Target problems, Second order BSDEs, and probabilistic numerical methods for fully nonlinear PDEs

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Outline

The standard model in frictionless markets

• $(\Omega, \mathcal{F}, \mathbb{P})$, W Brownian motion in $\mathbb{R}^d$, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\} = \mathbb{F}^W$
• The financial market consists of a riskless asset $S^0 \equiv 1$, and a risky asset with price process

$$dS_t = \operatorname{diag}[S_t]\left(\mu_t dt + \sigma_t dW_t\right)$$

μ, σ adapted, σ invertible $+ \ldots$
• Portfolio Z^i_t : amount invested in asset i time t :

$$\{Z_t, t \geq 0\} \qquad \mathbb{F}-\text{adapted with values in } \mathbb{R}^d$$

• Self-financing condition $\Longrightarrow$ dynamics of portfolio value :

$$dY_t = Z_t \cdot \operatorname{diag}[S_t]^{-1} dS_t$$

Super-hedging problem of $\mathcal{F}_T-$measurable $G \geq 0$

$$V_0 := \inf\{Y_0 : \ Y_T \geq G \text{ a.s. for some } Z \in \mathcal{A}\}$$

Solution : the Black-Scholes model

- We may assume $\mu \equiv 0$: equivalent change of measure
- Then for $Y_0 > V_0$, $\mathbb{E}[Y_T] \geq \mathbb{E}[G] \Longrightarrow V_0 \geq \mathbb{E}[G]$
- From the martingale representation in Brownian filtration

$$\hat{Y}_t := \mathbb{E}[G|\mathcal{F}_t] = \mathbb{E}[G] + \int_0^T \phi_t \cdot dW_t = \hat{Y}_0 + \int_0^T \hat{Z}_t \cdot \sigma_t dW_t$$

Since $Y_T = G$, we deduce that $\mathbb{E}[G] \geq V_0$

Hence $V_0 = \mathbb{E}[G]$ and $Y_T = G$ a.s. for some portfolio $Z \in \mathcal{A}$

Stochastic target problems

- Controlled process

$$dX_t = \mu(t, X_t, \nu_t)dt + \sigma(t, X_t, \nu_t)dW_t$$

where the control process $\nu \in \mathcal{U}$ takes values in $U \subset \mathbb{R}^k$

- Given a Borel set $\Gamma_0 \subset \mathbb{R}^d$, find

$$\mathcal{V}_0 := \left\{X_0 \in \mathbb{R}^d : X_T \in \Gamma_0 \text{ for some } \nu \in \mathcal{U}\right\}$$

- If $X = (S, Y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ where Y is increasing in Y_0, find

$$V_0 := \inf\{Y_0 : X_T = (S_T, Y_T) \in \Gamma_0 \text{ for some } \nu \in \mathcal{U}\}$$

Main ingredient for target problems

- Define the dynamic problems $\mathcal{V}_t$ and V_t

Geometric Dynamic Programming for any stopping time θ valued in $[t, T]$

$$\mathcal{V}_t = \{X_t : X_\theta \in \mathcal{V}_\theta \text{ for some } \nu \in \mathcal{U}\}$$

if Y is increasing in Y_0 :

Geometric Dynamic Programming for any stopping time θ valued in $[t, T]$

$$V_t = \inf \{Y_t : Y_\theta \geq V_\theta \text{ for some } \nu \in \mathcal{U}\}$$

Dynamic Programming Equation for V

• If $U = \mathbb{R}^k$. Assume that V is locally bounded. Then $V(t,s)$ is a (discontinuous) viscosity solution of

$$-\frac{\partial V}{\partial t}(t,s) - \mathcal{L}^{\nu_0(t,s)} V(t,s) + \mu^Y\left(t,s,V(t,s),\nu_0(t,s)\right) = 0$$

where

$$\mathcal{L}^{\nu} V(t,s) = \mu^S\left(t,s,V(t,s),\nu\right) \cdot DV(t,s) + \frac{1}{2}\mathrm{Tr}\left[\sigma^S \sigma^{S^*} D^2 V(t,s)\right]$$

and

$$\sigma^Y\left(t,s,V(t,s),\nu_0(t,s)\right) = \sigma^S\left(t,s,V(t,s),\nu_0(t,s)\right)\ DV(t,s)$$

• If $U \neq \mathbb{R}^k$: similar PDE, with gradient constraint, boundary layer...

