Kinetic equation method for compressible Navier-Stokes problems

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Notation

Let Ω be a bounded domain in $\mathbb{R}^3$ and $T \in (0, \infty)$. We denote by Q the cylinder with lateral surface S_T and by $\sqcup_T$ a parabolic boundary defined by

$$Q = \Omega \times (0, T), \quad S_T = \partial\Omega \times (0, T), \quad \sqcup_T = S_T \cup (\bar{\Omega} \times \{t = 0\}).$$

Let $\mathbf{U} \in C^3(\bar{Q})$ and $\mathbf{f} \in C(\bar{Q})$ be given vector fields,

$$\mathbf{U} : \bar{Q} \to \mathbb{R}^3, \quad \mathbf{f} : Q \to \mathbb{R}^3.$$

Let $\varrho_\infty \in L^\infty(\sqcup_T)$ be a given nonnegative function

$$\varrho_\infty : \sqcup_T \to \mathbb{R}^+.$$

Let the inlet $\Sigma_{\text{in}} \subset S_T$ be defined by

$$\Sigma_{\text{in}} = \{(x, t) \in \partial B \times (0, T) : \mathbf{U}(x, t) \cdot \mathbf{n}(x) < 0\},$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$.

Compressible NSE. Problem formulation.

Problem N-S. Find a velocity **u** and a density $\varrho \geq 0$ satisfying

$$\partial_t(\varrho\,\mathbf{u}) + \operatorname{div}(\varrho\,\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = \operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\,\mathbf{f} \quad \text{in } Q,$$

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } Q,$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \sqcup_T,$$
$$\varrho = \varrho_\infty \quad \text{on } \Sigma_{\mathrm{in}},$$
$$\varrho(x,0) = \varrho_\infty(x,0) \quad \text{in } \Omega,$$

where

$$\mathbb{S}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^\top + (\lambda - 1)\operatorname{div}\mathbf{u}, \quad \operatorname{div}\mathbb{S}(\mathbf{u}) = \Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u},$$

$$p(\varrho) = \varrho^\gamma.$$

Renormalized solution

A couple

$$\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega)), \quad \varrho \in L^\infty(0, T; L^\gamma(\Omega))$$

is said to be a *weak renormalized solution* to Problem (**N-S**) if $(\mathbf{u}, \varrho)$ satisfies

- The kinetic energy is bounded $\varrho|\mathbf{u}|^2 \in L^\infty(0, T; L^1(\Omega))$.
- The velocity satisfies the nonhomogeneous Dirichlet boundary condition $\mathbf{u} = \mathbf{U}$ on S_T.

Renormalized solution

- The integral identity

$$\int_Q \big(\varrho\mathbf{u}\cdot\partial_t\xi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\xi + p\,\mathrm{div}\,\xi - \mathbb{S}(\mathbf{u}) : \nabla\xi\big)\,dxdt + \int_Q \varrho\mathbf{f}\cdot\xi\,dxdt + \int_\Omega (\varrho_\infty\mathbf{U}\cdot\xi)(x,0)\,dx = 0$$

holds for all vector fields $\xi \in C^\infty(Q)$ equal to 0 in a neighborhood of the lateral side S_T and of the top $\Omega\times\{t = T\}$.

Renormalized solution

- The integral identity

$$\int_Q \big(\varphi(\varrho)\partial_t\psi + \varphi(\varrho)\mathbf{u}\cdot\nabla\psi + \psi(\varphi(\varrho) - \varphi'(\varrho)\varrho)\,\mathrm{div}\,\mathbf{u}\big)\,dxdt$$
$$= \int_{\Sigma_{\mathrm{in}}} \psi\varphi(\varrho_\infty)\mathbf{U}\cdot\mathbf{n}\,d\Sigma - \int_\Omega (\varphi(\varrho_\infty)\psi)(x,0)\,dx$$

holds for all $\psi \in C^\infty(Q)$ vanishing in a neighborhood of the surface $S_T \setminus \Sigma_{\mathrm{in}}$ and in a neighborhood of the top $\Omega \times \{t = T\}$, and for all smooth functions $\varphi : [0,\infty) \to \mathbb{R}$ such that

$$\limsup_{\varrho\to\infty}(|\varphi(\varrho)| + |\varrho\varphi'(\varrho)|) < \infty.$$

This means that φ has minimal admissible smoothness and φ is bounded at infinity.

