Mean Field Games: Numerical Methods

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I. A short review

Mean field games: infinite horizon

Find $u \in \mathcal{C}^2(\mathbb{T})$, $m \in W^{1,p}(\mathbb{T})$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{cases} -\nu \Delta u + H(x, \nabla u) + \lambda = V[m], \\ -\nu \Delta m - \mathrm{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0, \\ \displaystyle\int_{\mathbb{T}} u dx = 0, \qquad \int_{\mathbb{T}} m dx = 1, \quad \text{and} \quad m > 0 \quad \text{in } \mathbb{T}. \end{cases} \qquad (*)$$

- $\mathbb{T}$ unit torus of $\mathbb{R}^d$
- $\nu > 0$
- H is a $\mathcal{C}^1$ Hamiltonian (convex):

$$H(x, p) = \sup_{\alpha \in \mathbb{R}^d} \left(p \cdot \alpha - L(x, \alpha) \right), \quad \text{with} \quad \lim_{|\alpha| \to \infty} \inf_x \frac{L(x, \alpha)}{|\alpha|} = +\infty$$

- V is an operator **from the space of probability measures on $\mathbb{T}$ into a bounded set of Lipschitz functions on $\mathbb{T}$** such that

  $V[m_n]$ converges uniformly on $\mathbb{T}$ to $V[m]$ if m_n weakly converges to m.

  Typical examples for V include nonlocal smoothing operators.

Local operators

$$V[m](x) = f(m(x), x)$$

may be considered as well.

($*$) has been obtained by J-M. Lasry and P-L. Lions by passing to the limit **in stochastic differential games involving a very large number N of identical rational agents** (or players) with a **(limited) global information**

- Dynamics:

$$dX_t^i = \sqrt{2\nu}\, dW_t^i - \alpha^i dt, \quad X_0^i = x^i \in \mathbb{R}^d$$

- Cost:

$$J^i(\alpha^1, \dots, \alpha^N) = \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left(\int_0^T \left(L(X_t^i, \alpha_t^i) + V \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X_t^j} \right] (X_t^i) \right) dt \right)$$

- $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash point if $\quad \forall i = 1, \dots, N,$

$$\bar{\alpha}^i = \operatorname*{Argmin}_{\alpha^i} J^i(\bar{\alpha}^1, \dots, \bar{\alpha}^{i-1}, \alpha^i, \bar{\alpha}^{i+1}, \dots, \bar{\alpha}^N).$$

- Structure assumption on H: there exists $\theta \in (0,1)$ such that for $|p|$ large,

$$\inf_{x \in \mathbb{T}} \left(\frac{\partial H}{\partial x} \cdot p + \frac{\theta}{d\nu} H^2 \right) > 0$$

(makes it possible to obtain Lipschitz estimate on u with Bernstein inequality).

- There is a system of $2N$ PDEs
  - N HJB equations for the value functions
  - N Kolmogorov equations for the stationary measures of $(X_t^i)_i$

  whose solutions yield Nash equilibria.

- $N \to \infty$, pass to the limit...

Uniqueness for the mean field problem

By contrast with the system of PDEs for N players, the system (*) is well posed under some assumptions:

Theorem (Lasry-Lions) If V is strictly monotone, i.e.

$$\int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2)dx \leq 0 \Rightarrow m_1 = m_2,$$

then the solution of the mean field system (*) is unique.

Proof

Consider two solutions of (*): (λ_1, u_1, m_1) and (λ_2, u_2, m_2):

• multiply the first equation by $m_1 - m_2$

$$\int_{\mathbb{T}} \Big(\nu \nabla (u_1 - u_2) \cdot \nabla (m_1 - m_2) + (H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) \Big) dx$$

$$= \int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2) dx$$

• multiply the second equation by $u_1 - u_2$

$$0 = \int_{\mathbb{T}} \nu \nabla (u_1 - u_2) \cdot \nabla (m_1 - m_2) dx$$

$$+ \int_{\mathbb{T}} \left(m_1 \frac{\partial H}{\partial p}(x, \nabla u_1) - m_2 \frac{\partial H}{\partial p}(x, \nabla u_2) \right) \cdot \nabla (u_1 - u_2) dx.$$

• subtract:

$$0 = \begin{cases} \int_{\mathbb{T}} m_1 \left(H(x, \nabla u_1) - H(x, \nabla u_2) - \frac{\partial H}{\partial p}(x, \nabla u_1) \cdot \nabla (u_1 - u_2) \right) dx \\ + \int_{\mathbb{T}} m_2 \left(H(x, \nabla u_2) - H(x, \nabla u_1) - \frac{\partial H}{\partial p}(x, \nabla u_2) \cdot \nabla (u_2 - u_1) \right) dx \\ + \int_{\mathbb{T}} (V[m_1] - V[m_2])(m_1 - m_2) dx \end{cases}$$

Since H is convex and V is monotone, the 3 terms vanish.

The strict monotonicity of V implies that $m_1 = m_2$.

The identities $u_1 = u_2$ and $\lambda_1 = \lambda_2$ come from the uniqueness for the HJB equation:

$$-\nu \Delta u + H(x, \nabla u) + \lambda = f \quad \text{with} \quad \int_{\mathbb{T}} u = 0.$$

Finite horizon Nash equilibrium with N players

The N players initial conditions are random, independent, with the same probability distribution m^0.

Cost of the player i at time t:

$$\mathbb{E}\left(\int_t^T \left(L^i(X_s^i, \alpha_s^i) + V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}\right]\right) ds + V_0\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_T^j}\right]\right)$$

$N \to \infty$: with the change of variable $t \to T - t$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m], \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \\ \displaystyle\int_{\mathbb{T}} m dx = 1, \quad \text{and} \quad m > 0 \quad \text{in } \mathbb{T}, \\ u(t=0) = V_0[m(t=0)], \qquad m(t=T) = m_\circ. \end{cases} \qquad (**)$$

Existence for (**)

Theorem (Lasry-Lions)
If

- same kind of assumptions on V and V_0 as in the stationary case (V and V_0 are nonlocal smoothing operators).
- H is smooth on $\mathbb{T} \times \mathbb{R}^d$ and

$$\left| \frac{\partial H}{\partial x}(x,p) \right| \leq C(1+|p|), \quad \forall x \in \mathbb{T},\ \forall p \in \mathbb{R}^d,$$

then

(**) has at least a smooth solution.

Uniqueness for (**)

Theorem (Lasry-Lions)

If

the operators V and V_0 are strictly monotone, i.e.

$$\int_{\mathbb{T}} (V[m] - V[\tilde{m}])(m - \tilde{m}) \leq 0 \Rightarrow V[m] = V[\tilde{m}],$$
$$\int_{\mathbb{T}} (V_0[m] - V_0[\tilde{m}])(m - \tilde{m}) \leq 0 \Rightarrow V_0[m] = V_0[\tilde{m}],$$

then

(**) has a unique solution.

II. Finite Horizon: Numerical Methods

Take $d = 2$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m], \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0, \\ \displaystyle\int_{\mathbb{T}} m dx = 1, \qquad m > 0 \quad \text{in } \mathbb{T}, \\ u(t=0) = V_0[m(t=0)], \quad m(t=T) = m_\circ, \end{cases} \tag{**}$$

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_{ij} be a generic point in $\mathbb{T}_h$.
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$.
- The values of u and m at $(x_{i,j}, t_n)$ are resp. approximated by $U^n_{i,j}$ and $M^n_{i,j}$.

Goal: use a fully implicit scheme, robust when $\nu \to 0$, which guarantees existence, and possibly uniform bounds and uniqueness.

Notation:

- The discrete Laplace operator:

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1}).$$

- Right-sided finite difference formulas for $\partial_1 w(x_{i,j})$ and $\partial_2 w(x_{i,j})$:

$$(D_1^+ W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad \text{and} \quad (D_2^+ W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}.$$

- The set of 4 finite difference formulas at $x_{i,j}$:

$$[D_h W]_{i,j} = \Big((D_1^+ W)_{i,j}, (D_1^+ W)_{i-1,j}, (D_2^+ W)_{i,j}, (D_2^+ W)_{i,j-1}\Big).$$

Discrete HJB equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m]$$

$$\downarrow$$

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = \left(V_h[M^{n+1}]\right)_{i,j}$$

- $$\begin{aligned} & g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \\ =& g\Big(x_{i,j}, (D_1^+ U^{n+1})_{i,j}, (D_1^+ U^{n+1})_{i-1,j}, (D_2^+ U^{n+1})_{i,j}, (D_2^+ U^{n+1})_{i,j-1}\Big), \end{aligned}$$

- for instance,

$$(V_h[M])_{i,j} = V[m_h](x_{i,j}),$$

calling m_h the piecewise constant function on $\mathbb{T}$ taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \leq h/2$.