Dynamic Programming equation for $\mathcal{V}$

Set $u(t,x) := \mathbb{1}_{\mathcal{V}(t)^c}(x)$

Theorem Under some conditions, u is a (discontinuous) viscosity solution of the geometric equation

$$-\frac{\partial v}{\partial t}(t,x) + F\left(t,x,Dv(t,x),D^2v(t,x)\right) = 0$$

where

$$F(t,x,p,A) = \sup\left\{\mu(t,x,\nu)\cdot p + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(t,x,\nu)A\right] : \nu \in \mathcal{N}(t,x,p)\right\}$$

and

$$\mathcal{N}(t,x,p) := \left\{\nu \in U : \ \sigma(t,x,\nu)^T p = 0\right\}$$

$\Longrightarrow$ Stochastic representation for a class of geometric equations (exp : mean curvature flow)

Quantile target problems

- Controlling the probability of reaching the target :

$$V(t,s,p) \quad := \quad \inf\left\{y : \mathbb{P}\left[(S,Y)_T \in \Gamma_0\right] \geq p \text{ for some } \nu \in \mathcal{U}\right\}$$

- Introduce an additional controlled state :

$$dP_t \quad = \quad \alpha_t \cdot dW_t$$

Then

$$V(t,s,p) \quad := \quad \inf\left\{y : \mathbb{1}_{(S,Y)_T \in \Gamma_0} - P_T \geq 0 \text{ for some } (\alpha,\nu) \in \overline{\mathcal{U}}\right\}$$

thus converting the quantile target problem into a target problem

- $V(T,s,p)$!! <Bouchard, Elie, T.>

Hedging under liquidity costs (1)

<Çetin, Jarrow and Protter 2004, 2006>

- Risky asset price is defined by a supply curve :

$$\mathbf{S}\left(S_t, \nu\right) \quad : \qquad \text{price per share of } \nu \text{ risky assets}$$

$\mathbf{S}\left(S_t, 0\right) = S_t$

- X_t : holdings in cash, Z_t : holdings in risky asset (number of shares)

$$X_{t+dt} - X_t \; + \; \left(Z_{t+dt} - Z_t\right) \mathbf{S}\left(S_t, Z_{t+dt} - Z_t\right) \;\; = \;\; 0$$

$$\begin{aligned} \Longrightarrow \quad X_T \;\; &= \;\; X_0 - \sum \left(Z_{t+dt} - Z_t\right) \mathbf{S}\left(S_t, Z_{t+dt} - Z_t\right) \\ &= \;\; X_0 + \sum Z_t \left(S_t - S_{t+dt}\right) + \ldots . \end{aligned}$$

Hedging under liquidity costs (2)

Direct computation leads to

$$Y_T := X_T + Z_T S_T = Y_0 + \sum Z_t \left(S_{t+dt} - S_t\right) - \sum \left(Z_{t+dt} - Z_t\right) \left[\mathbf{S}\left(S_t, Z_{t+dt} - Z_t\right) - S_t\right]$$

Assume $\nu \longmapsto \mathbf{S}(S_t, \nu)$ is smooth, then :

$$Y_T = Y_0 + \int_0^T Z_t dS_t - \int_0^T \frac{\partial \mathbf{S}}{\partial \nu}(S_t, 0)\, d < Z^c >_t - \sum_{t \le T} \Delta Z_t \left[\mathbf{S}\left(S_t, \Delta Z_t\right) - S_t\right]$$

Super-hedging problem

$$V_0 := \inf\{y : Y_T \geq g(S_T) \text{ a.s. for some } Y \in \mathcal{A}\}$$

Second order target problems

- The controlled state is defined by

$$dY_t = f(t, S_t, Y_t, Z_t, \Gamma_t)\,dt + Z_t \cdot dS_t$$

and the control Z satisfies the dynamics

$$dZ_t = dA_t + \Gamma_t dS_t$$

- Given a function g, find

$$V_0 := \inf\{y : Y_T \geq g(S_T) \text{ for some } Z \in \mathcal{A}\}$$

Theorem $V(t,s)$ is a (discontinuous) viscosity solution of

$$-\frac{\partial V}{\partial t} - \mathcal{L}^S V(t,s) - \hat{f}\left(t, s, V(t,s), DV(t,s), D^2V(t,s)\right) = 0$$

where $\hat{f}(t,s,r,p,A) := \sup_{\beta \geq 0} f(t,s,r,p,A+\beta)$ (elliptic envelope)

