# De la théorie de Perron-Frobenius à la programmation dynamique et vice versa

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Quelques résultats empruntés à des travaux avec M. Akian, J. Cochet, J. Gunawardena, B. Lemmens, R. Nussbaum, C. Walsh,. . . illustrés par deux applications: J. Clairambault, B. Perthame; E. Goubault, S. Zennou.

# Historical background: classical Perron-Frobenius theory

Perron (1907) proved the following.

Let $A \in \mathbb{R}^{n\times n}$, with $A_{ij} > 0\ \forall i, j$. Then,

1. $\exists u \in \mathbb{R}^n$, $u_i > 0\ \forall i$, $Au = \rho(A)u$, with $\rho(A) := \max\{|\lambda| \mid \lambda$ eigenval. of $A\}$.

2. The eigenvalue $\rho(A)$ is algebraically simple, a fortiori, $u$ is unique up to a multiplicative constant.

Frobenius (1912) showed that the same is true when $A_{ij} \geq 0$, with $G := \{(i, j) \mid\ A_{ij} > 0\}$ strongly connected ($A$ irreducible).

Let $c$ be the cyclicity of $G$ (=gcd lengths of circuits), and $\omega = \exp(i2\pi/c)$.

Then,

3. the whole spectrum of $A$ is invariant by multiplication by $\omega$, and $\omega^j \rho(A)$, $j = 0, \ldots, c-1$ are the only eigenvalues of maximal modulus (all algebraically simple)

4. so $\rho_{\max}(A)^{-kc} A^{kc}$ converges as $k$ tends to $\infty$.

Kreĭn and Rutman (1948) considered more generally a linear operator $F$ leaving invariant a (closed, convex, pointed) cone $C$ in a Banach space $E$.

$C$ invariant iff $F$ preserves the order: $x \leq y$ when $y - x \in C$. For $E = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$, we recover Perron-Frobenius theory.

When $F$ is non-linear, we must assume that $x \leq y \implies F(x) \leq F(y)$, and some homogeneity condition, e.g. $F(\lambda x) = \lambda F(x)$, $\lambda > 0$.

Kreĭn and Rutman again, Krasnoselskiĭ's school, Morishima, . . . , Nussbaum (AMS memoirs, 80), . . .

applications: population dynamics / biology (monotone systems); combinatorial matrix theory, diffusion on fractals, . . .

Let's use **logarithmic glasses**, i.e. log-log plots ! (Viro).

When $E = \mathbb{R}^n$, $C = \mathbb{R}^n_+$, or $E = \mathcal{C}(K)$, $C = \mathcal{C}^+(K)$,

$$G(x) = \log(F(\exp(x))) \ .$$

$G$ is a **dynamic programming operator**:

$u \leq v \implies G(u) \leq G(v)$ (M)

$G(\mu 1 + u) = \mu 1 + G(u)$, $\mu \in \mathbb{R}$ (AH)

(M)+(AH) $\implies$ (N): $\|G(u) - G(v)\|_\infty \leq \|u - v\|_\infty$.

The importance of these axioms was recognized by several authors, including: Blackwell, . . . Crandall, Tartar (PAMS 80),. . . Neyman, Sorin, . . .

(unfortunately, logarithmic glasses seem less powerful for the cone of semidefinite positive matrices).

When $G(x) = Px$ is linear,

(M) $\implies P_{ij} \geq 0$,

(AH) $\implies \sum_j P_{ij} = 1$,

so $P$ is a Markov matrix.

In general, $G$ may be thought of as a *non-linear Markov operator*.

We may replace (AH) by (SAH):

$G(\mu 1 + u) \leq \mu 1 + G(u)$, $\mu \in \mathbb{R}_+$ (sub-Markov case modelling a termination probability).

Fundamental example: *dynamic operators of zero-sum repeated games* with state space $\{1,\ldots,n\}$ are of the form:

$$G:\mathbb{R}^n\to\mathbb{R}^n \qquad G_i(x)=\inf_{a\in A(i)}\sup_{b\in B(i,a)}(r_i^{ab}+P_i^{ab}x)$$

$P_i^{ab}:=(P_{ij}^{ab})$, proba. of moving $i\to j$, $\sum_j P_{ij}^{ab}\leq 1$.

$r_i^{ab}$: instantaneous payment of Player I to player II. Undiscounted value defined in terms of

$$\mathbb{E}\left(r_{i_1}^{a_1b_1}+\cdots+r_{i_k}^{a_kb_k}+v_{i_{k+1}}\right) \ .$$

$(G^k(v))_i$ gives the value of the game in horizon $k$ when the initial state is $i$.

Kolokoltsov has shown that any order preserving sup-norm nonexpansive map can be represented by such a game (Rubinov and Singer showed: even with deterministic transition probabilities $P_i^{ab}$).

Hamilton-Jacobi-Isaacs equations

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}), \quad v(t=0) \text{ given}$$

come from stochastic games (players control a diffusion process):

$X \preceq Y \implies H(x,p,X) \leq H(x,p,Y)$.

The evolution semigroup $S^t : v(0,\cdot) \to v(t,\cdot)$ is M AH.

