Coupling method and Monge-Kantorovich distance

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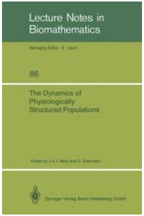
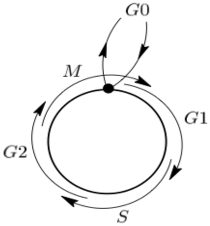
with Nicolas Fournier





Organisation of the talk

1. Coupling method : the heat equation

2. Scattering and Boltzmann Eq.

3. Structured equations : why ?

4. Structured equations : coupling method

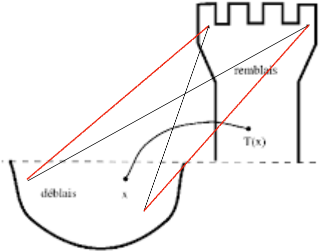
Monge-Kantorovich distance

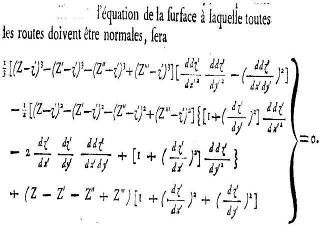
- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x,y) \geq 0,\ x,y \in \Omega, \qquad c(0,0) = 0$
- $n_1,\ n_2$ probability measures on Ω

- **Monge :** $\qquad T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$c(x,y) = |x-y|$$

$$d_1(n_1,n_2) := \min_T \int_\Omega c(x, T(x)) n_1(x) dx$$





l'équation de la ſurface à laquelle toutes les routes doivent être normales, ſera

$$\left.\begin{array}{l} \frac{1}{3}[(Z-\zeta')^3-(Z'-\zeta')^3-(Z''-\zeta')^3+(Z'''-\zeta')^3][\frac{dd\zeta'}{dx'^2}\frac{dd\zeta'}{dy'^2}-(\frac{dd\zeta'}{dx'dy'})^2] \\ -\frac{1}{2}[(Z-\zeta')^2-(Z'-\zeta')^2-(Z''-\zeta')^2+(Z'''-\zeta')^2]\{[1+(\frac{d\zeta'}{dy'})^2]\frac{dd\zeta'}{dx'^2} \\ -2\frac{d\zeta'}{dx'}\frac{d\zeta'}{dy'}\frac{dd\zeta'}{dx'dy'}+[1+(\frac{d\zeta'}{dx'})^2]\frac{dd\zeta'}{dy'^2}\} \\ +(Z-Z'-Z''+Z''')[1+(\frac{d\zeta'}{dx'})^2+(\frac{d\zeta'}{dy'})^2] \end{array}\right\} = 0.$$

Monge-Kantorovich distance

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$$d_c(n_1, n_2) := \min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

• **Kantorovich :**

$$d_c(n_1, n_2) := \min_{v(\cdot,\cdot)} \int_\Omega c(x,y) v(x,y) dx dy$$

$$\int_\Omega v(x,y) dy = n_1(x), \qquad \int_\Omega v(x,y) dx = n_2(y)$$

• Monge is equivalent to choose $v(x,y) = n_1(x)\delta(y = T(x))$

• Kantorovich can be seen as $v(x,y) = n_1(x) \sum a_i \delta(y = T_i(x))$

Monge-Kantorovich distance

- **Monge :** $T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$d_c(n_1, n_2) := \min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

- **Kantorovich :** $d_c(n_1, n_2) := \min_\nu \int_\Omega c(x,y) \nu(x,y) dxdy$

$$\int_\Omega \nu(x,y) dy = n_1(x), \qquad \int_\Omega \nu(x,y) dx = n_2(y)$$

- **Brenier :** For $\Omega = \mathbb{R}^d$, $c(x,y) = |x-y|^2$, n_i 'smooth', T is optimal if and only if

$$T(x) = \nabla \Phi(x) \qquad \text{with} \qquad \Phi : \mathbb{R}^d \to \mathbb{R} \;\; \text{convex}$$

$$\det D^2 \Phi(x) = \frac{n_1(x)}{n_2(T(x))} \qquad \text{Monge-Ampere eq.}$$

(Evans-Gangbo, Cafarelli-Feldman-McCann, Trudinger-Wang, Ambrosio)

Coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \; t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^0, n_2^0)$$

Proof : Consider v solution of

$$\frac{\partial v}{\partial t} - \Delta_x v - \Delta_y v - 2\nabla_x \cdot \nabla_y v = 0, \quad x, y \in \mathbb{R}^d, \; t \geq 0$$

with a compatible initial data

$$\int v^0(x,y)dy = n_1^0(x) \qquad \int v^0(x,y)dx = n_2^0(y)$$

Step 1. $v \geq 0$ because $\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ is nonnegative

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$$\int v^0(x,y)dy = n_1^0(x) \qquad \int v^0(x,y)dx = n_2^0(y)$$

Step 2. Marginals are correct. Integrate in y :

$$\frac{\partial v_1(x,t)}{\partial t} - \Delta_x v_1 = 0$$

Coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \ t \geq 0$$

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Proof : Consider v solution of

$$\frac{\partial v}{\partial t} - \Delta_x v - \Delta_y v - 2\nabla_x \cdot \nabla_y v = 0, \quad x, y \in \mathbb{R}^d, \ t \geq 0$$