Outline

Backward SDE : Definition

Find an $\mathbb{F}^W$−**adapted** (Y, Z) satisfying :

$$Y_t = G + \int_t^T F_r(Y_r, Z_r)dr - \int_t^T Z_r \cdot dW_r$$

$$\text{i.e.} \quad dY_t = -F_t(Y_t, Z_t)dt + Z_t \cdot dW_t \quad \text{and} \quad Y_T = G$$

where the generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, and

$$\{F_t(y, z),\ t \in [0, T]\} \text{ is } \mathbb{F}^W - \text{adapted}$$

If F is Lipschitz in (y, z) uniformly in (ω, t), and $G \in \mathbb{L}^2(\mathbb{P})$, then there is a unique solution satisfying

$$\mathbb{E} \sup_{t \le T} |Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt < \infty$$

Markov BSDE's

Let $X_.$ be defined by the (forward) SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

and $$F_t(y,z) = f(t, X_t, y, z), \; f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$G = g(X_T) \in \mathbb{L}^2(\mathbb{P}), \quad g : \mathbb{R}^d \longrightarrow \mathbb{R}$$

If f continuous, Lipschitz in (x, y, z) uniformly in t, then there is a unique solution to the BSDE

$$dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t \cdot \sigma(t, X_t)dW_t, \qquad Y_T = g(X_T)$$

Moreover, there exists a measurable function V :

$$Y_t = V(t, X_t), \qquad 0 \le t \le T$$

BSDE's and semilinear PDE's

- By definition,

$$\begin{aligned} Y_{t+h} - Y_t &= V(t+h, X_{t+h}) - V(t, X_t) \\ &= -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

- If $V(t,x)$ is smooth, it follows from Itô's formula that :

$$\begin{aligned} \int_t^{t+h} \mathcal{L}V(r, X_r)dr + \int_t^{t+h} DV(r, X_r) \cdot \sigma(X_r)dW_r \\ = -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

where $\mathcal{L}$ is the Dynkin operator associated to X :

$$\mathcal{L}V = V_t + b \cdot DV + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$$

Stochastic representation of solutions of a semilinear PDE

• Under some conditions, the semilinear PDE

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^T(x)D^2V(t,x)\right] - f\left(x, V(t,x), DV(t,x)\right) = 0$$
$$V(T,x) = g(x)$$

has a unique solution which can be represented as $V(t,x) = Y_t^{t,x}$ where $Y^{t,x}$ solves the BSDE

$$Y_T = g(X_T), \qquad dY_s = -f(X_s, Y_s, Z_s)ds + Z_s \cdot \sigma(X_s)dW_s$$
$$X_t = x, \qquad dX_s = \sigma(X_s)dW_s, \; t \le s \le T$$

• Extension to semilinear PDEs with obstacle is available by introducing Reflected BSDEs

• For $f \equiv 0$, we recover the Feynman-Kac formula

Second order BSDEs : Definition

$$\hat{f}(x, y, z, \gamma) := f(x, y, z, \gamma) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(x)\gamma]$$ non-decreasing in γ

Consider the 2nd order BSDE :

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t \\ dY_t &= -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z_t\sigma(X_t)dW_t\,, \quad Y_T = g(X_T) \\ dZ_t &= \alpha_t\, dt + \Gamma_t\, \sigma(X_t)dW_t \end{aligned}$$

A solution of (2BSDE) is

a process (Y, Z, α, Γ) with values in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$

Question : existence ? uniqueness ? in which class ?
<Cheridito, Soner, Touzi and Victoir CPAM 2007>

Second order BSDEs : Main technical tool

(i) Suppose a solution exists with $Y_t = V(t, X_t)$, then

$$
\begin{aligned}
Y_{t+h} - Y_t &= V(t+h, X_{t+h}) - V(t, X_t) \\
&= -\int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r)dr + \int_t^s Z_r \cdot dW_r \\
&= -\int_t^{t+h} f(X_r, Y_r, Z_r, \Gamma_r)dr \\
&\quad + \int_t^{t+h} \left(Z_t + \int_t^r \alpha_u du + \int_t^r \Gamma_u dW_u \right) \cdot dW_r
\end{aligned}
$$

($\sigma(.) =$ Identity matrix for simplification)

