options and corporate liabilities" [BS1973]

> Thus the risk in the hedged position is zero if the short position in the option is adjusted continuously. If the position is not adjusted continuously, the risk is small, and consists entirely of risk that can be diversified away by forming a portfolio of a large number of such hedged positions.

- This was the insight that led to the phenomenal growth of the financial derivatives industry.
- This and related work earned Robert Merton and Myron Scholes the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997.

The Black-Scholes formula

The fair price of a European call option with strike price K and time to expiration τ when the stock price is S is given by[2]

$$C_{\rm BS}(S, K, \tau, \sigma) = S\, N\,(d_+) - K\, N(d_-)$$

where $N(\cdot)$ is the cumulative normal distribution and

$$d_\pm = \frac{-k}{\sigma\sqrt{\tau}} \pm \frac{\sigma\sqrt{\tau}}{2}.$$

Here the log-strike $k = \log K/S$ and σ is the (constant) *volatility*.

[2] We have ignored interest rates and dividends.

Delta hedging

- Option pricing formulas not only give the price, but also the sensitivity of the price with respect to model parameters.
  - In the case of the Bachelier and Black-Scholes formulae, this means sensitivity to the stock price.
- According to this strategy, to hedge a short option position,
  - buy the stock after the price goes up,
  - sell the stock after the price goes down.
- This sounds like a money-losing strategy!
  - The cost of this strategy is the option premium.

Black Monday

- Up until October 19, 1987, traders seemed pretty comfortable pricing and hedging options using the Black-Scholes formula.
- After this date, options were priced differently.
- Why?

Black Monday

Figure 10: On October 19, 1987, the SPX declined by 22.9%.

- The probability for this to happen under Black-Scholes assumptions is infinitesimally small.

Interactions, Rétroactions, Crises

- Black Monday is a perfect example of the theme of this lecture series.
- The debacle was caused by portfolio insurance – traders delta-hedging short put positions.
  - As the market falls, sell stock.
  - This makes the price fall futher.
  - Sell more stock, and so on...

Consequences for options pricing

- Traders who were long options made huge gains.
- Traders who were short options made huge losses.
  - Returns on out-of-the-money options were particularly huge.

- Conclusion: Options had been mis-priced and mis-hedged.

- Let's take a look at the data again.

Histogram of daily log returns

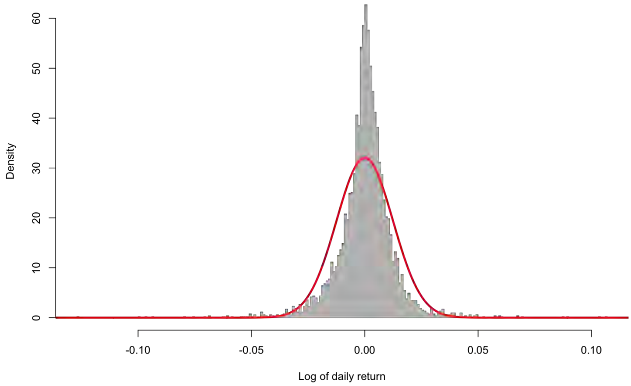


Figure 11: Histogram of SPX log-returns since 2000. The normal density with the sample standard deviation is superimposed.

SPX daily returns

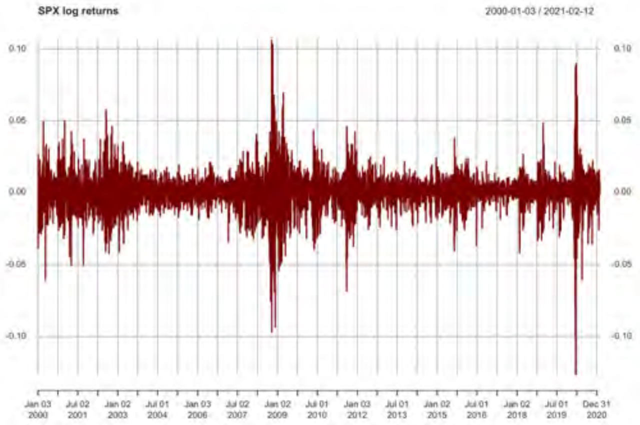


Figure 12: Time series of SPX daily returns since 2000.