Classical assumptions on the discrete Hamiltonian g

$$(q_1, q_2, q_3, q_4) \to g\left(x, q_1, q_2, q_3, q_4\right).$$

- **Monotonicity:** g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to to q_2 and q_4.
- **Consistency:**

$$g\left(x, q_1, q_1, q_3, q_3\right) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q = (q_1, q_3) \in \mathbb{R}^2.$$

- **Differentiability:** g is of class $\mathcal{C}^1$, and

$$\left|\frac{\partial g}{\partial x}\Big(x, (q_1, q_2, q_3, q_4)\Big)\right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

- **Convexity:** $(q_1, q_2, q_3, q_4) \to g\left(x, q_1, q_2, q_3, q_4\right)$ is convex.

The discrete version of

$$\frac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \qquad (\dagger)$$

It is chosen so that

- each time step leads to a linear system with a matrix
  - whose diagonal coefficients are negative,
  - whose off-diagonal coefficients are nonnegative,

  in order to hopefully use some **discrete maximum principle.**

- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian g should be used for $(\dagger)$ as well.**

Principle

Discretize

$$-\int_{\mathbb{T}} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

by

$$h^2 \sum_{i,j} m_{i,j} \nabla_q g(x_{i,j}, [D_h U]_{i,j}) \cdot [D_h W]_{i,j}.$$

This yields the scheme:

$$0 = \frac{M_{i,j}^{n+1} - M_{i,j}^{n}}{\Delta t} + \nu(\Delta_h M^n)_{i,j}$$

$$+\frac{1}{h}\left\{\begin{array}{l} M_{i,j}^{n}\,\dfrac{\partial g}{\partial q_1}\left(x_{i,j}, (D_1^+U^n)_{i,j}, (D_1^+U^n)_{i-1,j}, (D_2^+U^n)_{i,j}, (D_2^+U^n)_{i,j-1}\right) \\ -M_{i-1,j}^{n}\,\dfrac{\partial g}{\partial q_1}\left(x_{i-1,j}, (D_1^+U^n)_{i-1,j}, (D_1^+U^n)_{i-2,j}, (D_2^+U^n)_{i-1,j}, (D_2^+U^n)_{i-1,j-1}\right) \\ +M_{i+1,j}^{n}\,\dfrac{\partial g}{\partial q_2}\left(x_{i+1,j}, (D_1^+U^n)_{i+1,j}, (D_1^+U^n)_{i,j}, (D_2^+U^n)_{i+1,j}, (D_2^+U^n)_{i+1,j-1}\right) \\ -M_{i,j}^{n}\,\dfrac{\partial g}{\partial q_2}\left(x_{i,j}, (D_1^+U^n)_{i,j}, (D_1^+U^n)_{i-1,j}, (D_2^+U^n)_{i,j}, (D_2^+U^n)_{i,j-1}\right) \end{array}\right.$$

$$+\frac{1}{h}\left\{\begin{array}{l} M_{i,j}^{n}\,\dfrac{\partial g}{\partial q_3}\left(x_{i,j}, (D_1^+U^n)_{i,j}, (D_1^+U^n)_{i-1,j}, (D_2^+U^n)_{i,j}, (D_2^+U^n)_{i,j-1}\right) \\ -M_{i,j-1}^{n}\,\dfrac{\partial g}{\partial q_3}\left(x_{i,j-1}, (D_1^+U^n)_{i,j-1}, (D_1^+U^n)_{i-1,j-1}, (D_2^+U^n)_{i,j-1}, (D_2^+U^n)_{i,j-2}\right) \\ +M_{i,j+1}^{n}\,\dfrac{\partial g}{\partial q_4}\left(x_{i,j+1}, (D_1^+U^n)_{i,j+1}, (D_1^+U^n)_{i,j+1}, (D_2^+U^n)_{i,j+1}, (D_2^+U^n)_{i,j}\right) \\ -M_{i,j}^{n}\,\dfrac{\partial g}{\partial q_4}\left(x_{i,j}, (D_1^+U^n)_{i,j}, (D_1^+U^n)_{i-1,j}, (D_2^+U^n)_{i,j}, (D_2^+U^n)_{i,j-1}\right) \end{array}\right.$$

Classical discrete Hamiltonians g can be chosen.

For example, if the Hamiltonian is of the form

$$H(x, \nabla u) = \psi(x, |\nabla u|),$$

a possible choice is the **Godunov scheme**

$$g(x, q_1, q_2, q_3, q_4) =$$
$$\psi\left(x, \sqrt{\min(q_1, 0)^2 + \max(q_2, 0)^2 + \min(q_3, 0)^2 + \max(q_4, 0)^2}\right).$$

If $\psi(x, w)$ is convex and nondecreasing w.r.t. w, then g is a convex function of (q_1, q_2, q_3, q_4); g is nonincreasing w.r.t. q_1 and q_3 and nondecreasing w.r.t. q_2 and q_4.

Finally, it can be proven that the global scheme is consistent if H is smooth enough.

Kushner-Dupuis scheme

More generally, if H is given by

$$H(x,p) = \sup_{\alpha\in\mathbb{R}^2} \left(p\cdot\alpha - L(x,\alpha)\right), \quad x\in\mathbb{T}, p\in\mathbb{R}^2,$$

then one may choose g as

$$g(x,q_1,q_2,q_3,q_4) = \sup_{\alpha\in\mathbb{R}^2} \left(-\alpha_1^- q_1 + \alpha_1^+ q_2 - \alpha_2^- q_3 + \alpha_2^+ q_4 - L(x,\alpha)\right).$$

The numerical Hamiltonian g is clearly convex, nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to q_2 and q_4.

Existence for the discrete problem

Theorem Assume that $M^{N_T} \geq 0$ and that $h^2 \sum_{i,j} M_{i,j}^{N_T} = 1$. Under the assumptions above on V, V_0 and g, **the discrete problem has a solution and there is a Lipschitz estimate on U_h^n uniform in n, h and Δt.**

Strategy of proof

$$\mathcal{K} = \left\{ (M_{i,j})_{0 \leq i,j < N} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \geq 0 \right\}.$$

Apply Brouwer fixed point theorem to a well chosen mapping

$$\begin{array}{llll} \chi : & \mathcal{K}^{N_T} & \longrightarrow & \mathcal{K}^{N_T}, \\ & (M^n)_n \rightarrow & (U^n)_n \rightarrow & (M^n)_n. \end{array}$$

Proof: a fixed point method in $\mathcal{K}^{N_T}$,

Step 1: a map $\Phi: \quad (M^n)_n \to (U^n)_n$.

Given $(M_{i,j}^{N_T})$, define the map Φ: $(M^n)_{0\le n<N_T} \in \mathcal{K}^{N_T} \to (U^n)_{0\le n\le N_T}$:

$$\begin{cases} \dfrac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = \big(V_h[M^{n+1}]\big)_{i,j}, \\ U_{i,j}^0 = V_0[m_h^0](x_{i,j}). \end{cases}$$

- Existence is classical: (Leray-Schauder fixed point theorem at each time step, making use of the monotonicity of g, the uniform boundedness assumption on V and of $H(\cdot, 0)$).
- Uniqueness stems from the monotonicity of g.

Step 2: estimates

• There exists a constant C independent of $(M^n)_n$ and h s.t.

$$\|U^n\|_\infty \leq C(1+T).$$

• The map Φ is continuous, from the continuity of V and well known results on continuous dependence on the data for monotone schemes.