Renormalized solutions

The latter identity means that

$$\frac{\partial \varphi(\varrho)}{\partial t} + \mathrm{div}\,(\varphi(\varrho)\mathbf{u}) + (\varphi'(\varrho)\varrho - \varphi(\varrho))\mathrm{div}\,\mathbf{u} = 0.$$

$$\varphi(\varrho) = \varphi(\varrho_\infty) \text{ on } \Sigma_{\mathrm{in}} \text{ and on } \Omega \times \{0\}.$$

Existence Theorem

The following theorem constitutes the existence of a weak renormalized solution to Problem (N-S),
(Girinon - 2011, P.& Sokolowski-2010, 2012).

Theorem

Assume that $\gamma > 3/2$. *Then Problem (***N-S***) has a weak renormalized solution which satisfies the estimate*

$$\|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c,$$

$$\|\varrho|\mathbf{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c$$

$$\|\varrho^\gamma\|_{L^\infty(0,T;L^1(\Omega))} \leq c.$$

Moreover, there is $\theta > 0$ *such that for any* $Q' \Subset Q$,

$$\int_{Q'} \varrho^{\gamma+\theta} \leq c(Q')$$

Oscillating Data

Let us consider Problem (**N-S**) with rapidly oscillating boundary data

$$\varrho_\infty^\epsilon \rightharpoonup \overline{\varrho}_\infty \quad \text{weakly}^\star \text{ in } L^\infty(\sqcup).$$

The sequence of boundary and initial data $\varrho_\infty^\varepsilon$ is only weakly convergent, and it may be rapidly oscillating as $\epsilon \to 0$. The example of a rapidly oscillating sequence is

$$\varrho_\infty^\epsilon = R\Big(x, t, \frac{x}{\epsilon}, \frac{t}{\epsilon}\Big),$$

where $R(x, t, y, \tau)$ is a smooth function periodic in y and τ. In particular,

$$\varrho_\infty^\epsilon = \text{const.} + A(x, t) \sin\Big(\frac{\omega t - \mathbf{k} \cdot x}{\epsilon}\Big).$$

Oscillating Data

In view of the mass transport equation, oscillations in boundary and initial data are transferred inside the flow domain along fluid particle trajectories. Hence we can expect that these oscillations induce rapid oscillations of the density in the flow domain, and the propagation of such oscillations will now be under discussion. It is worth noting that density oscillations can be regarded as sound waves, studied in acoustics. Rapid oscillations appear if the wavelength of the sound is small compared to the diameter of the flow domain.

Weak limits

Let $(\mathbf{u}_\epsilon, \varrho_\epsilon)$ be solutions to Problem (**N-S**). After passing to a subsequence, we can assume that

$$\begin{aligned}
\mathbf{u}_\epsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\\
\varrho_\epsilon &\rightharpoonup \overline{\varrho} && \text{weakly}^\star \text{ in } L^\infty(0,T;L^\gamma(\Omega)),\\
\varrho_\epsilon &\rightharpoonup \overline{\varrho} && \text{weakly in } L^{\gamma+\theta}(Q') \qquad \text{for all } Q' \Subset Q,\\
p(\varrho_\epsilon) &\rightharpoonup \overline{p} && \text{weakly in } L^{1+\theta/\gamma}(Q') \quad \text{for all } Q' \Subset Q
\end{aligned}$$

Weak Limits

The limits satisfy the equations

$$\partial_t(\overline{\varrho}\,\mathbf{u}) + \mathrm{div}(\overline{\varrho}\,\mathbf{u}\otimes\mathbf{u}) + \nabla\overline{p}$$
$$= \operatorname{div}\mathbb{S}(\mathbf{u}) + \overline{\varrho}\,\mathbf{f} \quad \text{in } Q,$$

$$\partial_t\overline{\varrho} + \mathrm{div}(\overline{\varrho}\mathbf{u}) = 0 \quad \text{in } Q,$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } S_T,$$
$$\overline{\varrho} = \overline{\varrho}_\infty \quad \text{on } \Sigma_{\mathrm{in}},$$
$$\mathbf{u}(x,0) = \mathbf{U}(x,0) \quad \text{in } \Omega,$$
$$\overline{\varrho}(x,0) = \varrho_\infty(x,0) \quad \text{in } \Omega,$$

Problem

Now we establish the connection between $\overline{\varrho}$ and $\overline{p}$.
The main ingredients of our method are the kinetic equation method (Lions, Perthame, Tadmor 1994) and the Young measures theory (Tartar 1979).