Monotone (sup-norm stable) discretisation schemes yield discrete stochastic game operators, as above.

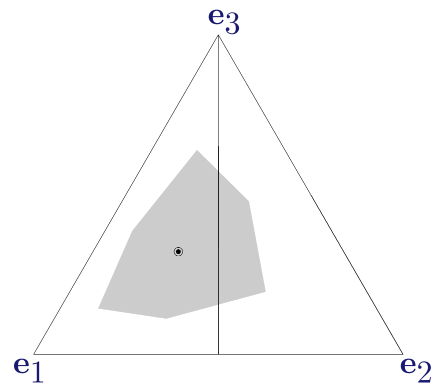
Some notions become natural with log-glasses.

(Garrett) Birkhoff (1957) approached Perron-Frobenius theory by means of Hilbert's geometry. *Hilbert's projective metric* is defined by

$$d_H(x,y) = \log\inf\{\frac{\beta}{\alpha} \mid \ \alpha y \leq x \leq \beta y,\ \alpha,\beta > 0\} \ .$$

It defines a metric on the set of rays included in the interior of $C$, i.e. on $\{\mathbb{R}_+ u \mid \ u \in \operatorname{int} C\}$, because $d(x,y) = 0$ iff $x = \beta y$. If $C$ is normal, meaning that $0 \leq x \leq y \implies \|x\| \leq \gamma \|y\|$ for some constant $\gamma$, the latter metric space is complete.

Here is the intersection of a ball in Hilbert metric with the simplex, when $C = \mathbb{R}^3_+$:




When $C = \mathbb{R}^n_+$, setting $\log(x) := (\log(x_i))_{1 \le i \le n}$,

$$d_H(x, y) = \|\log x - \log y\|_H \text{ where } \|z\|_H := \max_i z_i - \min_i z_i$$

One may also consider *Thompson's metric*:

$$d_T(x, y) := \|\log x - \log y\|_\infty$$

Birkhoff showed that if $A$ is linear preserving $C$ and if the diameter $\Delta$ of $A(C - \{0\})$ in Hilbert's metric is finite, then

$$d_H(Ax, Ay) \leq \gamma d_H(x, y) \qquad \forall x, y \in \operatorname{int} C, \qquad \gamma := \tanh(\frac{\Delta}{4}) \ .$$

Perron's theorem is a corollary.

In the non-linear case, if $F$ is M and $F(\lambda x) = \lambda F(x)$, $\lambda > 0$, then $F$ is still nonexpansive in Hilbert's and Thompson metric (it may not be contracting in Hilbert metric).

Making simple things complicated ? looking at a linear monotone population dynamics with log glasses.

$x(k+1) = Mx(k)$, $M_{ij} \geq 0$, $v(k) = \log x(k)$.

$v(k+1) = G(v(k))$

$G_i(v) = \log(\sum_j M_{ij} e^{v_j})$

$G_i$ is convex (as the log of a Laplace transform of a positive measure / by Hölder)

$G_i(v) = \sup_{p \geq 0, \sum_i p_i = 1} (p \cdot v - S_i(p; M))$

$S_i(p, M) := \sum_j p_j \log(p_j / M_{ij})$

So, $v_m(k+1) = \log x_m(k)$ is the value of a stochastic control problem in horizon $k$, with initial state $m$ (state space $\{1, \ldots, n\}$).

When in state $i$, the player choose a probability vector $p \ll M_{i\cdot}$, so that $S_i(p; M) < \infty$, she receives $-S_i(p; M)$, and moves to the next state according to the probability $p$.

The Collatz-Wielandt formula tells that

$$\begin{aligned}\rho(M) &= \inf\{\mu > 0 \mid Mx \leq \mu x,\ x \in \operatorname{int} \mathbb{R}^n_+\} \\ &= \inf_{x \in \operatorname{int} \mathbb{R}^n_+} \sup_i (Mx)_i / x_i\end{aligned}$$

We recover *Kingmans' inequality*: $\log \circ \rho \circ \exp$ is convex (the exp is entrywise, $\exp(A) = (\exp(A_{ij}))$). I.e. $\rho((A_{ij}B_{ij})^{1/2}) \leq \rho((A_{ij}))^{1/2}\rho((B_{ij}))^{1/2}$.