• There exists a constant $L(T)$ independent of $(M^n)_n$ and h s.t.

$$\|D_h U^n\|_\infty \leq L(T), \quad \forall n,$$

proved by using the assumption

$$\left|\frac{\partial g}{\partial x}(x, q_1, q_2, q_3, q_4)\right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

Step 3: A map $\chi : \quad (M^n)_n \to (\widetilde{M}^n)_n$

- Choose a positive constant $\mu > 0$ large enough.
- For $(U^n)_n = \Phi((M^n)_n)$, **backward linear** parabolic problem for $\widetilde{M}^n$:

$$\left\{ \begin{array}{rcl} \widetilde{M}^{N_T} & = & M^{N_T}, \\ -\mu M^n_{i,j} & = & \dfrac{\widetilde{M}^{n+1}_{i,j} - \widetilde{M}^n_{i,j}}{\Delta t} - \nu(\Delta_h \widetilde{M}^n)_{i,j} - \mu \widetilde{M}^n_{i,j} \\ & & + \dfrac{1}{h} \left(\begin{array}{l} \widetilde{M}^n_{i,j} \dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U^n]_{i,j}) - \widetilde{M}^n_{i-1,j} \dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U^n]_{i-1,j}) \\ + \widetilde{M}^n_{i+1,j} \dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U^n]_{i+1,j}) - \widetilde{M}^n_{i,j} \dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U^n]_{i,j}) \end{array} \right) \\ & & + \dfrac{1}{h} \left(\begin{array}{l} \widetilde{M}^n_{i,j} \dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U^n]_{i,j}) - \widetilde{M}^n_{i,j-1} \dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U^n]_{i,j-1}) \\ + \widetilde{M}^n_{i,j+1} \dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U^n]_{i,j+1}) - \widetilde{M}^n_{i,j} \dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U^n]_{i,j}) \end{array} \right) \end{array} \right.$$

From the previous estimates on $(U^n)_n$, one can find μ large enough and independent of $(M^n)_n$ such that the iteration matrix is the opposite of a M-matrix, thus **there is a discrete maximum principle.**

Therefore, there exists a unique solution $(\widetilde{M}^n)_n$.
Moreover,

$$\begin{aligned} M^n \geq 0 \quad &\Rightarrow \quad \widetilde{M}^n \geq 0, & \forall n, \\ h^2 \sum_{i,j} M^n = 1 \quad &\Rightarrow \quad h^2 \sum_{i,j} \widetilde{M}^n = 1, & \forall n. \end{aligned}$$

Thus $\widetilde{M}^n \in \mathcal{K}$ for all n. Define the map

$$\chi : \quad \begin{aligned} \mathcal{K}^{N_T} \quad &\mapsto \quad \mathcal{K}^{N_T}, \\ (M^n)_{0\leq n<N_T} \quad &\to \quad (\widetilde{M}^n)_{0\leq n<N_T} \end{aligned}$$

Step 4: existence of a fixed point of χ

From the boundedness and continuity of the mapping Φ, and from the fact that g is $\mathcal{C}^1$, we obtain that $\chi : \mathcal{K}^{N_T} \mapsto \mathcal{K}^{N_T}$ is continuous.

From Brouwer fixed point theorem, χ has a fixed point, which yields a solution of (**).

Discrete weak maximum principle-1

The discrete version of

$$\frac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0$$

can be written

$$-M^n - \Delta t A^n M^n = -M^{n+1}.$$

Introduce the semi-norm $|||W|||$:

$$|||W|||^2 = \sum_{i,j} \left((D_1^+ W)_{i,j}^2 + (D_2^+ W)_{i,j}^2 \right).$$

Discrete Gårding inequality: from the estimates on U^n, there exists $\gamma \geq 0$ (independent of h and Δt) s.t.

$$(A^n W, W)_2 \geq \frac{\nu}{2} |||W|||^2 - \gamma \|W\|_2^2, \qquad \forall W, \quad \forall n.$$

Discrete weak maximum principle-2

We have also

$$-(A^n W, W^-)_2 \geq \frac{\nu}{2} |||W^-|||^2 - \gamma \|W^-\|_2^2, \qquad \forall W.$$

Taking $(M^n)^-$ as a test-function in the discrete equation yields

$$(1 - 2\gamma\Delta t) \left\|(M^n)^-\right\|_2^2 + \nu\Delta t |||(M^n)^-|||^2 \leq \left\|(M^{n+1})^-\right\|_2^2,$$

so there is a discrete weak maximum principle if $2\gamma\Delta t < 1$.

Uniqueness

Theorem Same assumptions as above on V, V_0, H and g. Assume also that the operators V_h and $V_{0,h}$ are strictly monotone, i.e.

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \quad \Rightarrow \quad V_h[M] = V_h[\widetilde{M}],$$
$$\left(V_{0,h}[M] - V_{0,h}[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \quad \Rightarrow \quad V_{0,h}[M] = V_{0,h}[\widetilde{M}].$$

If $2\gamma\Delta t < 1$, then the discrete problem (slightly modified) has a unique solution.

Proof The choice of the scheme makes it possible to mimic the proof used in the continuous case: uses the convexity assumption on g and the discrete weak maximum principle.

III. Infinite Horizon: A numerical method

$$\left\{\begin{array}{l}
-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}\,, \\
\left(\begin{array}{l}
-\nu(\Delta_h M)_{i,j} \\
-\dfrac{1}{h}\left(\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j}) \\
+M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j})
\end{array}\right) \\
-\dfrac{1}{h}\left(\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1}) \\
+M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j})
\end{array}\right)
\end{array}\right) = 0, \\
M_{i,j} \geq 0,
\end{array}\right.$$

and

$$h^2 \sum_{i,j} M_{i,j} = 1, \quad \text{and} \quad \sum_{i,j} U_{i,j} = 0.$$

Existence for the discrete problem: strategy of proof

- Use Brouwer fixed point theorem in the set of discrete probability measures for a mapping $\chi : \quad M \to U \to M$.

- The map $\Phi : \quad M \to U$ consists of solving

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j} \,, \\ \sum_{i,j} U_{i,j} = 0 \end{cases}$$

- (U, λ) is obtained by considering the ergodic approximation:

$$-\nu(\Delta_h U^{(\rho)})_{i,j} + g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) + \rho U^{(\rho)}_{i,j} = (V_h[M])_{i,j} \,,$$

  and passing to the limit as $\rho \to 0$.

- We need estimates on $U^{(\rho)} - U^{(\rho)}_{0,0}$ uniform in ρ and h.

Difficulty

Continuous problem: the a priori estimate $\|\nabla u\|_\infty \leq C$ was obtained with the Bernstein method.

Discrete case: this argument is difficult to reproduce.

We make more restrictive assumptions on H and g to obtain bounds uniform in h.

Possible assumptions on the Hamiltonian

$$H(x,p) = \max_{\alpha \in \mathcal{A}} \Big(p \cdot \alpha - L(x,\alpha) \Big),$$

where

- $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$,
- L is a $\mathcal{C}^0$ function on $\mathbb{T} \times \mathcal{A}$,

Kushner-Dupuis scheme:

$$g(x, q_1, q_2, q_3, q_4) = \sup_{\alpha \in \mathcal{A}} \Big(-\alpha_1^- q_1 + \alpha_1^+ q_2 - \alpha_2^- q_3 + \alpha_2^+ q_4 - L(x,\alpha) \Big).$$

Estimates on the discrete ergodic approximation

Proposition (using Kuo-Trudinger(1992) and Camilli-Marchi(2008))
Consider a grid function V and make the assumptions:

- L is a continuous function, $\mathcal{A}$ is compact,
- $q \mapsto g(x, q)$ is a $\mathcal{C}^1$ function on $\mathbb{R}^4$,
- $\|V\|_\infty$ is bounded uniformly w.r.t h.

For any real number $\rho > 0$, there exists a unique grid function U^ρ such that

$$\rho U^\rho_{i,j} - \nu(\Delta_h U^\rho)_{i,j} + g(x_{i,j}, [D_h U^\rho]_{i,j}) = V_{i,j},$$

and there exist two constants δ, $\delta \in (0, 1)$ and C, $C > 0$, uniform in h and ρ s.t.

$$|U^\rho(\xi) - U^\rho(\xi')| \le C|\xi - \xi'|^\delta, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Estimates on the discrete ergodic approximation

Proposition (using Krylov(2007) and Camilli-Marchi(2008))
Same assumptions as before, and furthermore

- L is uniformly Lipschitz continuous w.r.t. x,
- $\|D_h V\|_\infty$ is bounded uniformly w.r.t h,

For any real number $\rho > 0$, there exists a unique grid function U^ρ such that

$$\rho U^\rho_{i,j} - \nu(\Delta_h U^\rho)_{i,j} + g(x_{i,j}, [D_h U^\rho]_{i,j}) = V_{i,j},$$

and there exists a constant C, $C > 0$, uniform in h and ρ s.t.

$$|U^\rho(\xi) - U^\rho(\xi')| \leq C|\xi - \xi'|, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

The map Φ: $M \to U$

Proposition

Under the first set of assumptions, there exists a unique grid function U and a real number λ such that

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}, \\ \sum_{i,j} U_{i,j} = 0, \end{cases}$$

and there exist two constants δ, $\delta \in (0,1)$ and C, $C > 0$, uniform in h s.t.

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'|^{\delta}, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Under the second set of assumptions,

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'|, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Existence and uniqueness for the stationary problem

Theorem Under the above assumptions on V and g, the discrete stationary problem has at least a solution and we have either a uniform Hölder or a Lipschitz estimate on u_h, depending on the assumptions.