There exist subsequences, still denoted by ϱ_ε, ϱ^ϵ_∞, and the Young measures $\mu \in L^\infty_w(Q; \mathcal{M}(\mathbb{R}))$, $\mu^\infty \in L^\infty_w(\sqcup_T; \mathcal{M}(\mathbb{R}))$, with the following properties:

- For any continuous function $\varphi \in C_0(\mathbb{R})$,

$$\begin{aligned} \varphi(\varrho_\epsilon) &\rightharpoonup \overline{\varphi} \quad \text{weakly}^\star \text{ in } L^\infty(Q), \\ \varphi(\varrho^\epsilon_\infty) &\rightharpoonup \overline{\varphi}_\infty \quad \text{weakly}^\star \text{ in } L^\infty(\sqcup_T), \end{aligned}$$

- $$\overline{\varphi}(x,t) = \langle \mu_{xt}, \varphi \rangle, \quad \overline{\varphi}_\infty(x,t) = \langle \mu^\infty_{xt}, \varphi \rangle.$$

In this framework rapidly oscillating sequences are associated with some Young measures. At this point it is worth noting that a Young measure determines a random function (depending on the spatial variables and the time variable).

Definition. Let $(\mathcal{E}, \mathcal{A}, \pi)$ be a probability space, i.e., $\mathcal{A}$ is a σ-algebra on the set $\mathcal{E}$ and $\pi : \mathcal{A} \to \mathbb{R}$ is a probability measure, $\pi(\mathcal{E}) = 1$. A *random variable* is a Borel map $\varsigma : \mathcal{E} \to \mathbb{R}$. Recall that ς is *Borel* if $\varsigma^{-1}(B) \in \mathcal{A}$ for any Borel set $B \subset \mathbb{R}$. The *probability distribution* of the random variable ς is the probability measure μ on the real line defined by $\mu(B) = \pi(\varsigma^{-1}(B))$ for all Borel sets $B \subset \mathbb{R}$. The *distribution function* (cumulative distribution function) of ς is defined by $f(\lambda) = \mu(-\infty, \lambda]$, $\lambda \in \mathbb{R}$. A family of random variables ς_{xt} labeled by points $(x, t) \in Q$ is called a *random function* (random field) on Q.

Examples of distribution function

For

$$u_\varepsilon = \sin\Big(\frac{x_1}{\epsilon}\Big),$$

there is $f(s) = H(s)$, where

$$H(s) = 0 \;\text{ for }\; s < -1, \quad H(s) = 1 \;\text{ for }\; s > 1$$

$$H(s) = \frac{1}{\pi}\int_{-1}^{s}(1 - z^2)^{-1/2}\,dz \;\text{ for }\; -1 \le s \le 1.$$

For

$$u_\epsilon = \sin\Big(\frac{\mathbf{k}\cdot x - \omega t}{\epsilon}\Big)$$

there is $f(x, t, s) = H(s)$. Finally, for

$$u_\epsilon = A(x, t)\sin\Big(\frac{\mathbf{k}\cdot x - \omega t}{\epsilon}\Big)$$

there is $f(x, t, s) = H(s/A(x, t))$.

Distribution function

We can consider rapidly oscillating sequences ϱ_ϵ and ϱ_∞^ϵ as random functions on the cylinder Q and on $\sqcup_T$ with the associated Young measures μ_{xt} and μ_{xt}^∞, respectively. Introduce the corresponding cumulative distribution functions

$$f(x,t,s) = \mu_{xt}(-\infty, s], \quad f_\infty(x,t,s) = \mu_{xt}^\infty(-\infty, s].$$

For a.e. $(x,t) \in Q$ (resp. $(x,t) \in \sqcup_T$), the functions $f(x,t,s)$ and $f_\infty(x,t,s)$ are monotone and right continuous in s. They tends to 1 as $s \to \infty$ and vanish for $s < 0$.

$$\langle \mu_{xt}, \varphi \rangle = \int_{\mathbb{R}} \varphi(s)\, d_s f(x,t,s), \quad \langle \mu_{xt}^\infty, \varphi \rangle = \int_{\mathbb{R}} \varphi(s)\, d_s f_\infty(x,t,s).$$

In particular,

$$\overline{\varrho} = \int_{\mathbb{R}} s\, d_s f(x,t,s), \quad \overline{p} = \int_{\mathbb{R}} p(s)\, d_s f(x,t,s).$$

Now, the so-called *kinetic equation* for the cumulative probability distribution f is derived.

