Toward a Statistical Neuroscience

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From Medical Images to Computational Medicine
Collège de France
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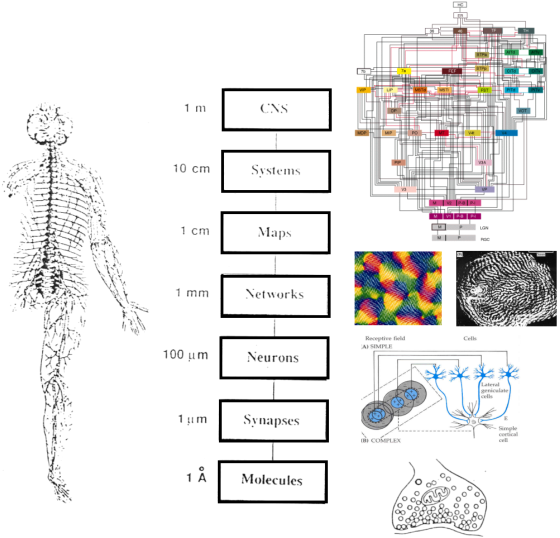
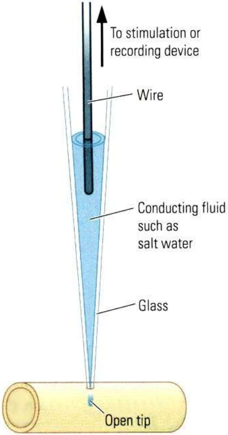
Scales in the CNS

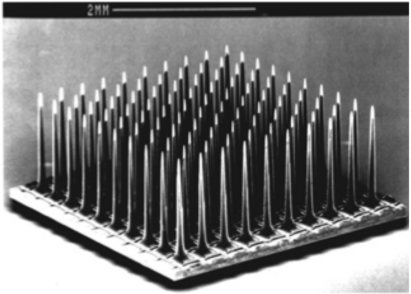


Measuring at different scales

Microelectrode

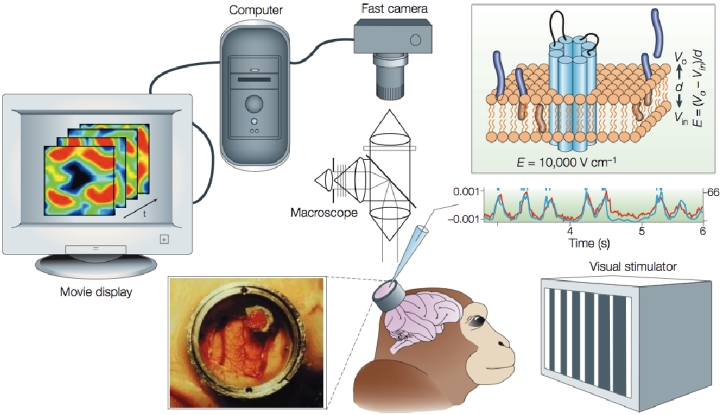


Utah Multi Electrode Array (MEA)



Mesoscopic and macroscopic models

Optical imaging



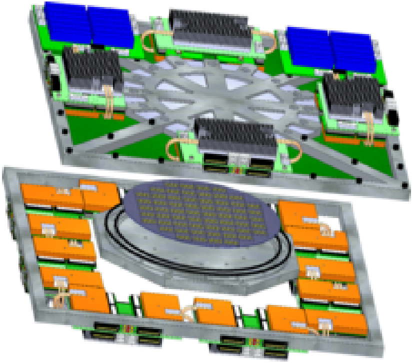
Grinvald-Hildesheim Nature Reviews Neuroscience 04

Mesoscopic and macroscopic models

Electroencephalography (EEG) and Magnetoencephalography (MEG)

Mesoscopic and macroscopic models

The Brain-Scales project hardware



Mesoscopic and macroscopic models

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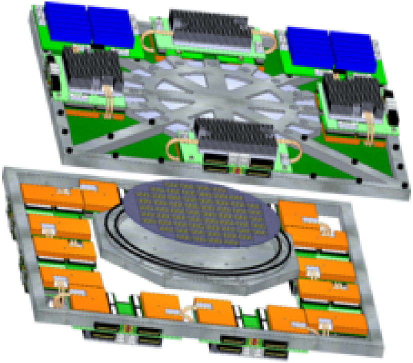
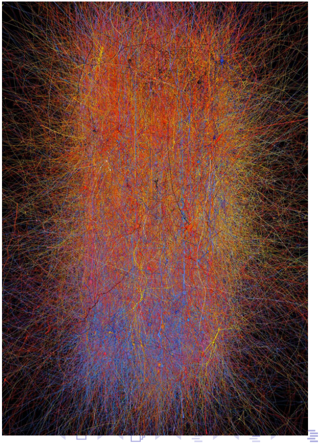