SPX daily returns

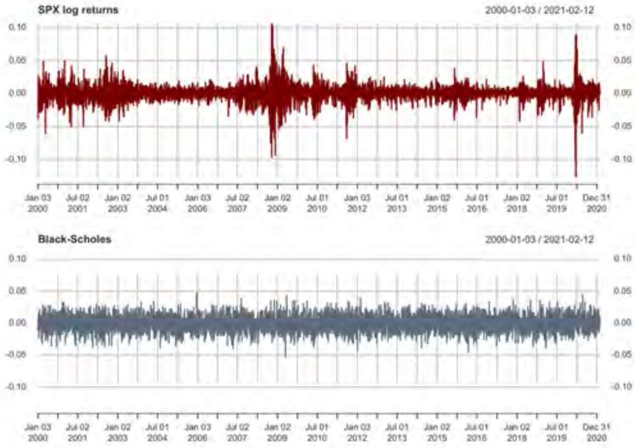


Figure 13: Top: Time series of SPX daily returns since 2000. Bottom: The Black-Scholes equivalent.

Quote from Benoit Mandelbrot

The following quote is from Benoit Mandelbrot's 1963 paper "The variation of certain speculative prices" [Man1963]

> In other words, large changes tend to be followed by large changes — of either sign — and small changes tend to be followed by small changes.

- Volatility is a proxy for the magnitude of price changes.
- Volatility seems to be a stochastic process.

SPX realized variance

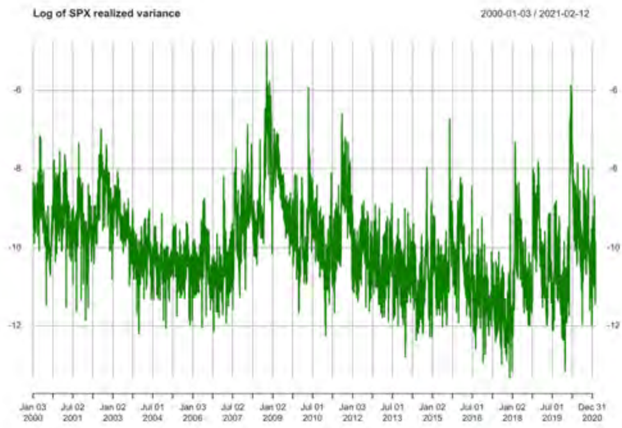


Figure 14: Time series of the log of SPX daily realized variance (a proxy for $\log \sigma_t^2$) since 2000.

The leverage effect

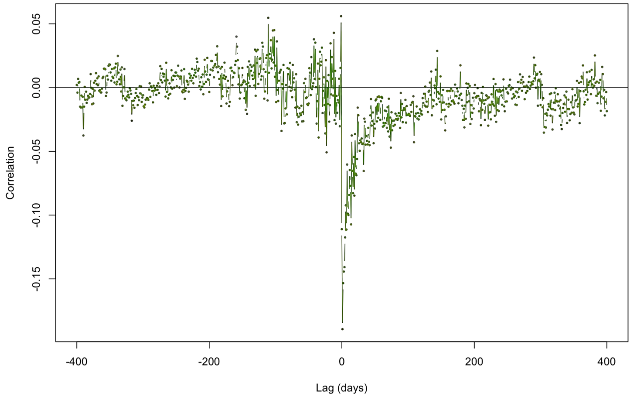


Figure 15: The correlation between $\Delta \log S_t$ and $\sigma^2_{t+\ell}$ for $\ell \in [-400, 400]$.

Stochastic volatility

- From 1987, option traders began to price options with stochastic volatility models.
- Under stochastic volatility,

$$\begin{aligned} dS_t &= S_t\,\sigma_t\,dW_t \\ d\sigma_t &= \alpha_t\,dt + \beta_t\,dZ_t. \end{aligned}$$

- W and Z are negatively correlated Brownian motions.
  - The negative correlation models the leverage effect - volatility tends to increase when the stock price falls.
- Volatility is a continuous stochastic process!
  - When volatility is high, there are large moves in either direction - consistent with Mandelbrot's observation.