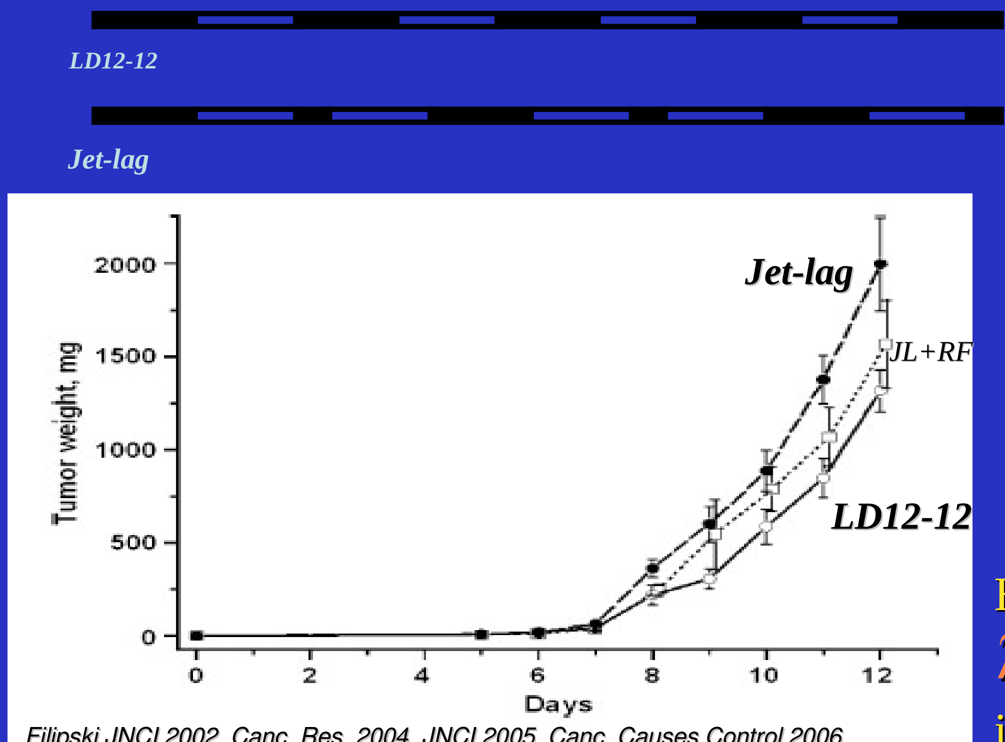
Indeed, $\log \rho(M) = \inf_{v \in \mathbb{R}^n} \sup_i \log(\sum_j e^{\log M_{ij} + v_j - v_i})$ is the marginal (minimum) of a convex function of the $v_i$, $\log M_{ij}$

This allows one to approach optimisation problems for the Perron eigenvalue, when the matrix $M$ depends of parameters.

Application: Clairambault, Perthame, SG (CRAS07): inequalities for Floquet multipliers.

# 1) A question from human and animal physiopathology: tumour growth and circadian clock disruption

*Observation*: a circadian rhythm perturbation by chronic jet-lag-like light entrainment (phase advance) enhances GOS tumour proliferation in B6D2F$_1$ mice



Filipski JNCI 2002, Canc. Res. 2004, JNCI 2005, Canc. Causes Control 2006

Here, clearly:

$\lambda$(jet-lag) > $\lambda$(LD 12-12)

if $\lambda$ is a growth exponent

How can this be accounted for in a mathematical model of tumour growth?

Major public health stake! (does shift work enhance incidence of cancer?)

*(The answer is yes, cf. e.g. Davis, S., Cancer Causes Control 2006)*

Following Clairambault, Michel, Perthame (CRAS06), we model our population of cells by a Partial Differential Equation for the density $n_i(t,x) \geq 0$ of cells with age $x$ in the phase $i = 1, \ldots, I$ at time $t$,

$$
\begin{cases}
\frac{\partial}{\partial t} n_i(t,x) + \frac{\partial}{\partial x} n_i(t,x) + [d_i(t,x) + K_{i \to i+1}(t,x)] n_i(t,x) = 0 \ , \\
n_i(t, x=0) = \displaystyle\int_{x' \geq 0} K_{i-1 \to i}(t,x') \, n_{i-1}(t,x') \, dx', \quad 2 \leq i \leq I, \\
n_1(t, x=0) = 2 \displaystyle\int_{x' \geq 0} K_{I \to 1}(t,x') \, n_I(t,x') \, dx' \ .
\end{cases}
$$

$d_i(t,x) \geq 0$ the apoptosis rate, $K_{i \to i+1}(t,x)$ the transition rates from one phase to the next. These coefficients can be constant in time or time $T$-periodic in order to take into account the circadian rhythm.

The Floquet eigenvalue $\lambda_{\text{per}}$ is such that there is a solution $n_i(t,x) = N_i(t,x) e^{\lambda_{\text{per}} t}$ with $N_i(t,x)$ $T$-periodic and $N_i > 0$.

Construct a stationnary system with some time-averaged coefficients $\bar{K}(x)$, $\bar{d}(x)$. Can we choose the averages such that $\lambda_{\mathrm{per}} \leq \lambda_s$ ?

$\lambda_s$ is such that there is a solution $n_i(t,x) = N_i(x)e^{\lambda_s t}$ with $N_i > 0$.

**Theorem** (Clairambault, SG, Perthame, CRAS07). *If we take the arithmetical mean of coefficients in infinitesimal terms, and the geometric mean of the coefficients in the integral terms, the averaged system satisfies* $\lambda_s \leq \lambda_{\mathrm{per}}$.