The Blue Brain project simulator (EPFL)



Mesoscopic and macroscopic models

- These two projects continue with the Human Brain Project (HBP) EC Flagship
- and the Brain project in the US

Introduction

Motivations: Sparse representations, emerging phenomena

Large Deviations

Take home messages

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- Understand the role of **randomness**

An example: a toy model of V1

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- You end up with

An example: a toy model of V1

$$\begin{cases} dV_t^i &= \left(V_t^i - \frac{(V_t^i)^3}{3} - w_t^i + I(t) \right) dt + \frac{1}{N} \sum_{j=1}^N \bar{J}(V_t^i - V_{\rm rev}) y_t^j dt + \\ & \quad \frac{1}{N} \left(\sum_{j=1}^N \sigma(V_t^i - V_{\rm rev}) y_t^j \right) dB_t^i + \sigma_{\rm ext}\, dW_t^i \\ dw_t^i &= a\left(b\, V_t^i - w_t^i \right) dt \\ dy_t^i &= \left(a_r S(V_t^i)(1 - y_t^i) - a_d y_t^i \right) dt + \sigma(V_t^i, y_t^i) dW_t^{i,y} \\ J_{ij}(t) &= \frac{\bar{J}}{N} + \frac{\sigma}{N} \xi^i(t) \end{cases}$$

$\simeq$ **10 millions times!**

What can be done?

- Develop a statistical neuroscience

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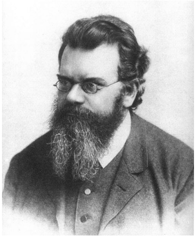

- Develop a statistical neuroscience
- Ludwig Boltzmann: Inventor of statistical mechanics

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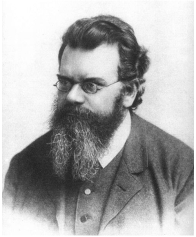
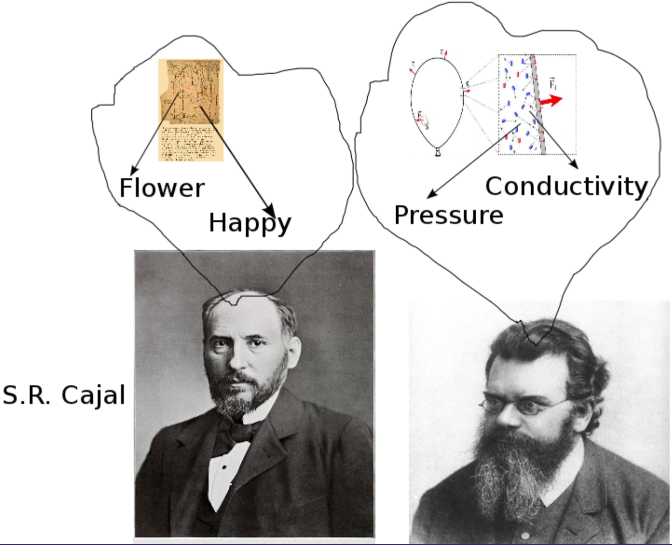


- Develop a statistical neuroscience

- Ludwig Boltzmann: Inventor of statistical mechanics
- Explains and predicts how the properties of atoms (mass, charge, and structure . . .) determine the visible properties of matter (viscosity, thermal conductivity, diffusion . . .)

Statistical mechanics and statistical neuroscience



Introduction

Motivations: Sparse representations, emerging phenomena

Large Deviations

Take home messages

Large Deviation Principle in a nutshell

- Deals with rare events,
- and the asymptotic computation of their probability on a log scale.
- Requires a rate function H.

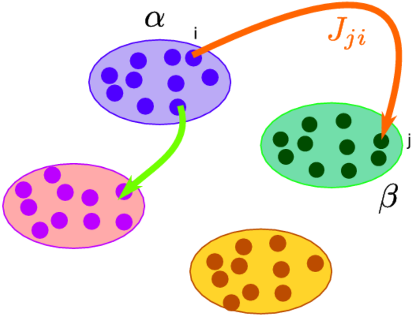
An intuitive approach

- Consider a sequence of random variables $\{X_n\}_{n \geq 0}$.
- It satisfies a Large Deviation Principle with rate function H if

$$P(X_n \simeq x) \simeq e^{-nH(x)}$$

- The random variables X_n concentrate on the points x such that $H(x) = 0$.
- If H reaches 0 at a unique point x^* then the law of X_n converges in law toward the Dirac mass δ_{x^*}: **concentration of measure** phenomenon..