Pricing and hedging

- There is no nice closed-form formula for the option price in most stochastic volatility models.
  - There is a quasi-closed form formula in the Heston model [Hes1993], which sadly has unrealistic dynamics.
- Numerical techniques are required.
- In the simplest case of a one-factor stochastic volatility model, options may be hedged using stock and one other option.
  - Management of portfolios is conceptually no harder than before.

Histogram of daily log returns again

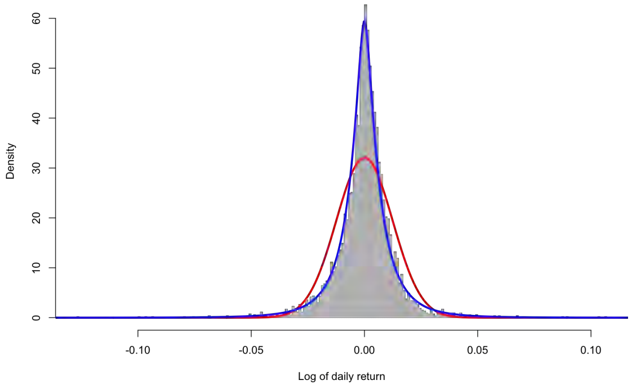


Figure 16: Histogram of SPX log-returns since 2000. A normal density fit is superimposed in red and a lognormal stochastic volatility model fit in blue.

Implied volatility

- The Black-Scholes formula $C_{\rm BS}(S,K,\tau,\sigma)$ gives the fair value of a call option assuming volatility is constant.
  - We can invert the market price of an option to get so-called *implied volatility* $\sigma_{\rm BS}$.

$$C_{mkt}(S,K,\tau) =: C_{\rm BS}(S,K,\tau,\sigma_{\rm BS}(K,\tau)).$$

- Fixing time to expiration τ, the graph of $\sigma_{\rm BS}(K,\tau)$ is known as the *volatility smile*.
  - This is a feature of option markets generated by trading – recall Nassim Taleb's "teaching birds how to fly"!
- Almost all stochastic volatility models generate prices consistent with an individual volatility smile.
  - But not with the collection of all volatility smiles.

An SPX volatility smile: 30-Dec-2020

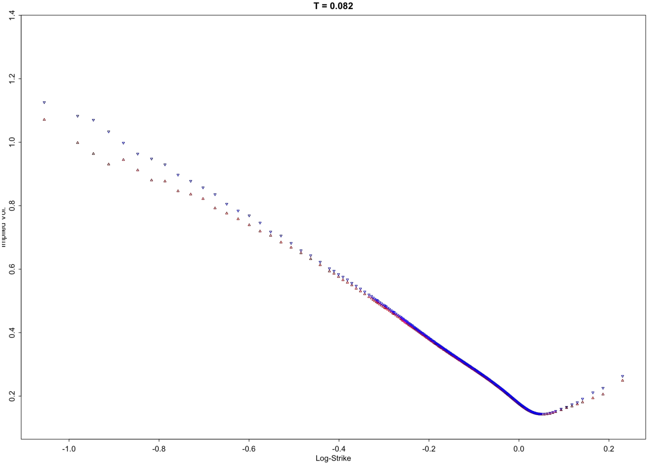


Figure 17: The volatility smile for $\tau = 0.077$ as of 30-Dec-2020.

The SPX volatility surface: 30-Dec-2020

Graphing all 35 slices together, we get:

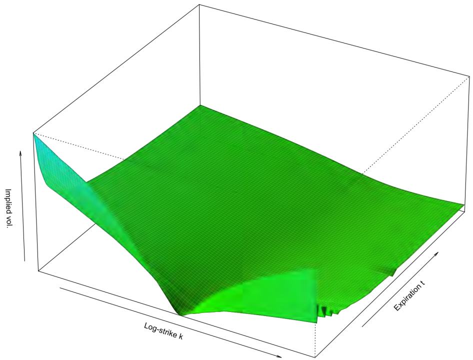


Figure 18: The SPX volatility surface as of 30-Dec-2020.

Remarks on the volatility surface

- The volatility surface has remained approximately the same shape since 1987.
  - Up to changes in level and slight changes in orientation.
- Until recently, no simple stochastic volatility model was able to reproduce this simple-looking shape.
  - This is a problem for option traders because they need accurate models that are consistent with the time series to minimize pricing errors and optimize hedges.