"Variability increases the growth rate". This is the opposite of the experimental inequality. (So the absence of circadian control cannot be explained by general convexity techniques).

$$\langle d_i(x)\rangle_a = \frac{1}{T}\int_0^T d_i(t,x)dt, \qquad \langle K_{i\to i+1}(t,x)\rangle_a = \frac{1}{T}\int_0^T K_{i\to i+1}(t,x)dt$$

$$\langle K_{i\to i+1}(t,x)\rangle_g = \exp\left(\frac{1}{T}\int_0^T \log\big(K_{i\to i+1}(t,x)\big)dt\right) \quad .$$

These averages define the Perron eigenvalue $\lambda_s \in \mathbb{R}$:

$$\frac{\partial}{\partial x}\bar{N}_i(x) + [\langle d_i(x)\rangle_a + \lambda_s + \langle K_{i\to i+1}(t,x)\rangle_a]\bar{N}_i = 0 \quad ,$$

$$\bar{N}_i(x=0) = \int_{x'\geq 0} \langle K_{i-1\to i}(t,x')\rangle_g \bar{N}_{i-1}(x')dx',\ i \neq 1 \quad ,$$

$$\bar{N}_1(x=0) = 2\int_{x'\geq 0} \langle K_{I\to 1}(t,x')\rangle_g \bar{N}_I(x')dx'$$

Let's prove the discrete analogue.

$\dot{x}(t) = A(t)x(t)$, $A$ $T-$ periodic, $A_{ij} \geq 0$ for $i \neq j$.

$\dot{x}(t) = A(t)x(t) - \lambda_{\mathrm{per}} x(t)$ with $x$ $T$-periodic.

Define $\bar{A}_{ii} = \langle A_{ii} \rangle_a$ and $\bar{A}_{ij} = \exp\left(\langle \log(A_{ij}) \rangle_a\right)$ for $i \neq j$.

Let $\lambda_s$ denote the spectral abscissa of $\bar{A}$, let's show $\lambda_s \leq \lambda_{\mathrm{per}}$.

$$v_i(t) := \log x_i(t) \quad \text{and for } i \neq j, \quad \log A_{ij}(t) := \log A_{ij}(t) \ .$$

$$\begin{aligned}\dot{v}_i(t) &= \sum_j x_i^{-1}(t) A_{ij}(t) x_j(t) - \lambda_{\mathrm{per}} \\ &= \sum_{j \neq i} \exp(-v_i(t) + \log A_{ij}(t) + v_j(t)) + A_{ii}(t) - \lambda_{\mathrm{per}} \quad .\end{aligned}$$

Taking the arithmetic mean on $[0, T]$, and using Jensen:

$$0 = \left\langle \sum_{j \neq i} \exp(-v_i(t) + \log A_{ij}(t) + v_j(t)) \right\rangle_a + \langle A_{ii}(t) \rangle_a - \lambda_{\mathrm{per}} \quad ,$$

$$0 \geq \sum_{j \neq i} \exp(- \langle v_i(t) \rangle_a + \langle \log A_{ij}(t) \rangle_a + \langle v_j(t) \rangle_a) + \langle A_{ii}(t) \rangle_a - \lambda_{\mathrm{per}} \quad .$$

Setting $\bar{x}_i := \exp(\langle v_i(t) \rangle_a)$,

$$0 \geq \sum_j \bar{x}_i^{-1} \bar{A}_{ij} \bar{x}_j - \lambda_{\mathrm{per}} \quad ,$$

$$\bar{A}\bar{x} \leq \lambda_{\mathrm{per}} \bar{x} \quad .$$

Using Collatz-Wielandt, $\lambda_s = \min\{r;\ \exists Y \in \operatorname{int} \mathbb{R}_+^d, \bar{A}Y \leq rY\} \leq \lambda_{\mathrm{per}}$.

# Some results of non-linear Perron-Frobenius theory

**Convergence to periodic orbits**.

**Theorem** (Akian, SG, Lemmens, Nussbaum, Math. Proc. Camb. Phil. Soc.,06). *Let $C$ be a polyhedral cone with $N$ facets in a finite dimensional vector space $X$. If $F: C \to C$ is a continuous order preserving subhomogeneous map and the orbit of $x \in C$ is bounded, then $\lim_{k\to\infty} F^{kp}(x)$ exists with*

$$p \leq \max_{q+r+s=N} \frac{N!}{q!r!s!} = \frac{N!}{\lfloor \frac{N}{3} \rfloor ! \lfloor \frac{N+1}{3} \rfloor ! \lfloor \frac{N+2}{3} \rfloor !}.$$

Subhomogeneous means that $F(\lambda x) \leq \lambda F(x)$ for $\lambda > 1$.

This comes after a long series of works on periodic orbits of nonexpansive maps when the norm is polyhedral: Ackoglu and Krengel, Weller, Martus, Nussbaum, Sine, Scheutzow, Verdyun-Lunel,. . .

Some ingredients of the proof: Reduce $C = \mathbb{R}^n_+$. If the orbit of $x$ stays in the interior of $C$, we can look at it with logarithmic glasses, i.e., consider $G := \log \circ F \circ \exp$, which is order preserving and nonexpansive in the sup-norm. Then a result of Lemmens and Scheutzow (Erg. Th. Dyn. S. 05) shows that the orbit length is at most

$$\frac{N!}{\lfloor \frac{N}{2} \rfloor ! \lfloor \frac{N+1}{2} \rfloor !}$$

Conjecture (Nussbaum). If $G$ is non-expansive in the sup-norm (but not order preserving) the bound becomes $2^N$.