(ii) $2\times$ Itô's formula to V, identify terms of different orders

$\Longrightarrow$ Need short time asymptotics of double stochastic integrals

$$\int_0^t \int_0^r b_u \, dW_u \cdot dW_r \ , \quad t \geq 0$$

Second order BSDE : Uniqueness Assumptions

Assumption (f) $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d(\mathbb{R})\longrightarrow\mathbb{R}$ *continuous, Lipschitz in* y *uniformly in* (t,x,z,γ)*, and for some* $C,p>0$ *:*

$$|f(t,x,y,z)| \;\le\; C\;(1+|y|+|x|^p+|z|^p+|\gamma|^p)$$

Assumption (Comp) *If* w *(resp.* u*) :* $[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}$ *is a l.s.c. (resp. u.s.c.) viscosity supersolution (resp.subsolution) of (E) with*

$$w(t,x)\;\ge\;-C(1+|x|^p)\,,\quad\text{and}\quad u(t,x)\;\le\;C(1+|x|^p)$$

then $w(T,.)\ge u(T,.)$ *implies that* $w\ge u$ *on* $[0,T]\times\mathbb{R}^d$

Second order BSDE : Class of solutions

Let $\mathcal{A}^m_{t,x}$ be the class of all processes Z of the form

$$Z_s = z + \int_t^s \alpha_r \, dr + \int_t^s \Gamma_r \, dX^{t,x}_r , \quad s \in [t, T]$$

where $z \in \mathbb{R}^d$, α and Γ are respectively $\mathbb{R}^d$ and $\mathcal{S}_d(\mathbb{R}^d)$ progressively measurable processes with

$$\max\left\{|Z_s|, \|\alpha\|_b, |\Gamma_s|\right\} \le m\left(1 + \left|X^{t,x}_s\right|^p\right) ,$$

$$|\Gamma_r - \Gamma_s| \le m\left(1 + \left|X^{t,x}_r\right|^p + \left|X^{t,x}_s\right|^p\right)\left(|r-s| + \left|X^{t,x}_r - X^{t,x}_s\right|\right)$$

We shall look for a solution (Y, Z, α, Γ) of (2BSDE) such that

$$Z \in \mathcal{A}_{t,x} := \cup_{m\ge 0}\mathcal{A}^m_{t,x}$$

Scond Order BSDE : The Uniqueness Result

Theorem *Suppose that the nonlinear PDE (E) satisfies the comparison Assumption* Com*. Then, under Assumption* (f)*, for every* g *with polynomial growth, there is at most one solution to (2BSDE) with*

$$Z \in \mathcal{A}_{t,x}$$

2BSDE : Idea of proof of uniqueness

Define the stochastic target problems

$$V(t,x) := \inf\left\{y \;:\; Y_T^{t,y,Z} \geq g\left(X_T^{t,x}\right) \text{ a.s. for some } Z \in \mathcal{A}_{t,x}\right\}$$

(Seller super-replication cost in finance), and

$$U(t,x) := \sup\left\{y \;:\; Y_T^{t,y,Z} \leq g\left(X_T^{t,x}\right) \text{ a.s. for some } Z \in \mathcal{A}_{t,x}\right\}$$

(Buyer super-replication cost in finance)

- By definition : $V(t,X_t) \leq Y_t \leq U(t,X_t)$ for every solution (Y,Z,α,Γ) of (2BSDE) with $Z \in \mathcal{A}_{0,x}$
- Main technical result : V is a (discontinuous) viscosity super-solution of the nonlinear PDE (E)

$\Longrightarrow$ U is a (discontinuous) viscosity subsolution of (E)

- Assumption Com $\Longrightarrow$ $V \geq U$

Second order BSDE : Existence

• Consider the fully nonlinear PDE (with $\mathcal{L}V = V_t + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$)

$$
(\mathrm{E}) \quad \begin{aligned} -\mathcal{L}v(t,x) - f\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) &= 0 \\ v(T,x) = g(x) & \end{aligned}
$$

• If (E) has a smooth solution, then

$$
\begin{aligned} \bar{Y}_t &= v(t,X_t), & \bar{Z}_t &:= Dv(t,X_t), \\ \bar{\alpha}_t &:= \mathcal{L}Dv(t,X_t), & \bar{\Gamma}_t &:= V_{xx}(t,X_t) \end{aligned}
$$

is a solution of (2BSDE), immediate application of Itô's formula

• Existence is an open problem, is there a weak theory of existence ? ?