The VIX

- Assuming that paths of S and σ are continuous, the fair value of expected variance $\mathbb{E}_t \left[\int_t^T \sigma_s^2 \, ds \right]$ can be computed from the prices of options with expiry T.
- A good approximation to this number is updated in real time.
  - Roughly speaking

$$VIX_t = \sqrt{\frac{1}{\Delta} \mathbb{E}_t \left[\int_t^{t+\Delta} \sigma_s^2 \, ds \right]},$$

  with $\Delta = 30$ days.
- By inspection of the formula, VIX ("volatility index") is a proxy for implied volatility.
- VIX futures and options on VIX are traded.

The VIX time series

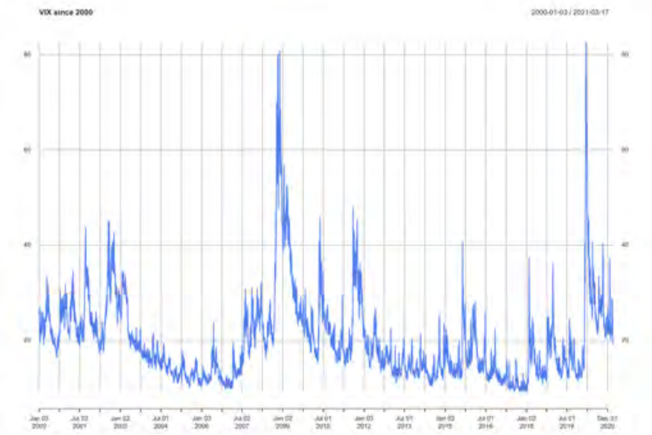


Figure 19: The VIX since 2000.

The VIX and RV time series

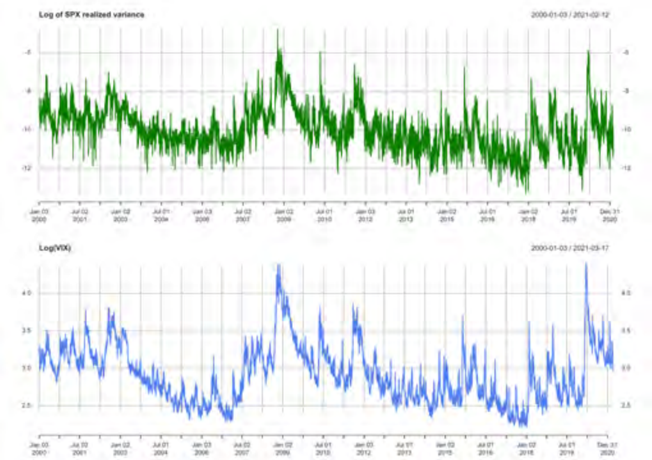


Figure 20: VIX looks qualitatively like RV.

Trading VIX

- Trading VIX is like trading (implied) volatility.
  - Sensitivity to implied volatility is called *vega*.
- Since 2013, more vega has been traded in VIX options and futures than in SPX options, still the world's biggest equity options market.
- We can easily compute VIX implied volatility and the VIX volatility surface.

A VIX volatility smile: 30-Dec-2020

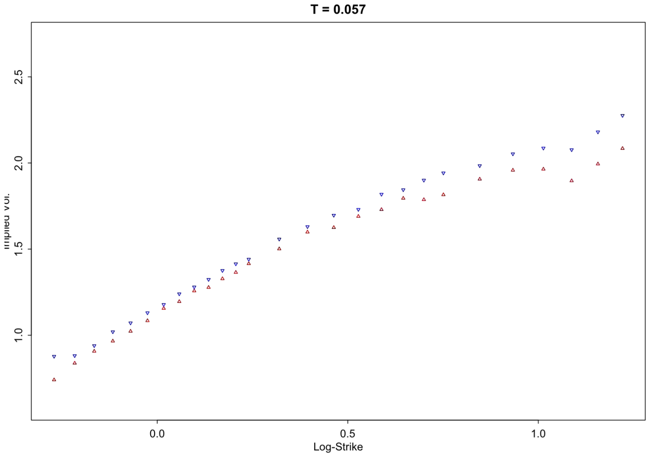


Figure 21: The VIX volatility smile for $\tau = 0.057$ as of 30-Dec-2020.