The network



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- Coupled dynamics

$$U^i(t) = \gamma U^i(t-1) + \sum_{j=1}^{N} J_{ij}^N f(U^j(t-1)) + B^i(t-1) \quad t = 1 \cdots T$$

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- f is a sigmoid defining the firing rate

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- A solution of the networks equations is a (random) probability measure $Q^N(J^N)$ on $\mathcal{T}^N$
- The goal is to characterize the limit (if it exists) of

$$Q^N = \mathbb{E}^{J^N}[Q^N(J^N)],$$

when $N \to \infty$

The tools of the trade

- Let $\mathcal{T}^{\mathbb{Z}}$ be the set of doubly infinite sequences of trajectories

$$u \in \mathcal{T}^{\mathbb{Z}}, \quad u^i \quad \text{its} \quad i\text{th coordinate} \ \in \mathcal{T}$$

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- $\mathcal{M}_1^+(\mathcal{T}^{\mathbb{Z}})$ the set of probability measures on $\mathcal{T}^{\mathbb{Z}}$
- $\mathcal{M}_{1,S}^+(\mathcal{T}^{\mathbb{Z}})$ the set of stationary probability measures on $\mathcal{T}^{\mathbb{Z}}$ (shift invariant)

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- **What it is:**
  a mapping from $\mathcal{T}^N$ to $\mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}})$

The search for a neuronal LDP

- **Key mathematical idea:**
  Consider the image law Π_N of Q^N through

$$\hat{\mu}^N : \mathcal{T}^N \to \mathcal{M}_{1,S}^{+}(\mathcal{T}^{\mathbb{Z}})$$

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  Pushed forward measure

- It is a probability measure on $\mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}})$ (**probability measure on a set of probability measures!**):

$$\forall B \in \mathcal{B}(\mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}})), \quad \Pi^N(B) = Q^N(\hat{\mu}^N \in B)$$

Main result

Theorem (O.F., J. MacLaurin)

Π^N *is governed by a large deviation principle with a good rate function* H.

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It generalizes previous work by

- H. Sompolinsky (Hebrew University)
- G. BenArous and A. Guionnet (Courant Institute and MIT)
- O. Moynot and M. Samuelides (Toulouse University)

The unique minimum of the rate function

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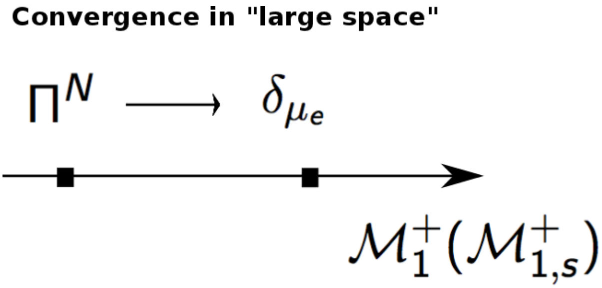
- μ_e can be effectively computed: numerical experiments are possible!
- It is non-Markov

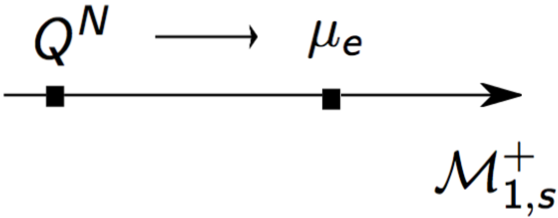
Consequences: Convergence results



This is the concentration of measure phenomenon.

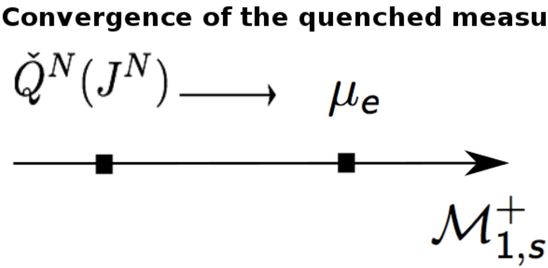
Consequences: Convergence results

Convergence of the averaged measure



The convergence holds when one averages over **all possible networks**

Consequences: Convergence results



The convergence is for **almost all synaptic weights**: it is unnecessary to average over many different networks.

Conclusion: LDP

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  1. Proving the existence of a limit law for the dynamics
  2. Characterizing the law
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- Large Deviations are essential in:
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  2. Characterizing the law
  3. Establishing averaged and quenched results
- Situation is a natural generalization of the i.i.d. case:
  1. Maximum correlation distance of the synaptic weights is d.
  2. The mean-field neurons are uncorrelated if they are separated by more than d: the representation is **sparse**
  3. If $d = 0$, then we observe propagation of chaos.

Take home messages

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- it is somewhat unrealistic.
- Taking into account correlations between synaptic weights is more realistic **but**
- the mathematical description is more complex.
- **Emerging phenomena** may be contained in the non-Markov description

Other potential applications of Large Deviations techniques

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  3. gene networks.