Remark Existence is still OK if for $\gamma > 1$,

$$g(x, q_1, q_2, q_3, q_4) \geq \alpha((q_1)_-^2 + (q_2)_+^2 + (q_3)_-^2 + (q_4)_+^2)^{\gamma/2} - C,$$

but no bounds on u_h uniform in h.

Uniqueness: Ok if

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \Rightarrow M = \widetilde{M}.$$

Convergence

The same method used for uniqueness can be used for proving convergence of the discrete scheme under some assumptions on consistency and stronger assumptions on V_h.

Example

If there exist $s > 0$ such that

$$h^2 \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \geq c \| V_h[M] - V_h[\widetilde{M}] \|_\infty^s,$$

then uniform convergence for u, convergence of λ and a convergence related to V for m.

Uses the Hölder or Lipschitz estimates on U_h uniform w.r.t. h.

The case when V is a local operator

$$V[m](x) = F(m(x), x),$$

Same assumptions on H, g as above.

- Existence for the discrete problem: OK
- If F is a bounded and $\mathcal{C}^1$ function on $\mathbb{R} \times \mathbb{T}$, uniform bounds for some Hölder norm of u_h.

IV. Infinite Horizon: long time approximation

Long time approximation (Eductive strategy, see Guéant-Lasry)

$$\begin{cases} \dfrac{\partial \tilde u}{\partial t} - \nu \Delta \tilde u + H(x, \nabla \tilde u) & = & V[\tilde m], \\ \dfrac{\partial \tilde m}{\partial t} - \nu \Delta \tilde m - \operatorname{div}\left(\tilde m \dfrac{\partial H}{\partial p}(x, \nabla \tilde u) \right) & = & 0, \\ \tilde u(0,x) = \tilde u_0(x), \quad \tilde m(0,x) & = & \tilde m_0(x), \end{cases}$$

with $\int_{\mathbb{T}} \tilde m_0 = 1 \quad \text{and} \quad \tilde m_0 \geq 0.$

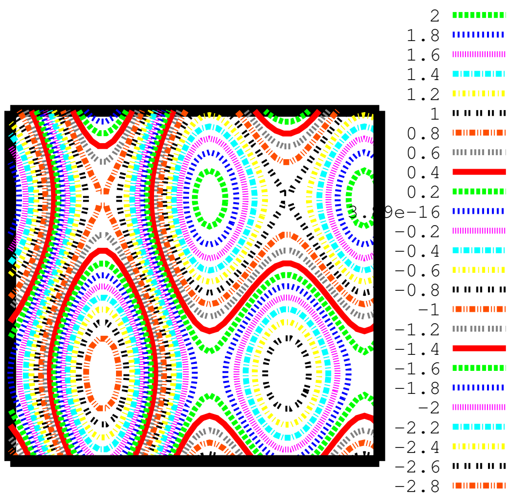
We expect that

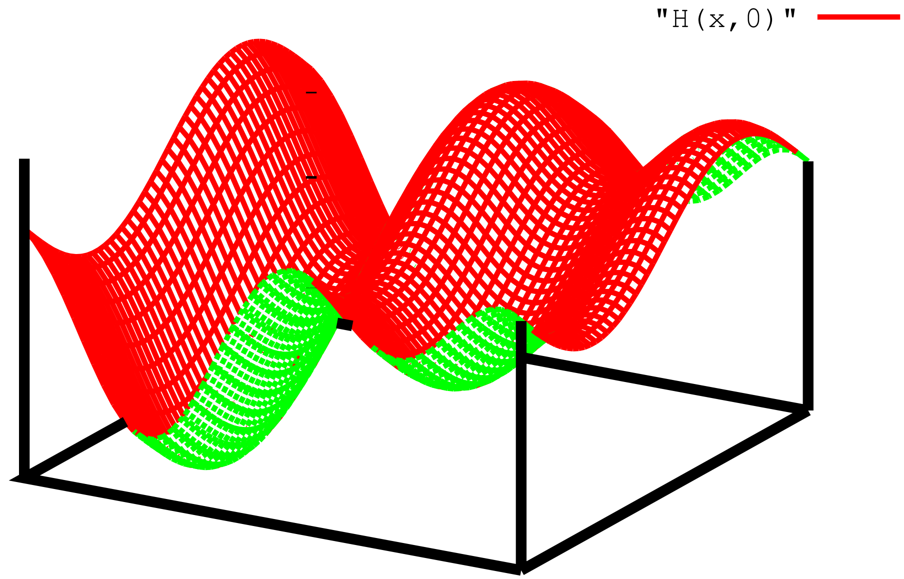
$$\lim_{t\to\infty} (\tilde u(t,x) - \lambda t) = u(x), \qquad \lim_{t\to\infty} \tilde m(t,x) = m(x),$$

Same thing at the discrete level.

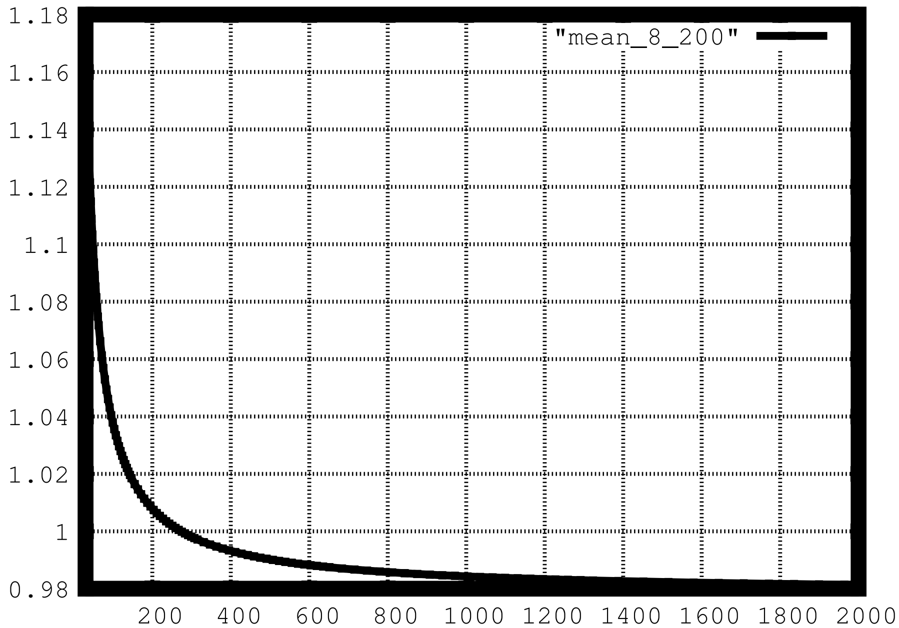
We use a semi-implicit linearized scheme. It requires the numerical solution of a linearized problem. Linearizing must be done carefully and is not always possible. In such cases, an explicit method can be used.