Application. Assume that $G : \mathbb{R}^n \to \mathbb{R}^n$ is order preserving and additively homogeneous: $G(\alpha + x) = \alpha + G(x)$ (think to undiscounted games). Here, $\alpha + x := (\alpha + x_i)_{1 \leq i \leq n}$.

If $G(u) \leq u$, $u \in \mathbb{R}^n$, then $\lim_k G^{kp}(x)$ exists in $(\mathbb{R} \cup \{-\infty\})^n$, for all $x \in \mathbb{R}^n$.

If $G(u) = u$, $u \in \mathbb{R}^n$, then $\lim_k G^{kp}(x)$ exists in $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$.

Assume that $G$ has an additive eigenvector, so $G(u) = \lambda + u$, $\lambda \in \mathbb{R}$, $u \in \mathbb{R}^n$, then, for all $x \in \mathbb{R}^n$,

$$G^k(x) = k\lambda + \text{asymp. periodic. term in } k$$

$\neq$ Difficult case in which $G_i^k(x) - G_j^k(x) \to \infty$: Neyman, Sorin, Rosenberg. . .

## **Existence of eigenvectors**.

**Theorem** (SG, Gunawardena, TAMS 04). *Assume that $G$ is order preserving and additively homogeneous, and that the recession function*

$$\hat{G}(x) := \lim_{t\to\infty} t^{-1} G(tx)$$

*exists. If*

$$\hat{G}(x) = x \implies x_1 = \cdots = x_n$$

*then*

$$\exists u \in \mathbb{R}^n,\ \exists \lambda \in \mathbb{R}, \qquad G(u) = \lambda + u$$

Application. A game inspired by Richman games / discretisation of infinity Laplacian.

Let

$$G_i(x) = \frac{1}{2}\Big(\min_{(i,j)\in G} A_{ij} + x_j + \max_{(i,j)\in G} A_{ij} + x_j\Big) \ .$$

Two players. One flips a coin to decide who plays. Player MIN plays $A_{ij}$ to Player MAX if the move is $i \to j$.

$$\hat{G}(x) = \frac{1}{2}\Big(\min_{(i,j)\in G} x_j + \max_{(i,j)\in G} x_j\Big)$$

If $x_i = m := \max_k x_k$, and $x = \hat{G}(x)$, $(i,j) \in G \implies x_j = m$. So $x = \hat{G}(x) \implies x_1 = \cdots = x_n$ if $G$ is strongly connected.

## Representation of the fixed point set

Let $P$ be a Markov matrix: an harmonic function, $v = Pv$, is determined uniquely by its value on recurrent classes, and its restriction on a recurrent class is a constant.

Precise results are available when $G : \mathbb{R}^n \to \mathbb{R}^n$ is order preserving, nonexpansive in the sup-norm, and *convex*.

Then, by Legendre-Fenchel duality

$$G(x) = \sup_{P \in S_n^+} (Px - G^*(P))$$

where $S_n^+$ denote the set of substochastic matrices, and $G^*(P) \in (\mathbb{R} \cup \{+\infty\})^n$.

Compare with the dynamic programming operator of a stochastic control problem with state space $\{1, \ldots, n\}$:

$$G_i(x) = \sup_{a \in A(i)} (r_i^a + P_i^a x)$$

The previous expression is a canonical form of $G$: when in state $i$, the player chooses the substochastic vector $P_i \in \operatorname{dom} G_i^*$, receives the payment $-G_i^*(P)$ when in state $i$, and moves to $j$ with probability $P_{ij}$.

The ergodic control problem consists in finding $P \in \operatorname{dom} G^*$ such that $-mG^*(P)$ is maximal, where $m$ is an invariant measure of $P$. If $G(u) = \lambda + u$ with $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$, this maximum is equal to $\lambda$.

Normalise $G$ and assume that $G(u) = u$. We say that $i$ is *critical* if it belongs to a recurrence class of some matrix $P$ such that $G(u) = Pu - G^*(P)$ (in other words, if it is recurrent for a stationnary strategy which is optimal for the ergodic problem). Let $C$ denote the set of critical nodes. We can write $C = C_1 \cup \cdots \cup C_s$ where the $C_i$ are maximal recurrent classes of randomised optimal strategies.
**Theorem** (Akian, SG, NLA TMA 03). *The restriction $x \mapsto x_C$ is a sup-norm isometry from $E := \{x \mid G(x) = x\}$ to a convex set $K$, and for all $x, y \in E$, $i \mapsto x_i - y_i$ is constant on each class $C_k$, and so $\dim K \leq s$.*

"An harmonic function is defined uniquely by its value on the boundary". Here, $C$ plays the role of the boundary, it is a discrete version of the Aubry set arising in Fathi weak KAM's theory.