Outline

Discrete-time approximation of BSDEs

<Bally-Pagès SPA03, Zhang AAP04, Bouchard-Touzi SPA04>

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods
Start from Euler discretization : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ is given, and

$$Y^n_{t_{i+1}} - Y^n_{t_i} = -f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i + Z^n_{t_i} \cdot \sigma\left(X^n_{t_i}\right)\Delta W_{t_{i+1}}$$

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$$\mathbb{E}_i^n[\qquad \longrightarrow Y^n_{t_{i+1}} - Y^n_{t_i} = -f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i + Z^n_{t_i} \cdot \sigma\left(X^n_{t_i}\right)\Delta W_{t_{i+1}}$$

$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ and

$$Y^n_{t_i} = \mathbb{E}_i^n\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \quad , \; 0 \le i \le n-1 \; ,$$

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$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \; 0 \le i \le n-1 \\ Z^n_{t_i} &= \left(\Delta t_i\right)^{-1} \, \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}\right] \end{aligned}$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} \; = \; g\left(X^n_{t_n}\right)$ and

$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \; 0 \le i \le n-1 \\ Z^n_{t_i} &= (\Delta t_i)^{-1} \; \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}\right] \end{aligned}$$

$\Longrightarrow$ Similar to numerical computation of **American options**

Discrete-time approximation of BSDEs, continued

$$\pi : 0 = t_0 < t_1 < \ldots < t_n = T, \ |\pi| = \max_{1 \le i \le n} |t_{i+1} - t_i|$$

Theorem Assume f and g are Lipschitz. Then :

$$\limsup_{n \to \infty} n^{1/2} \left\{ \sup_{0 \le t \le 1} \| Y_t^n - Y_t \|_{\mathbb{L}^2} + \| Z^n - Z \|_{\mathbb{H}^2} \right\} < \infty$$

Theorem <Gobet-Labart 06> Under additional regularity conditions :

$$\limsup_{n \to \infty} n \| Y_0^n - Y_0 \|_{\mathbb{L}^2} \quad < \quad \infty$$

Weak error

Approximation of conditional expectations

Main observation : in our context all conditional expectations are regressions, i.e.

$$\begin{aligned} \mathbb{E}\left[Y^n_{t_{i+1}}|\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}}|X_{t_i}\right] \\ \mathbb{E}\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}|\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}|X_{t_i}\right] \end{aligned}$$

Classical methods from statistics :

• Kernel regression <Carrière>

• Projection on subspaces of $\mathbb{L}^2(\mathbb{P})$ <Longstaff-Schwarz, Gobet-Lemor-Warin AAP05>

from numerical probabilistic methods

• quantization... <Bally-Pagès SPA03>

Integration by parts <Lions-Reigner 00, Bouchard-Touzi SPA04>

Simulation of Backward SDE's

1. Simulate trajectories of the forward process X (well understood)

2. Backward algorithm :

$$\left\| \begin{array}{lcl} \hat{Y}^n_{t_n} & = & g\left(X^n_{t_n}\right) \\ \hat{Y}^n_{t_{i-1}} & = & \widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, \hat{Y}^n_{t_{i-1}}, \hat{Z}^n_{t_{i-1}}\right)\Delta t_i\ , \quad 1 \le i \le n\ , \\ \hat{Z}^n_{t_{i-1}} & = & \frac{1}{\Delta t_i}\,\widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\Delta W_{t_i}\right] \end{array} \right.$$

(truncation of $\hat{Y}^n$ and $\hat{Z}^n$ needed in order to control the $\mathbb{L}^p$ error)

Simulation of BSDEs : bound on the rate of convergence

Error estimate for the Malliavin-based algorithm, $|\pi| = n^{-1}$

Theorem For $p > 1$:

$$\limsup_{n\to\infty} \max_{0\le i\le n} n^{-1-d/(4p)} N^{1/2p} \left\| \hat{Y}^n_{t_i} - Y^n_{t_i} \right\|_{\mathbb{L}^p} < \infty$$

For the time step $\frac{1}{n}$, and limit case $p = 1$:

rate of convergence of $\frac{1}{\sqrt{n}}$
if and only if
$$n^{-1-\frac{d}{4}} N^{1/2} = n^{1/2}, \quad \text{i.e. } N = n^{3+\frac{d}{2}}$$

A probabilistic numerical scheme for fully nonlinear PDEs

By analogy with BSDE, we introduce the following discretization for 2BSDEs :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right) , \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i \ , \quad 1 \le i \le n \ , \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i} \frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i} \frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Intuition From Greeks Calculation

- First, use the approximation $f''(x) \sim_{h=0} \mathbb{E}[f''(x+W_h)]$
- Then, integration by parts shows that

$$\begin{aligned} f''(x) &\sim \int f''(x+y)\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\ &= \int f'(x+y)\frac{y}{h}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \mathbb{E}\left[f'(x+W_h)\frac{W_h}{h}\right] \\ &= \int f(x+y)\frac{y^2-h}{h^2}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \mathbb{E}\left[f(x+W_h)\left(\frac{W_h^2-h}{h^2}\right)\right] \end{aligned}$$

- Connection with Finite Differences : $W_h \sim \sqrt{h}\left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}\right)$

$$\mathbb{E}\left[\psi(x+W_h)\frac{W_h}{h}\right] \sim \frac{\psi(x+\sqrt{h})-\psi(x-\sqrt{h})}{2h}$$ Centered FD !