$$\nu = 1, \quad H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2$$



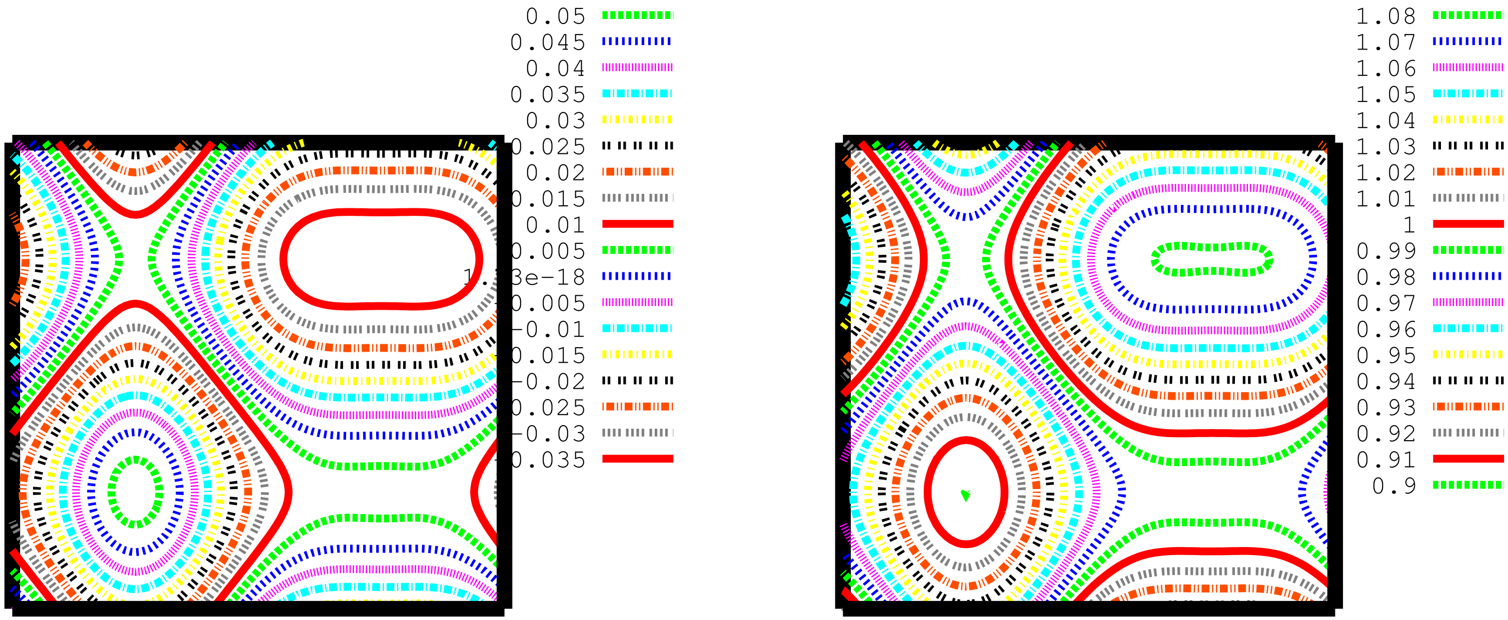


$$H(x,0) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1).$$

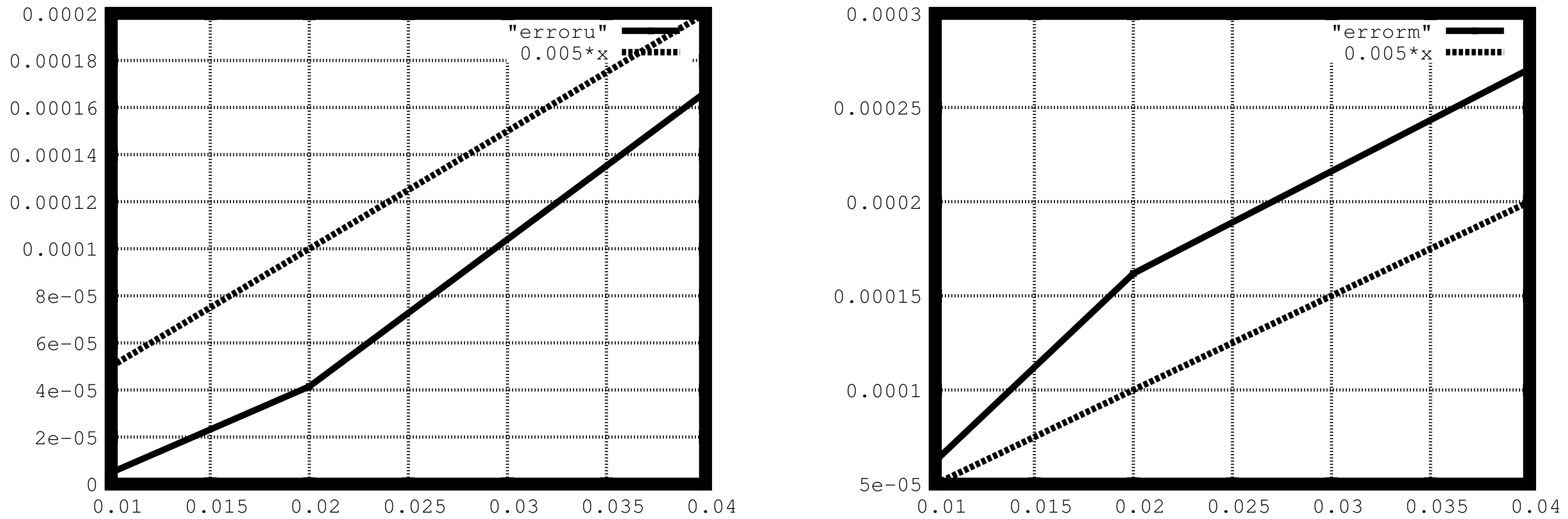


$\nu = 1,$ Convergence $\frac{1}{t}\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$.

Very long time steps are used near convergence.



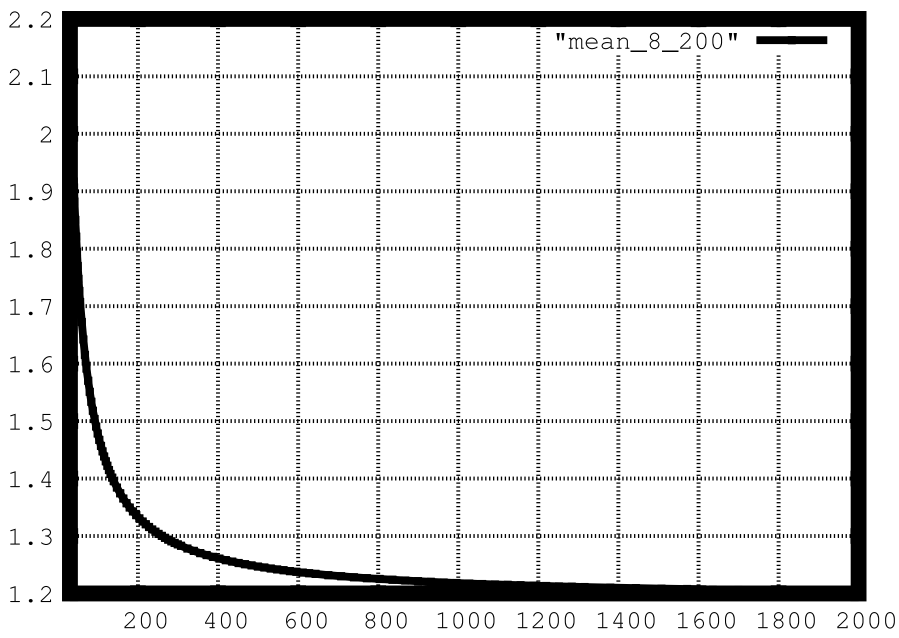
$\nu = 1,$ left: u, right m.



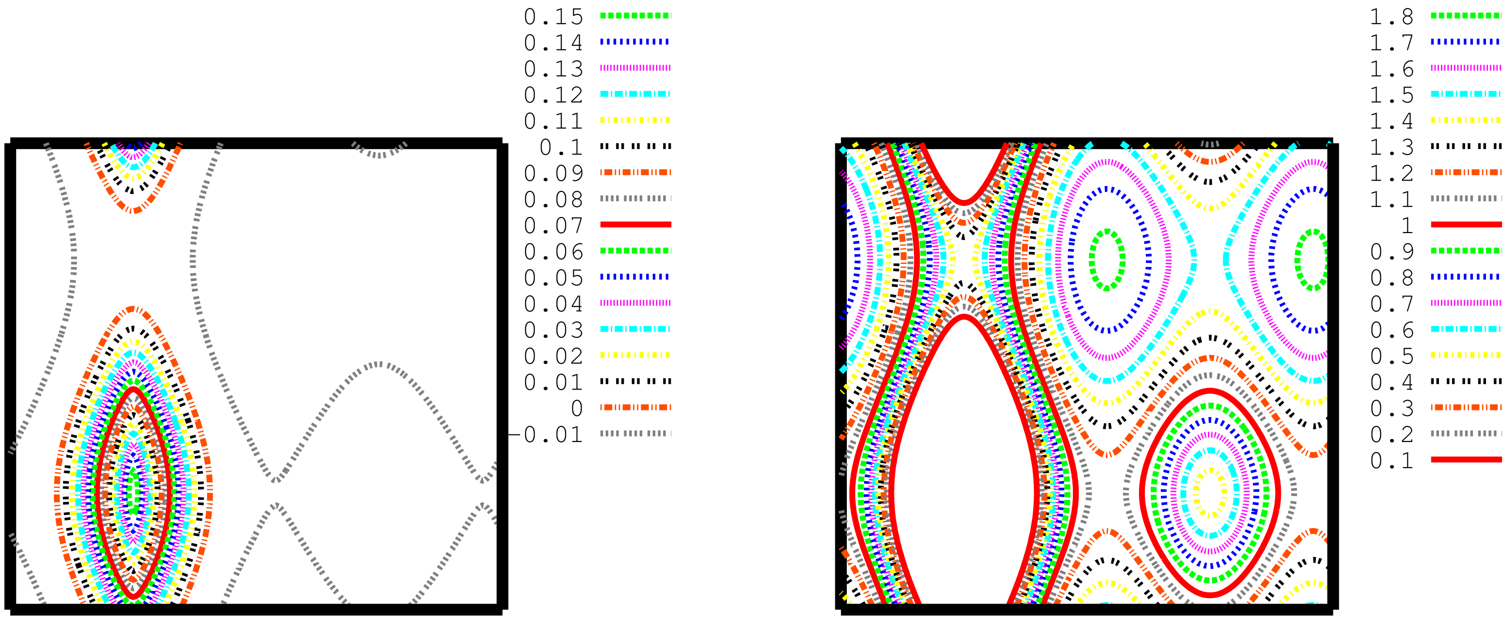
Convergence as $h \to 0$

$$\nu = 0.01,$$

$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2.$$



$\nu = 0.01,$ Convergence $\frac{1}{t}\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$