The VIX volatility surface: 30-Dec-2020

Graphing the 9 slices together, we get:

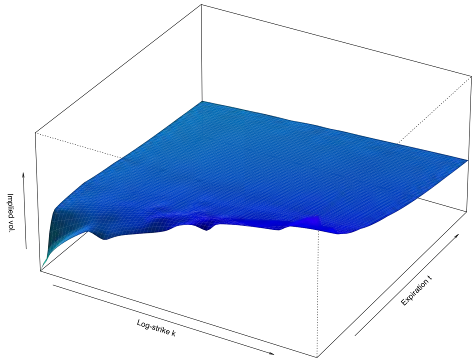


Figure 22: The VIX volatility surface as of 30-Dec-2020.

Consistent modeling of SPX and VIX

- The consistent modeling of SPX and VIX is a very active area of research.
  - Can we find a (tractable) model that is consistent with SPX and VIX smiles?
- The existence of such a model would permit the efficient hedging of VIX options with SPX options and likely reduce VIX option spreads.

- Another example of a feature of market prices that is not yet well understood.
  - Traders do not need to be taught how to trade!

Term structure of SPX ATM skew: 30-Dec-2020

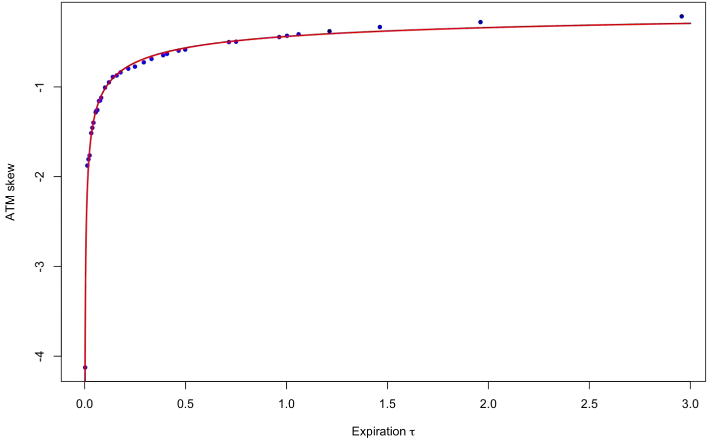


Figure 23: Term structure of ATM skew as of 30-Dec-2020, with power law fit $\tau^{-0.37}$ superimposed in red.

A theoretical idea

- Conventional stochastic volatility models cannot generate this power-law term structure.
- [ALV07] and [Fuk11] showed that power-law skew could be generated by a stochastic volatility model where volatility is a function of fractional Brownian motion (fBm).
    - Fractional Brownian motion (fBm) is a generalization of Brownian motion due to Benoit Mandelbrot.
- For such a model to offer a realistic description of the historical time series of instantaneous volatility, this time series would have to exhibit scaling properties consistent with fBm.

SPX realized variance

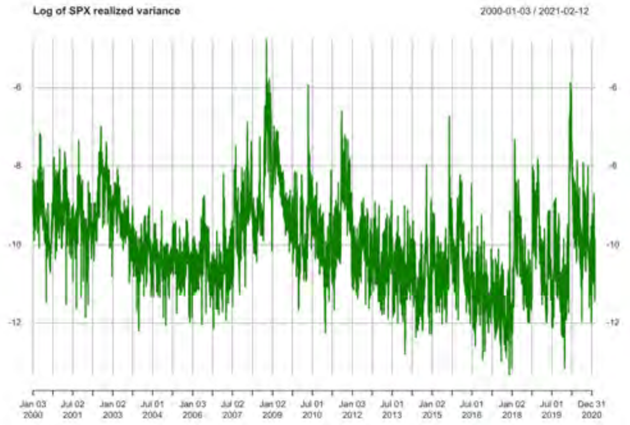


Figure 24: Time series of the log of SPX daily realized variance (a proxy for $\log \sigma_t^2$) since 2000.