Can we extend this to continuous time/second order PDE? special case in a recent work with Akian and David, control of a degenerate diffusion on the torus.

$$d\mathbf{x}_t = g(\mathbf{x}_t, \mathbf{u}_t)\ dt + \sigma(\mathbf{x}_t, \mathbf{u}_t)\ d\mathbf{b}_t$$

$$\mathrm{Min}_{\mathbf{u}}\, \mathbb{E}\left[\int_0^T L(\mathbf{x}_s, \mathbf{u}_s)\ ds + \phi(\mathbf{x}_T)\right],$$

Find $\lambda \in \mathbb{R}$, $\phi$ a function, such that $S^t\phi = \lambda t + \phi, \forall t \geq 0$.

$$\lambda - H(x, D\phi(x), D^2\phi(x)) = 0, \qquad x \in \mathbb{T}^n,$$

$$H(x, p, A) = \min_{u \in U} \left( \frac{1}{2}(\sigma(x,u)\sigma(x,u)^T A) + \langle p, g(x,u)\rangle + L(x,u) \right).$$

We assume that for $x \in \mathbb{T}^d \setminus \{x_1, \ldots, x_k\}$, and for all $u$, $L(x,u) > 0$ or $\sigma(x,u)\sigma^T(x,u) \succ 0$, whereas $\forall x_i, \exists u^i$, $L, g, \sigma$ vanish at $(x^i, u)$.

So the "Aubry set" is finite. Similar representation theorem (unfortunately, existence of Lyapunov functions is needed for the full representation).

In another recent work, we extend the discrete result to the case where $G$ is expansive (but still convex), i.e. to the case of a "negative discount rate".

Define the *semiderivative* $G'_v(y) = \sup_{P\in\partial G(v)} Py$, so that $G(v+y) = G(v) + G'_v(y) + o(\|y\|)$.

A fixed point $v$ of $G$ is $\star$-stable if every orbit of $G'_v$ is bounded above. This is weaker than Lypunov stability. Critical classes are defined as before, as the classes of the $P \in \partial G(v)$ which have spectral radius $1$ (such matrices must be stable - every orbit of $P$ is bounded).

**Theorem** (Akian, SG, Lemmens)**.** *A $\star$-stable fixed point is determined uniquely by its value on critical classes.*

Application: the representation of the fixed points has been used in Cochet,SG CRAS 06, policy iteration algorithm for stochastic games with ergodic payoff in degenerate cases (degenerate transition probabilities).

Open question: find a representation of fixed points in the case of games.

**Static analysis of programs by abstract interpretation**

*Cousot*: finding invariants of a program reduces to compute the smallest fixed point of a monotone self-map of a complete lattice $L$

To each breakpoint $i$ of the program, is associated a set $x^i \in L$ which is an overapproximation of the set of reachable values of the variables, at this breakpoint.

$x^i$ may be a cartesian product of intervals (one interval for each variable of the program)

The best $x$ is the smallest solution of a fixed point problem $x = f(x)$ with $f$ order preserving $L^n \to L^n$ ($n \leq \sharp$ breakpoints).

```
void main() {
  int x=0;           // 1
  while (x<100) {    // 2
    x=x+1;           // 3
  }                  // 4
}
```

$$\begin{aligned} x_1 &= [0,0] \\ x_2 &= ]-\infty, 99] \cap (x_1 \cup x_3) \\ x_3 &= x_2 + [1,1] \\ x_4 &= [100, +\infty[ \cap (x_1 \cup x_3) \end{aligned}$$

Let $x_2^+ := \max x_2$. After some elimination, we arrive at

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) \ .$$

The smallest $x_2^+$ is 99, it is the value of a zero-sum game with a stopping option.

*When does the fixed point problem of abstract intepretation reduce to a game problem ?*

*Does it work for*

*More general programs ?*

*More general domains ?*

## Some useful domains

**1. Zones** (Miné). Sets of the form

$Z = \{x \in \mathbb{R}^n \mid \ x_i - x_j \leq M_{ij}\}$

a zone is coded by the matrix $M \in (\mathbb{R} \cup \{+\infty\})^{n \times n}$.

by setting $x_0 := 0$ and projecting, we see that Zones $\supset$ Intervals.

**2. Polyhedra** (Cousot, Halbwachs 78. . . )

but the number of extreme points or faces may grow exponentially

$\longrightarrow$ not scalable

**3. Templates** S. Sankaranarayanan and H. Sipma and Z. Manna (VMCAI'05)

almost as expressive as polyhedra but scalable.

I'll give a convex analytic view of templates.

The **support function** $\sigma_X$ of $X \subset \mathbb{R}^n$ is defined by

$\sigma_X(p) = \sup_{x \in X} p \cdot x$

Legendre-Fenchel duality tells that $\sigma_X = \sigma_Y$ iff $X$ and $Y$ have the same closed convex hull.

$\sigma_X(\alpha p) = \alpha \sigma_X(p)$ for $\alpha > 0$, so it is enough to know $\sigma_X(p)$ for all $p$ in the unit sphere.

Idea: discretize the unit sphere and represent $X$ by $\sigma_X$ restricted to the discretization points.

So fix $\mathcal{P} \subset \mathbb{R}^n$ a finite set of directions.