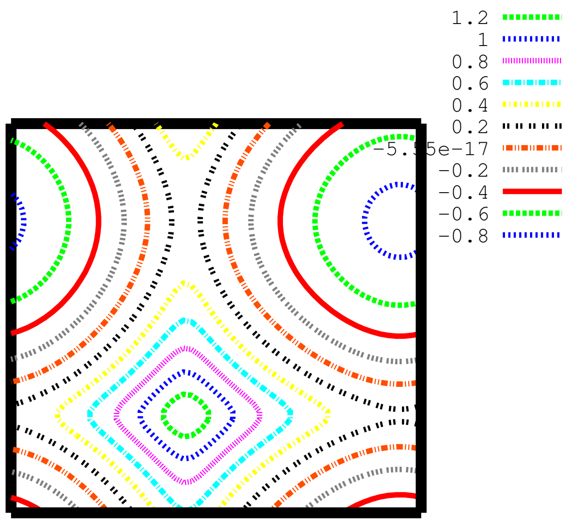


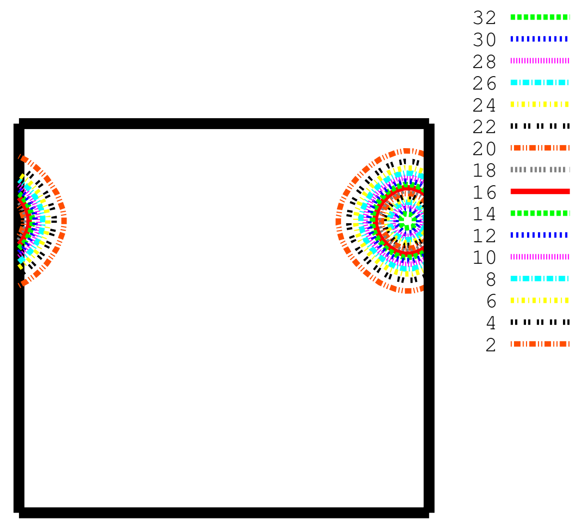
$\nu = 0.01,\qquad$ left: u, $\qquad$ right m.

Note that the supports of ∇u and of m tend to be disjoint as $\nu \to 0$.

$$V[m](x) = F(m(x)) = -\log(m(x)).$$

Same Hamiltonian as before. We now take $\nu = 0.1$.





left: u, right m.

The measure m_h concentrates near the minimum of u_h.

Deterministic limit $\nu \to 0$

Theorem (Lasry-Lions)
If

- $H(x,p) \geq H(x,0) = 0$,
- $V[m] = F(m) + f_0(x)$ where $F' > 0$,

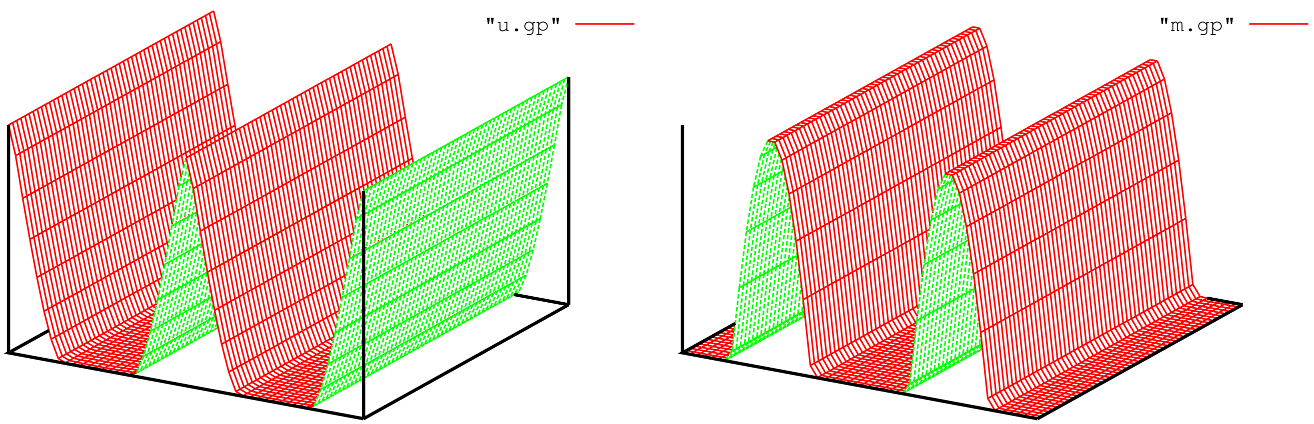
then

$$\lim_{\nu \to 0} (\lambda_\nu, m_\nu) = (\lambda, m),$$

where

$$m(x) = \big(F^{-1}(\lambda - f_0(x))\big)^+ \qquad \text{and} \qquad \int_{\mathbb{T}} m dx = 1.$$

$$\nu = 0.001,$$
$$H(x,p) = |p|^2,$$
$$V[m](x) = 4\cos(4\pi x) + m(x)$$

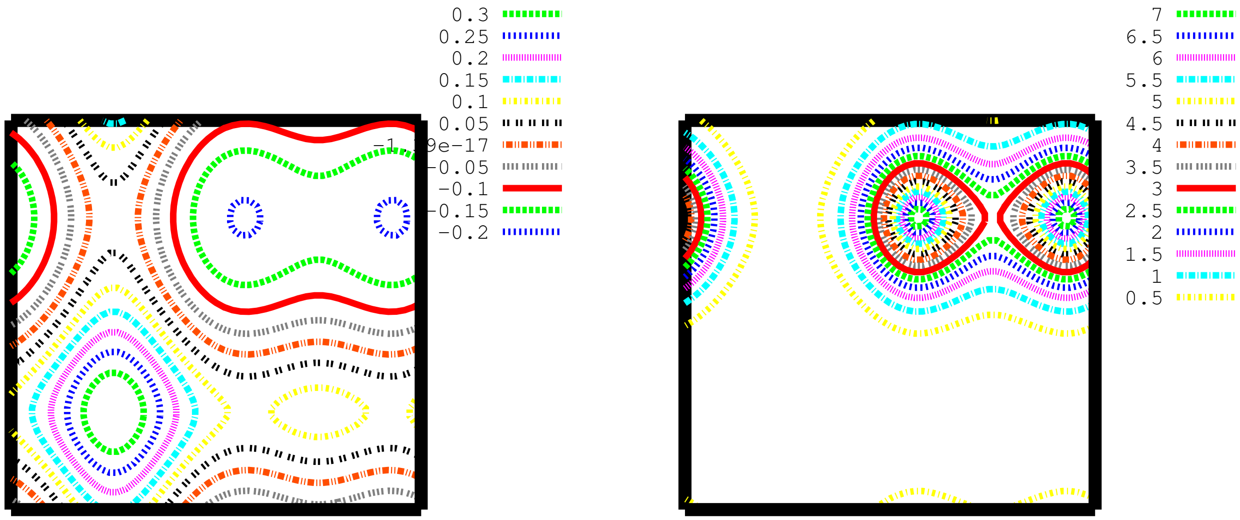


left: u, right m.

The supports of ∇u and of m tend to be disjoint.