Scaling of the volatility process

- The volatility process looks rough.
  - Recall that the variance of Brownian motion scales as τ.
- To study the roughness of the volatility process, we define the qth sample moment of differences of log-volatility at a given lag Δ.($\langle\cdot\rangle$ denotes the sample average):

$$m(q, \Delta) = \langle |\log \sigma_{t+\Delta} - \log \sigma_t|^q \rangle$$

  - For example

$$m(2, \Delta) = \langle (\log \sigma_{t+\Delta} - \log \sigma_t)^2 \rangle$$

  is just the sample variance of differences in log-volatility at the lag Δ.

SPX realized variance

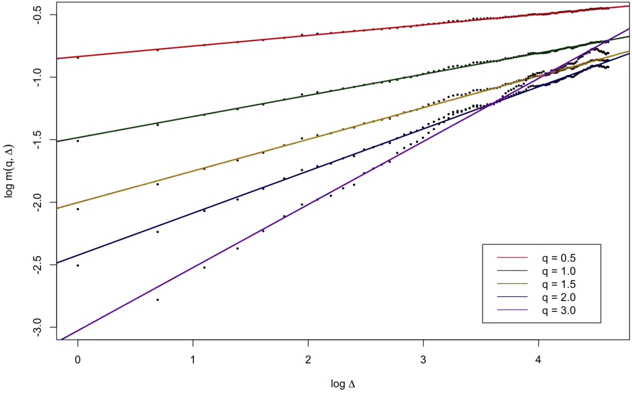


Figure 25: $\log m(q, \Delta)$ as a function of $\log \Delta$, SPX..

Scaling of ζ_q with q

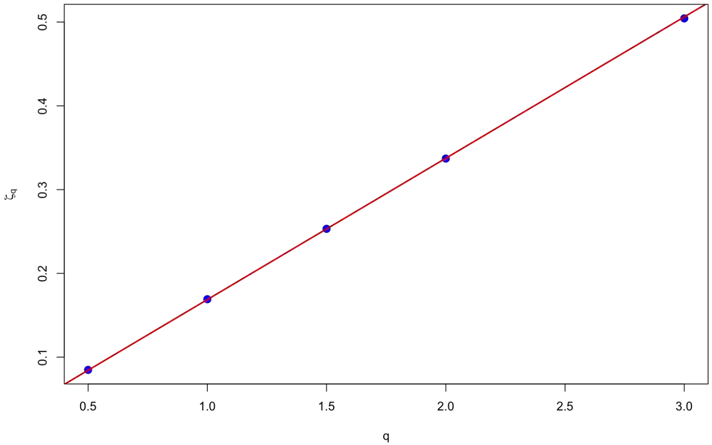


Figure 26: Scaling of ζ_q with q.

Monofractal scaling result

- From the log-log plot Figure 25, we see that for each q, $m(q, \Delta) \propto \Delta^{\zeta_q}$.
- And from Figure 26 the monofractal scaling relationship

$$\zeta_q = q\,H$$

  with $H \approx 0.167$.
  - This estimate of H is biased high: $H \approx 0.1$ is probably more reasonable.

- The unique continuous Gaussian process with this scaling property is fractional Brownian motion (fBm).
  - $\log \sigma_t$ looks like fBm!

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  - Thus $\log \sigma_t$ looks like fBm!

$\log \sigma_t$ looks like fBm

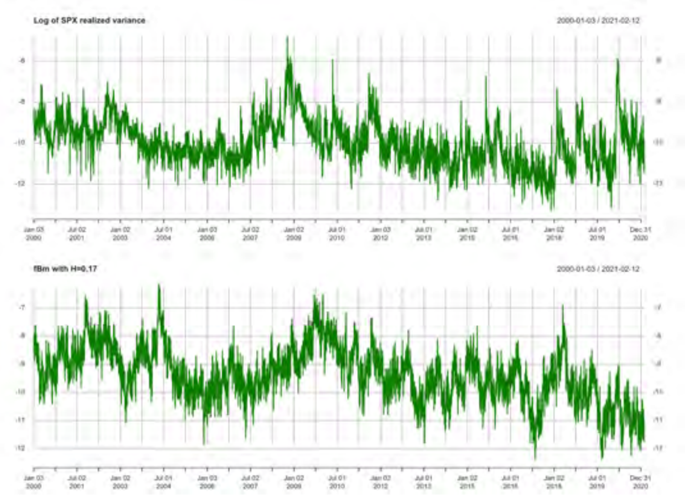

Figure 27: The top plot is $\log \sigma_t$; the bottom plot is fBm.