Kinetic Equation

Theorem *For a given distribution function* $f_\infty : \sqcup_T \times \mathbb{R} \to [0, 1]$, *the function* f *satisfies the kinetic equation*

$$\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u}) - \partial_s(sf\,\mathrm{div}\,\mathbf{u}) - \partial_s(s\mathcal{C}[f]) = 0 \quad \textit{in } Q \times \mathbb{R},$$

with the nonlinear operator $\mathcal{C}[f]$

$$\mathcal{C}[f] = \frac{1}{\lambda + 1} \int_{(-\infty,s]} (p(\tau) - \overline{p})\, d_\tau f(x, t, \tau),$$

and the boundary conditions

$$f = f_\infty \quad \text{on } \Sigma_{\mathrm{in}} \times \mathbb{R},$$

$$f(x, 0, s) = f_\infty(x, 0, s) \quad \text{on } \Omega \times \mathbb{R}.$$

The functions $\overline{\varrho}$ and $\overline{p}$ are given by

$$\overline{\varrho} = \int_{\mathbb{R}} s\, d_s f(x, t, s), \quad \overline{p} = \int_{\mathbb{R}} p(s)\, d_s f(x, t, s).$$

Remark on kinetic equation

- Kinetic equations for the Young measure distribution function contain some unknown measures in the right hand sides, thus it is not closed (see Perthame 2002). In the case of compressible **N-S** equations the kinetic equation is obtained in the closed form.
- Kinetic equation is nonlocal and nonlinear
- No memory effect.

Equations

$$\partial_t(\overline{\varrho}\,\mathbf{u}) + \mathrm{div}(\overline{\varrho}\,\mathbf{u}\otimes\mathbf{u}) + \nabla\overline{p} = \mathrm{div}\,\mathbb{S}(\mathbf{u}) + \overline{\varrho}\,\mathbf{f} \quad \text{in } Q,$$

$$\partial_t\overline{\varrho} + \mathrm{div}(\overline{\varrho}\mathbf{u}) = 0 \quad \text{in } Q,$$

$$\frac{\partial f}{\partial t} + \mathrm{div}(f\mathbf{u}) - \partial_s(sf\,\mathrm{div}\,\mathbf{u}) - \partial_s(s\,\mathcal{C}[f]) = 0 \quad \text{in } Q\times\mathbb{R}$$

$$\overline{\varrho} = \int_{\mathbb{R}} s\,d_sf(x,t,s), \quad \overline{p} = \int_{\mathbb{R}} p(s)\,d_sf(x,t,s).$$

Lemma. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *and* $\Phi(s) = \varphi'(s)s - \varphi(s)$. *Then*

$$\int_Q \overline{\varphi}(\partial_t\psi + \nabla\psi \cdot \mathbf{u})\,dxdt - \frac{1}{\lambda+1}\int_Q \psi\big(\overline{\Phi p} - \overline{\Phi}\,\overline{p}\big)\,dxdt$$
$$-\int_Q \psi\overline{\Phi}\,\mathrm{div}\,\mathbf{u}\,dxdt + \int_\Omega (\psi\overline{\varphi}_\infty)(x,0)\,dx - \int_{S_T} \psi\overline{\varphi}_\infty \mathbf{U}\cdot\mathbf{n}\,dSdt = 0$$

for all $\psi \in C^\infty(Q)$ *vanishing in a neighborhood of* $S_T \setminus \Sigma_{\mathrm{in}}$ *and of* $\Omega \times \{t = T\}$. *Here*

$$\overline{\varphi} = \int_{\mathbb{R}} \varphi(s)\,d_s f(x,t,s), \quad \overline{p} = \int_{\mathbb{R}} p(s)\,d_s f(x,t,s),$$
$$\overline{\varphi}_\infty = \int_{\mathbb{R}} \varphi(s)\,d_s f_\infty(x,t,s),$$
$$\overline{\Phi p} = \int_{\mathbb{R}} \Phi(s)p(s)\,d_s f(x,t,s), \quad \overline{\Phi} = \int_{\mathbb{R}} \Phi(s)\,d_s f(x,t,s).$$