$L(\mathcal{P})$ lattice of sets of the form

$Z = \{x \mid p \cdot x \leq \gamma(p),\ \forall p \in \mathcal{P}\}$, $\quad \gamma : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$.

$Z$ is coded by $\gamma := \sigma_Z \mid_{\mathcal{P}}$.

$Z$ is a polyhedron every facet of which is orthogonal to some $p \in \mathcal{P}$.

Specialization: $\mathcal{P} = \{\pm e_i,\ i = 1, \ldots, n\}$ gives intervals, $\mathcal{P} = \{\pm(e_i - e_j),\ 1 \leq i < j \leq n\}$ gives Miné's templates.

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$\gamma(e_1) = +\infty$$
$$\gamma(-e_1) = -1$$
$$\gamma(e_2) = 10$$
$$\gamma(-e_2) = -\infty$$
$$\gamma(e_1 - e_2) = 0$$
$$\gamma(e_1 + 2e_2) = 21$$
$$\gamma(-e_1 - 2e_2) = -21 \quad .$$

$\mathcal{P} = \{\pm e_1, \pm e_2, e_1 - e_2, \pm(e_1 + 2e_2)\}$, $\gamma$: breakpoint 1.

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$i \le +\infty$$

$$i \ge 1$$

$$j \le 10$$

$$j \ge -\infty$$

$$i \le j$$

$$i + 2j \le 21$$

$$i + 2j \ge 21$$

$(i, j) \in [(1, 10), (7, 7)]$ (exact result).

To reach this conclusion, we have to solve the fixed point problem:

$$\begin{aligned}
\gamma(p) &= ((1,10)\cdot p) \vee (\bar{\gamma}(p) + (2,-1)\cdot p), \qquad \forall p \in \mathcal{P} \setminus \{e_1 - e_2\} \\
\gamma(e_1 - e_2) &= 0 \wedge (-9 \vee (\bar{\gamma}(e_1 - e_2) - 3)) \\
\bar{\gamma} &= \text{convex hull}(\gamma)
\end{aligned}$$

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

**Correspondence theorem** (SG, Goubault, Taly, Zennou, ESOP'07) When the arithmetics of the program is *affine* (no product or division of variables), abstract interpretation over a lattice of templates reduces to finding the smallest fixed point of a map $f : (\mathbb{R} \cup \{+\infty\})^n \to (\mathbb{R} \cup \{+\infty\})^n$ of the form

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + M_i^{ab} x)$$

with $M_i^{ab} := (M_{ij}^{ab})$, $M_{ij}^{ab} \geq 0$, but possibly $\sum_j M_{ij}^{ab} > 1$

$\to$ game in infinite horizon with a "negative discount rate".

Sketch of proof.

$y = Ax + b$; If $x \in Z^1 := \{z \mid p \cdot z \leq \gamma^1(z),\ \forall p \in \mathcal{P}\}$, find the best $Z^2 := \{z \mid p \cdot z \leq \gamma^2(z),\ \forall p \in \mathcal{P}\}$ such that $y \in Z^2$.

$\gamma^2(p) = \sup_{x \in Z^1} p \cdot (Ax + b) = \sup p \cdot (Ax + b);\ p \cdot x \leq \gamma^1(p),\ \forall p \in \mathcal{P}$

by the strong duality theorem

$= \inf p \cdot b + \sum_{q \in \mathcal{P}} \lambda(q) \gamma^1(q);\ \ \lambda(q) \geq 0,\ A^T p = \sum_{q \in \mathcal{P}} \lambda(q) q$

The inf is attained at an extreme point of the feasible set, so this is in fact a min over a finite set.

$\sigma_{X \cap Y} =$ convex hull$(\inf(\sigma_X, \sigma_Y))$.

Convex hull reduces to a finite min by a similar argument.

Modelling the dataflow yields maxima, because $\sigma_{X \cup Y} = \sup(\sigma_X, \sigma_Y)$

**How to solve the fixed point problem ?**

Classically: Kleene (fixed point iteration) is slow or may even not converge, so widening and narrowing have been used, leading to an overapproximation of the solution.

An alternative: **Policy iteration.**

method developed by Howard (60) in stochastic control, extended by Hofman and Karp (66) to some special (nondegenerate) stochastic games. Extension to Newton method $\Longrightarrow$ fast. complexity still open.

extended by Costan, SG, Goubault, Martel, Putot, CAV'05) to fixed point problems in static analysis (difficulty: what are the strategies?)

experiments: PI often yields more accurate fixed points (because it avoids widening), small number of iterations.

A **strategy** is a map $\pi$ which to a state $i$ associates an action $\pi(i) \in A(i)$.

Consider the one player dynamic programming operator:

$$f_i^\pi(x) := \sup_{b \in B(i,\pi(i))} (r_i^{\pi(i)b} + M_i^{\pi(i)b} x)$$

$$f = \inf_\pi f^\pi$$

and the set $\{ f^\pi \mid \pi \text{ strategy } \}$ has a *selection*:

$$\forall v \in \mathbb{R}^n,\ \exists \pi \qquad f(v) = f^\pi(v) \ .$$

Since $f^\pi$ is convex and piecewise affine, finding the smallest finite fixed point of $f^\pi$ (if any) can be done by linear programming:

$$\min \sum_i v_i; \quad f^\pi(v) \leq v \enspace .$$

Can we compute the smallest fixed point of $f$ from the smallest fixed points of the $f^\pi$?