Microstructural foundation for rough volatility

- Rosenbaum et al. [EFR16] consider a simple model of order flow with the following properties:
  - The price jumps up(down) by one unit on arrival of a buy(sell) order.
    - The jump process is self-exciting: The arrival of an order increases the arrival rate of orders.
  - Liquidity asymmetry: The average impact of a sell order is greater than the impact of a buy order.
  - Splitting of metaorders motivates power-law decay of the excitation process.

The scaling limit of the microstructural model

- Cleverly rescaling this simple process in time and space leads to a rough stochastic volatility model.
  - A natural generalization of the Heston model – the rough Heston model.

- Later, [JR20] generalized the above by arguing that rough volatility is a direct consequence of no arbitrage plus the existence of market impact.

Another approach

- We need not rely on options prices to understand better the dynamics of prices.
- For example, Pr. Jean-Philippe Bouchaud and his collaborators at CFM have spent many years studying directly the time series of prices.
  - In particular, [BDB17] showed that Quadratic Hawkes (nonlinear self-exciting jump) models offer surprisingly good descriptions of the dynamics of financial prices.
- Also Julien Guyon has suggested that only explicitly path-dependent models (that we have not considered here) are capable of describing the joint dynamics of SPX and VIX.

The quadratic rough Heston model

- Lognormal dynamics are more reasonable.
- Inspired by [BDB17, DJR19], [GJR20] proposed the quadratic rough Heston model.
  - The QR Heston process is approximately lognormal.
  - The QR Heston model is explicitly path-dependent: Variance is a weighted function of the price path.
  - Moreover, it seems that this model calibrates to the SPX and VIX simultaneously!

SPX smiles as of 19-May-2017

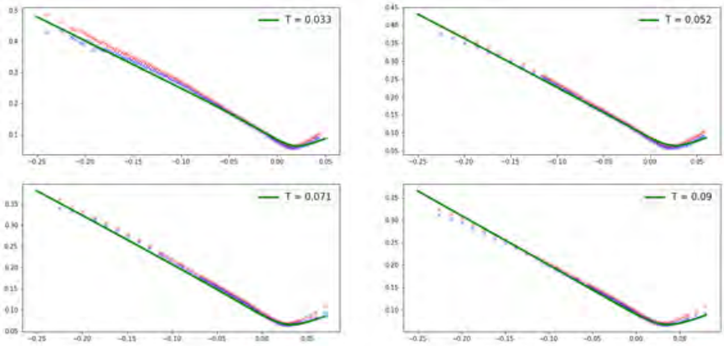

Figure 28: Some SPX smiles as of 19-May-2017.

VIX smiles as of 19-May-2017

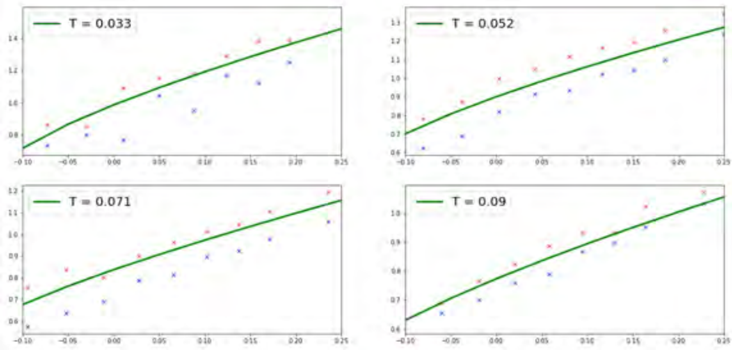

Figure 29: Some VIX smiles as of 19-May-2017.

Summary

- The complex dynamics of financial prices emerges from the interactions between many thousands of traders.
- The nature of these dynamics is still to be completely understood (and may never be).
  - The market pricing of options gives us sophisticated guidance towards a better understanding.
- As our understanding improves and leads to better risk management, spreads should decline further and more fascinating patterns in prices will likely emerge.

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