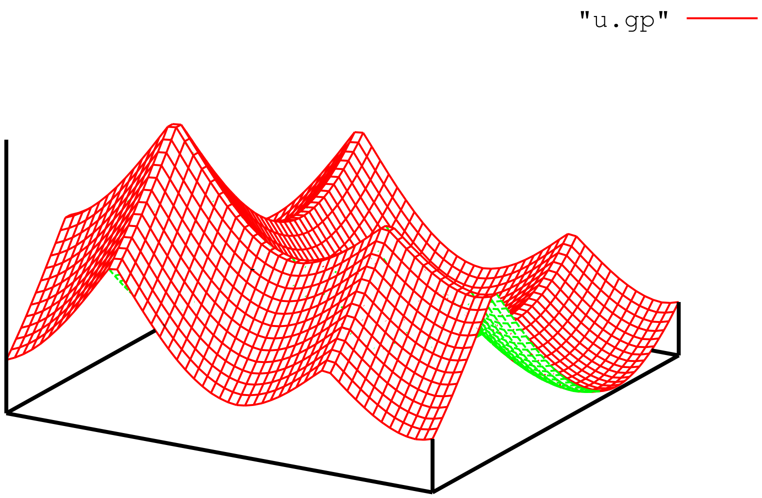
$$m(x) \approx (\lambda - 4\cos(4\pi x))^+$$

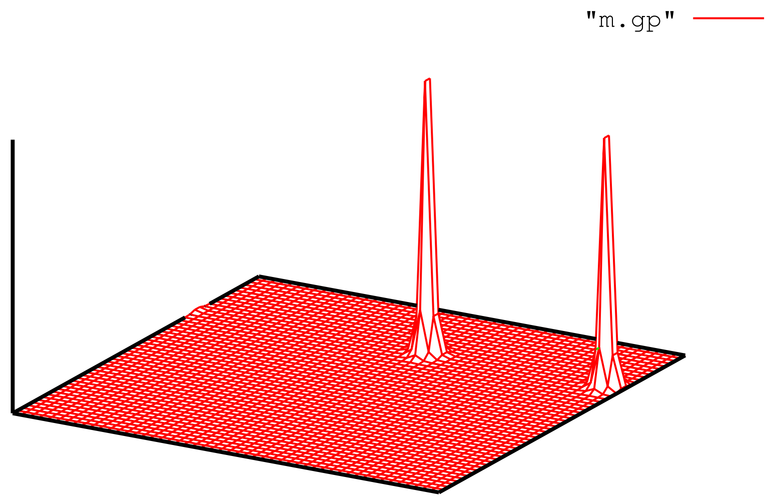
A nonlocal operator V



$$\nu = 0.1,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2},$$
$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

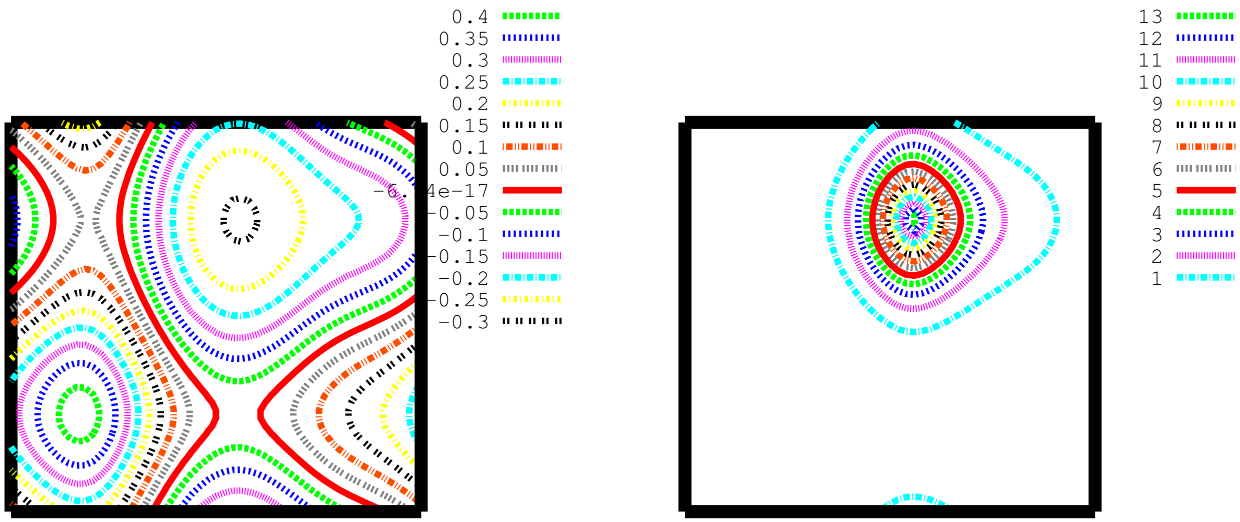
left: u, right m.





$$\nu = 0.001,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$
$$F(x,m) = (1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.



$$\nu = 0.1,$$

$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + (0.6 + 0.59\cos(2\pi x))|p|^{3/2},$$

$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.

V. Finite Horizon: a Newton method

Difficulty: time dependent problem with conditions at both initial and final times

$$\begin{cases} \mathcal{F}_U(\mathcal{U},\mathcal{M}) & = & 0, \\ \mathcal{F}_M(\mathcal{U},\mathcal{M}) & = & 0, \end{cases}$$

Solution procedure: Newton method

$$\begin{pmatrix} \mathcal{U}^{n+1} \\ \mathcal{M}^{n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{U}^{n} \\ \mathcal{M}^{n} \end{pmatrix} - \begin{pmatrix} A_{U,U}(\mathcal{U}^n,\mathcal{M}^n) & A_{U,M}(\mathcal{U}^n,\mathcal{M}^n) \\ A_{M,U}(\mathcal{U}^n,\mathcal{M}^n) & A_{M,M}(\mathcal{U}^n,\mathcal{M}^n) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{F}_U(\mathcal{U}^n,\mathcal{M}^n) \\ \mathcal{F}_M(\mathcal{U}^n,\mathcal{M}^n) \end{pmatrix}$$

where

$$\begin{array}{rclcrcl} A_{U,U}(\mathcal{U},\mathcal{M}) & = & D_{\mathcal{U}}\mathcal{F}_{\mathcal{U}}(\mathcal{U},\mathcal{M}), & & A_{U,M}(\mathcal{U},\mathcal{M}) & = & D_{\mathcal{M}}\mathcal{F}_{\mathcal{U}}(\mathcal{U},\mathcal{M}), \\ A_{M,U}(\mathcal{U},\mathcal{M}) & = & D_{\mathcal{U}}\mathcal{F}_{\mathcal{M}}(\mathcal{U},\mathcal{M}), & & A_{M,M}(\mathcal{U},\mathcal{M}) & = & D_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}(\mathcal{U},\mathcal{M}). \end{array}$$

The linear systems

The most time consuming part of the procedure lies in solving the system of linear equations

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{M} \end{pmatrix} = \begin{pmatrix} G_U \\ G_M \end{pmatrix}.$$

The matrix A_{UU} is block-lower triangular and block-bidiagonal.
The matrix A_{UM} is block-diagonal.
The matrix A_{MM} is block-upper triangular and block-bidiagonal.
The matrix A_{MU} is block-diagonal.

The chosen procedure is as follows:

1. solve first $A_{U,U}\widetilde{\mathcal{U}} = G_U$. This is done by sequentially solving

$$D_k \widetilde{U}^k = -L_k \widetilde{U}^{k-1} + G_U^k, \qquad (1)$$

   i.e. marching in time in the forward direction. (1) are solved with efficient direct solvers.

2. Introducing $\overline{\mathcal{U}} = \mathcal{U} - \widetilde{\mathcal{U}}$,

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \overline{\mathcal{U}} \\ \mathcal{M} \end{pmatrix} = \begin{pmatrix} 0 \\ G_M - A_{M,U}\widetilde{\mathcal{U}} \end{pmatrix},$$

   which implies

$$\left(A_{M,M} - A_{M,U} A_{U,U}^{-1} A_{U,M} \right) \mathcal{M} = G_M - A_{M,U}\widetilde{\mathcal{U}}. \qquad (2)$$

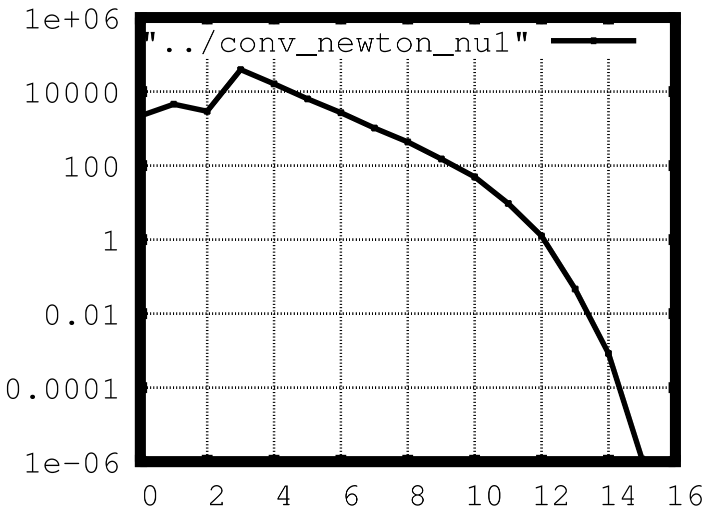
   (2) is solved by an iterative method, e.g. BiCGStab.