We denote by $f^-$ the smallest fixed point of a monotone self-map $f$ of a complete lattice $\mathcal{L}$, whose existence is guaranteed by Tarski's fixed point theorem.

**Theorem** (Costan, SG, Goubault, Martel, Putot CAV'05)**.** *Let $\mathcal{G}$ denote a family of monotone self-maps of a complete lattice $\mathcal{L}$ with a lower selection, and let $f = \inf \mathcal{G}$. Then $f^- = \inf_{g\in\mathcal{G}} g^-$ .*

The input of the following algorithm consists of a finite set $\mathcal{G}$ of monotone self-maps of a lattice $\mathcal{L}$ with a lower selection. When the algorithm terminates, its output is a fixed point of $f = \inf \mathcal{G}$.

1. *Initialization*. Set $k = 1$ and select any map $g_1 \in \mathcal{G}$.

2. *Value determination*. Compute a fixed point $x^k$ of $g_k$.

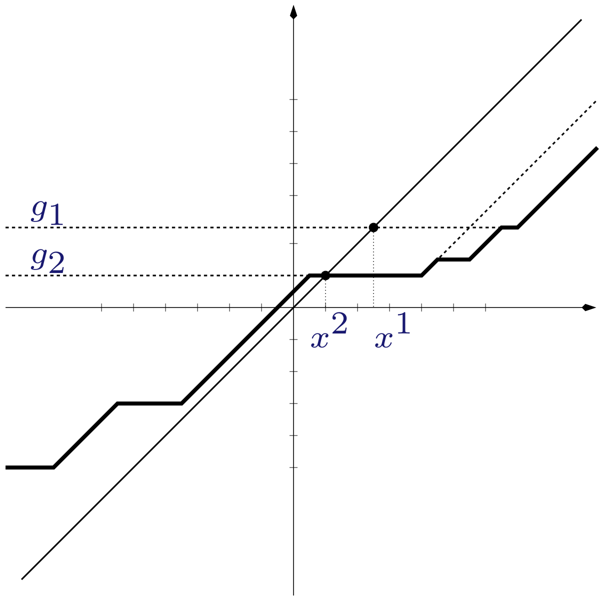
3. Compute $f(x^k)$.

4. If $f(x^k) = x^k$, return $x^k$.

5. *Policy improvement*. Take $g_{k+1}$ such that $f(x^k) = g_{k+1}(x^k)$. Increment $k$ and goto Step 2.

The algorithm does terminate when at each step, the smallest fixed-point of $g_k$, $x^k = g_k^-$ is selected.

*Example.* Take $\mathcal{L} = \overline{\mathbb{R}}$, and consider the self-map of $\mathcal{L}$, $f(x) = \inf_{1 \leq i \leq m} \max(a_i + x, b_i)$ , where $a_i, b_i \in \mathbb{R}$. The set $\mathcal{G}$ consisting of the $m$ maps $x \mapsto \max(a_i + x, b_i)$ admits a lower selection.



Experimentally fast, but the worst case complexity is not known. Condon showed: mean payoff games is in NP $\cap$ co-NP, same with positive discount. Much current work: (Zwick, Paterson, TCS 96), (Jurdziński, Paterson, Zwick, SODA'06), (Bjorklund, Sandberg, Vorobyov, preprint 04),

PI often more accurate than Klenne+widening/narrowwing:

```
i = 150;
j = 175;
while (j >= 100){
  i++;
  if (j<= i){
      i = i - 1;
      j = j - 2;
  }
}
```

$$
\begin{array}{lcl}
M_0 & = & \mathit{context\_initialization} \\
M_2 & = & (\mathit{Assignment}\ (i \leftarrow 150,\ \ j \leftarrow 175)(M_0))^* \\
M_3 & = & ((M_2 \sqcup M_8) \sqcap (j \geq 100))^* \\
M_4 & = & (\mathit{Assignment}\ (i \leftarrow i+1)(M_3))^* \\
M_5 & = & (M_4 \sqcap (j \leq i))^* \\
 & & \\
M_7 & = & (\mathit{Assignment}\ (i \leftarrow i-1,\ \ j \leftarrow j-2)(M_5))^* \\
M_8 & = & ((M_4 \sqcap (j > i))^* \sqcup M_7 \\
M_9 & = & ((M_2 \sqcup M_8) \sqcap (j < 100))^*
\end{array}
$$

$$
\text{IP} \left\{ \begin{array}{c} 150 \leq i \leq 174 \\ 98 \leq j \leq 99 \\ -76 \leq j - i \leq -51 \end{array} \right. \qquad \text{Mine's Octogon} \left\{ \begin{array}{c} 150 \leq i \\ 98 \leq j \leq 99 \\ j - i \leq -51 \\ 248 \leq j + i \end{array} \right.
$$

In the case of “negative” discount, no systematic efficient technique is known yet to determine the smallest fixed point!