Proof

We have

$$\int_Q \left(\varphi(\varrho_\epsilon)\partial_t\psi + \varphi(\varrho_\epsilon)\mathbf{u}_\epsilon \cdot \nabla\psi + \psi(\varphi(\varrho_\epsilon) - \varphi'(\varrho_\epsilon)\varrho_\epsilon)\,\mathrm{div}\,\mathbf{u}_\epsilon\right) dxdt$$
$$= \int_{\Sigma_{\mathrm{in}}} \psi\varphi(\varrho_\infty^\epsilon)\mathbf{U}\cdot\mathbf{n}\, d\Sigma - \int_\Omega (\varphi(\varrho_\infty^\epsilon)\psi)(x,0)\, dx$$

Letting $\epsilon \to 0$ and using the weak continuity of the viscous flux we obtain the desired identity.

The sequence ϱ_∞^ϵ converges to $\overline{\varrho}_\infty$ a.e. in $\sqcup_T$ if and only if

$$f_\infty(x,t,s) = 0 \ \text{ for } \ s < \overline{\varrho}_\infty(x,t),$$
$$f_\infty(x,t,s) = 1 \ \text{ for } \ s \geq \overline{\varrho}_\infty(x,t),$$

Definition *A Young measure (random function) μ_{xt} is deterministic if for a.e. (x,t) the measure μ_{xt} is concentrated at a single point, i.e. there is $\varrho : Q \to \mathbb{R}$ such that*

$$f(x,t,s) = 0 \quad \textit{for} \ \ s < \varrho(x,t),$$
$$f(x,t,s) = 1 \quad \textit{for} \ \ s \geq \varrho(x,t).$$

The Young measure is deterministic if and only if the sequence ϱ_ϵ converges strongly.

Noisiness

A *noisiness* $\mathcal{N}(t)$ of a distribution function f at the moment t is defined by

$$\mathcal{N}(t) = \int_{\Omega\times\mathbb{R}} f(x,t,s)(1 - f(x,t,s))\, dsdx.$$

$\mathcal{N}(t)$ is nonnegative. It vanishes if an only if the distribution function $f(\cdot, t, \cdot)$ is deterministic.

Question

The next question concerns the deterministic case. If the boundary data f_∞ is deterministic, is a solution to the kinetic equation deterministic?

This question is important because if f is deterministic, then obviously $\overline{p} = p(\overline{\varrho})$ and a solution of the kinetic equation becomes a weak renormalized solution of the mass balance equation.

Main Theorem

The theory of the kinetic equation is of independent interest aside from the theory of Navier-Stokes equations itself.

The following theorem on kinetic equations with deterministic data makes it possible, among other things, to prove compactness properties of solutions to compressible Navier-Stokes equations and to investigate the domain dependence of solutions to these equations.

Kinetic Problem

Let us consider the boundary value problem for the kinetic equation

Problem K.

$$\partial_t f + \operatorname{div}(f\mathbf{u}) - \partial_s(sf \operatorname{div}\mathbf{u} + s\mathcal{C}[f]) = 0 \quad \text{in } Q \times \mathbb{R},$$
$$f = f_\infty \quad \text{on } \Sigma_{\text{in}} \times \mathbb{R}, \quad f(x,0,s) = f_\infty(x,0,s) \quad \text{on } \Omega \times \mathbb{R}\,.$$

We emphasize that here **u** is a given vector field that has nothing to do with Navier-Stokes equations.

Assumptions

- The vector field $\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega))$ satisfies the boundary condition $\mathbf{u} = \mathbf{U}$ on $\sqcup_T$.
- Functions f and f_∞ are monotone and right continuous in s. Moreover, they tends to 1 as $s \to \infty$ and vanish for $s < 0$.
- $$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \Big\{ \int_{[0,\infty)} s^\gamma \, d_s f(x,t,s) \Big\} \, dx < \infty.$$
- $$\|\mathcal{H}\|_{L^{1+\gamma}(Q)} + \sup_{v\in\mathbb{R}^+} \|\mathcal{V}_v\|_{L^1(Q)} < \infty,$$

$$\mathcal{V}_v(x,t) = \int_{[0,\infty)} \min\{s, v\}(p(s) - \overline{p}) \, d_s f(x,t,s),$$

$$H(x,t) = \int_{[0,\infty)} f(x,t,s)(1 - f(x,t,s)) \, ds.$$

Result

Theorem 2.