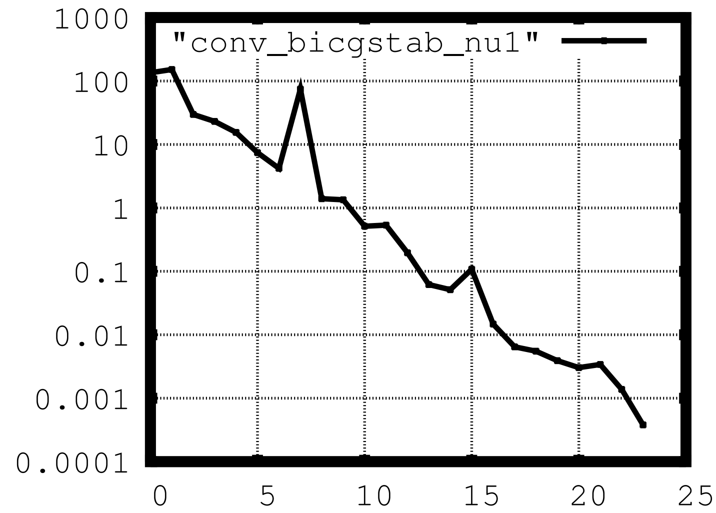
$$\nu = 1, \quad T = 1, \quad \Delta t = h = 1/50,$$

$$m(T) = 1$$

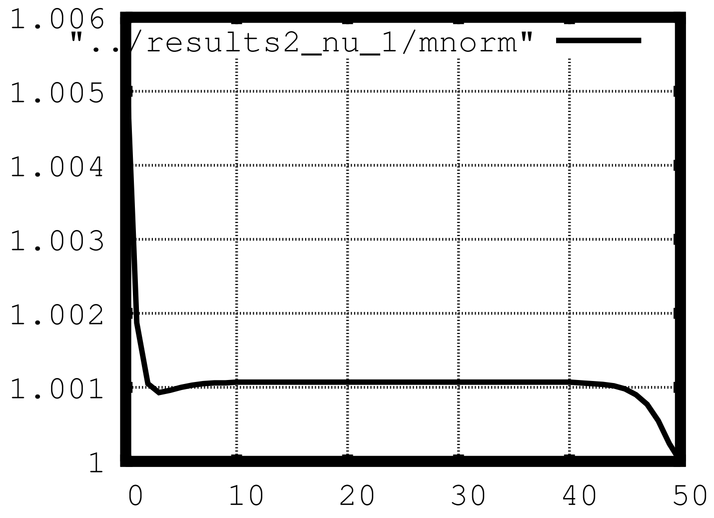
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$

$$F(x,m) = m^2, \qquad V_0[m](x) = m^2 + \cos(\pi x_1)\cos(\pi x_2).$$





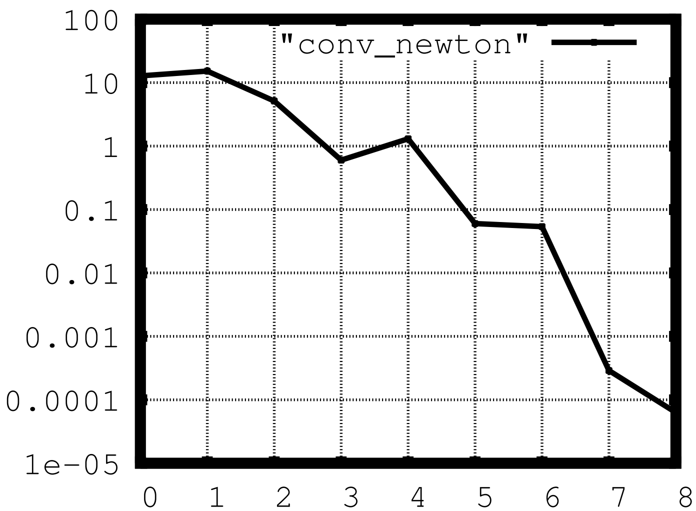
Convergence of the Newton method(left) and of a linear solver (right)

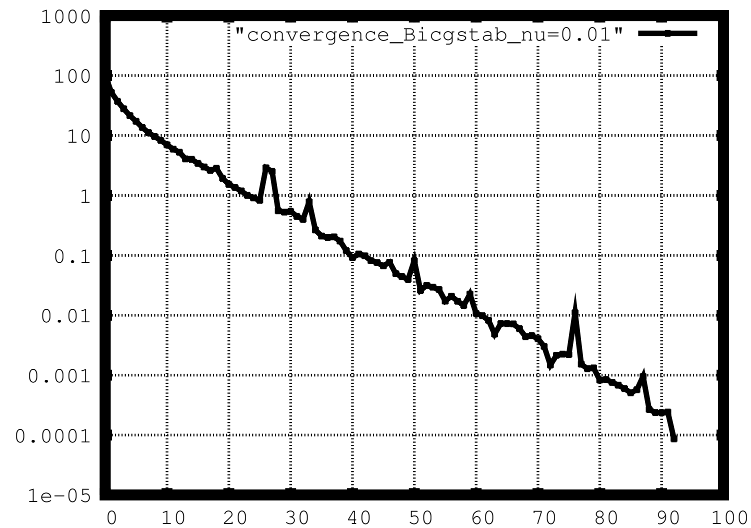


The L^2 norm of m_h (right) and u_h (left) vs. $50\times$ time

Same test except

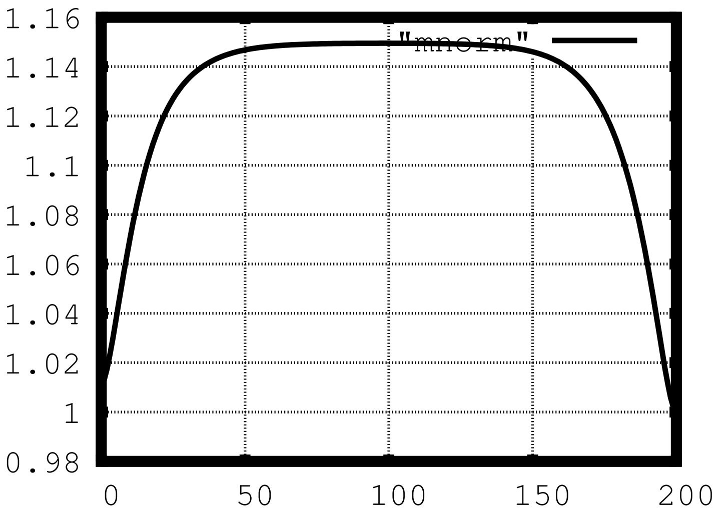
$$\nu = 0.01, \quad \Delta t = 1/200.$$

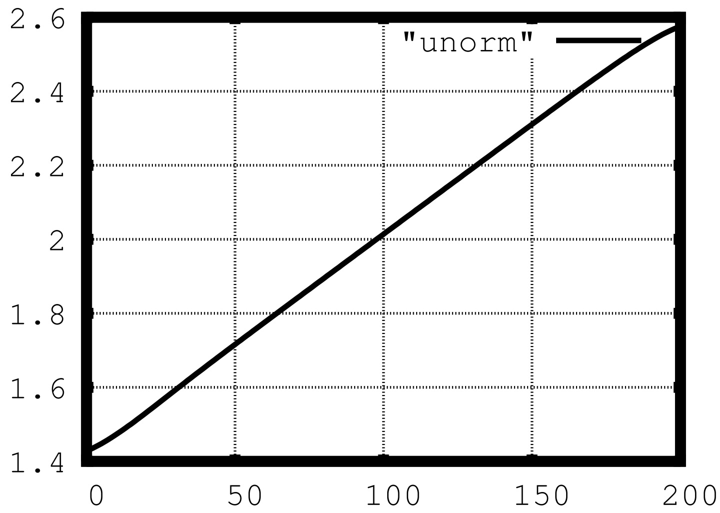




Convergence of the Newton method(left) and of a linear solver (right)

$500,000\ m$ unknowns in the nonlinear system.





The L^2 norm of m_h (right) and u_h (left) vs. $200\times$ time

Perspectives

- Obtain estimates on $\|D_h U\|_\infty$ uniform in h with more general assumptions in the stationary case (important for stability and convergence).
- When convergence is OK, prove error estimates?
- Better understand the Newton method in the finite horizon case.
- Tackle practically relevant situations.

A different strategy

Alternative numerical approach with a reformulation into an optimization problem (A.Lachapelle, J. Salomon, G.Turinici).