The Convergence Result

<Fahim and Touzi 2007>

Theorem Suppose in addition that f is Lipschitz and $\|f_\gamma\|_{\mathbb{L}^\infty} \leq \sigma$. Then

$$Y_0^n(t,x) \longrightarrow v(t,x) \qquad \text{uniformly on compacts}$$

where v is the unique viscosity solution of the nonlinear PDE.

• Proof : stability, consistency, monotonicity <Barles-Souganidis AA91>
• Bounds on the approximation error are available <Krylov, Barles-Jacobsen, Cafarelli-Souganidis>
• This convergence result is weaker than that of (first order) Backward SDEs...

Comments on the 2BSDE algorithm

• in BSDEs the drift coefficient μ of the forward SDE can be changed arbitrarily by Girsanov theorem (importance sampling...)

• in 2BSDEs both μ and σ can be changed (numerical results however recommend prudence...)

• The heat equation $v_t + v_{xx} = 0$ correspond to a BSDE with zero driver. Splitting the Laplacian in two pieces, it can also be viewed as a 2BSDE with driver $f(\gamma) = \frac{1}{2}\gamma$
$\longrightarrow$ numerical experiments show that the 2BSDE algorithm perform better than the pure finite differences scheme

Portfolio optimization (X. Warin)

With $U(x) = -e^{-\eta x}$, want to solve :

$$V(t,x) \quad := \quad \sup_{\theta} \mathbb{E}\left[U\left(x + \int_t^T \theta_u \sigma(\lambda du + dW_u)\right)\right]$$

- An explicit solution is available
- V is the characterized by the fully nonlinear PDE

$$-V_t + \frac{1}{2}\lambda^2 \frac{(V_x)^2}{V_{xx}} \;=\; 0 \quad \text{and} \quad V(T,.) = U$$

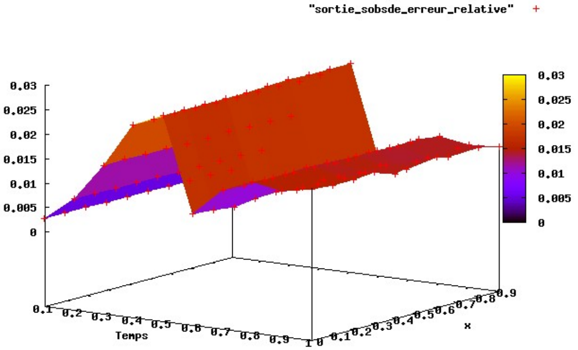

Fig.: Relative Error (Regression), dimension 1

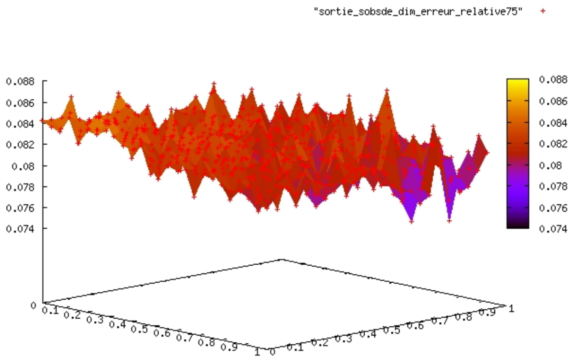

Fig.: Relative Error (Regression), dimension 2

Varying the drift of the FSDE

Drift FSDE	Relative error (Regression)
-1	0,0648429
-0,8	0,0676044
-0,6	0,0346846
-0,4	0,0243774
-0,2	0,0172359
0	0,0124126
0,2	0,00880041
0,4	0,00656142
0,6	0,00568952
0,8	0,00637239

Varying the volatility of the FSDE

Volatility FSDE	Relative error (Regression)	Relative error (Quantization)
0,2	0,581541	0,526552
0,4	0,42106	0,134675
0,6	0,0165435	0,0258884
0,8	0,0170161	0,00637319
1 0,	0124126	0,0109905
1,2	0,0211604	0,0209174
1,4	0,0360543	0,0362259
1,6	0,0656076	0,0624566