*Let Condition (**A**) be satisfied, f_∞ be deterministic and f be a solution to Problem **K**.*

Then f is deterministic, i.e. there is $\varrho \in L^\infty(0, T; L^\gamma(\Omega))$ such that

$$f(x,t,s) = 0 \quad \text{for } s < \varrho(x,t), \qquad f(x,t,s) = 1 \quad \text{for } s \geq \varrho(x,t).$$

Renormalization

The renormalization procedure was introduced by Di Perna & Lions (1991). Our case corresponds (in its local part) to Le Bris&Lions (2004). The main idea of the proof is a renormalization of Problem **K**. In other words we intend to derive an equation for a composite function $\Psi(f)$. We choose Ψ in such a way that it is concave and $\Psi(f)$ vanishes for every deterministic distribution function f. The simplest choice is

$$\Psi(f) = f(1 - f), \quad \Psi'(f) = 1 - 2f. \tag{6}$$

Because of the presence of nonlocal terms and delicate operations with Stiltjies integrals, the justification of renormalization procedure is complicated. We present the result for the case of $\mathbf{U} = 0$ (no-slip condition).

Proposition *Let* $\Psi = f(1-f)$. *Then for any* $\psi \in C^\infty(\mathbb{R}^d \times (0,T) \times \mathbb{R})$ *vanishing for* $t = T$ *and for sufficiently large* $|s|$, *we have*

$$\int_{Q\times\mathbb{R}} \{\Psi(f)(\partial_t\psi + \nabla\psi\cdot\mathbf{u} - s\partial_s\psi\ \mathrm{div}\,\mathbf{u}) - s\partial_s\psi\,\Psi'(f)\,\mathcal{C}[f]\}\,dxdtds$$

$$+2\int_{Q\times\mathbb{R}} s\psi\mathfrak{M}\,d_s f\,dxdt + \int\limits_{\Omega\times\mathbb{R}} \Psi(f_\infty(x,0,s))\psi(x,0,s)\,dxds = 0.$$

Here

$$\mathfrak{M}(x,t,s) = \frac{1}{2}\lim_{h\searrow 0}\mathcal{C}[f](x,t,s-h) + \frac{1}{2}\lim_{h\searrow 0}\mathcal{C}[f](x,t,s+h). \quad (7)$$

Corollary

The inequality

$$\int_{Q\times\mathbb{R}} \{\Psi(f)(\partial_t\Phi - s\partial_s\Phi\,\mathrm{div}\,\mathbf{u}) - s\partial_s\Phi\,\Psi'(f)\,\mathcal{C}[f]\}\,dxdtds \\ + 2\int_Q \Big\{\int_{\mathbb{R}} s\Phi\mathfrak{M}\,d_s f\Big\}\,dxdt \geq 0 \quad (8)$$

holds for any nonnegative function $\Phi \in C^\infty((0,T)\times\mathbb{R})$ vanishing for $t = T$ and for all large $|s|$.

Lemma

$$\mathfrak{M}(x,t,s) \leq 0.$$

Conclusion

$$\frac{\partial \mathcal{N}}{\partial t} \leq -2 \int_{\Omega \times \mathbb{R}} s|\mathfrak{M}|\, d_s f dx$$

Recall

$$\mathfrak{M}(x,t,s) = \frac{1}{2} \lim_{h \searrow 0} \mathcal{C}[f](x,t,s-h) + \frac{1}{2} \lim_{h \searrow 0} \mathcal{C}[f](x,t,s+h).$$

$$\mathcal{C}[f] = \frac{1}{\lambda + 1} \int_{(-\infty, s]} (p(\tau) - \overline{p})\, d_\tau f(x,t,\tau),$$

Conclusion

If

- $\mathbf{u}_\epsilon$ are uniformly bounded in $L^2(0, T; W^{1,2}(\Omega))$,
- ϱ_ϵ are bounded from above and are separated from zero,
- the magnitude of oscillations is separated from zero,

then

$$\mathcal{N}(t) \leq c \exp\left(-\frac{c}{\nu} t\right),$$

where ν is a gas viscosity.

Reference

The proofs of presented results are given in Chapter 7 of the monograph:

Plotnikov P.I., Sokolowski J.

Compressible Navier-Stokes equations.
Theory and shape optimization.

Monografie Matematyczne, Instytut Matematyczny Polskiej Akademii Nauk, New Series **73**, Birkhäuser Springer Basel AG, Basel, 2012.