Step 3. The distance diminishes

$$\frac{d}{dt}\int c(x-y)v(x,y,t)dxdy =$$
$$\int v(x,y,t)\Big(\Delta_x c(x-y) + \Delta_y c(x-y) + 2\nabla_x \nabla_y c(x-y|)\Big)dxdy$$
$$= \int v(x,y,t)\Big(\Delta_{x+y} c(x-y)\Big)dxdy = 0$$

Coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \ t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^0, n_2^0)$$

Step 4. Conclusion. For all initial coupling v^0

$$d_c(n_1(t), n_2(t)) \leq \int c(x-y)v(x,y,t)dxdy$$

$$\leq \int c(x-y)v^0(x,y)dxdy$$

$$\approx d_c(n_1^0, n_2^0)$$

Open question : Which costs for $\partial_t n - \Delta(A(x)n) = 0$?

Except dimension 1 for d_1

Transport equation

$$\partial_t n + \mathrm{div}\big(b(x)n(x,t)\big) = \Delta(An), \quad t \geq 0, \ x \in \mathbb{R}^d,$$

$$n^{init} \in P^2$$

and assume $A = \sigma.\sigma^t$, with σ Lipschitz,

$$(b(x) - b(y), x - y) \leq C|x - y|^2$$

Theorem. There is a unique 'solution' in the sense that

$$d_2(n, n_\varepsilon) \leq [d_2(n^{init}, n_\varepsilon^{init}) + C\varepsilon]e^{Ct}$$

for all solutions of

$$\partial_t n_\varepsilon + \mathrm{div}\big(b(x)n_\varepsilon(x,t)\big) = \Delta(An_\varepsilon) + \varepsilon\Delta(B_\varepsilon n_\varepsilon),$$

or any other compatible approximation.

See also Lebris-Lions,
$A = 0$: Bouchut, James, Mancini,Prignet, Eymard, Bianchini, Gloyer...

Organisation of the talk

1. Coupling method : the heat equation

2. Scattering and Boltzmann Eq.

3. Structured equations : why ?

4. Structured equations : coupling method

Scattering and Boltzmann Eq.

$$x \mapsto X = \Phi(x,h) \text{ distribution of jumps}$$
$$D_x\Phi(x,h) \text{ is an invertible matrix}$$
$$X \mapsto x = \Phi^{-1}(X,h) \text{ inverse in } x \text{ with } h \text{ fixed}$$

We consider the scattering problem

$$\partial_t n(x,t) = \int \Big[n(\Phi(x,h),t)\det(D_x\Phi(x,h)) - n(x,t)\Big] d\mu(h)$$

Integrating against a test function $\varphi(x)$,

$$\frac{d}{dt}\int \varphi(x)n(x,t)dx = \int n(X,t)[\varphi(\Phi^{-1}(X,h)) - \varphi(X)]dXd\mu(h)$$

Scattering and Boltzmann Eq.

Theorem Suppose that for all $X, Y \in \mathbb{R}^d$,

$$\int |\Phi^{-1}(X,h) - \Phi^{-1}(Y,h)|^p d\mu(h) \leq L|X-Y|^p$$

then for all $t \geq 0$,

$$d_p(n_1(t), n_2(t)) \leq e^{(L-1)t} d_p(n_1^0, n_2^0)$$

Proof. Introduce the coupling **with the same jumps**

$$\partial_t v(x,y,t) = \int \Big[v(\Phi(x,h), \Phi(y,h), t) \det(D_x\Phi(x,h)) \det(D_x\Phi(y,h)) \\ - v(x,y,t) \Big] d\mu(h)$$

- $v \geq 0$
- Marginales are correct

$$\frac{d}{dt}\int |x-y|^p v = \int |\Phi^{-1}(X,h) - \Phi^{-1}(Y,h)|^p |v(X,Y,t) DXdY \\ - \int |x-y|^p v$$

Scattering and Boltzmann Eq.

Homogeneous Boltzmann Eq.

$$\begin{cases} \partial_t f(v,t) = \int_{\mathbb{R}^3\times\mathbb{S}^2} [f(v',t)f(v'_*,t) - f(v,t)f(v_*,t)]B(\theta)dv_*d\sigma \\ v' = \frac{1}{2}(v+v_*) + \frac{1}{2}|v-v_*|\sigma, \quad v'_* = \frac{1}{2}(v+v_*) - \frac{1}{2}|v-v_*|\sigma \\ \cos(\theta) = \frac{v-v_*}{|v-v_*|}\cdot\frac{v'-v'_*}{|v'-v'_*|}, \qquad \int_0^\pi B(\theta)d\theta = 1 \end{cases}$$

Theorem (Tanaka,1978).

The Boltzmann equation is non-expansive for d_2.

Scattering and Boltzmann Eq.

Homogeneous Boltzmann Eq.

$$\frac{d}{dt}\int_{(\mathbb{R}^3)^2}\Psi(v,w)F(v,w,t)dvdw$$
$$=\int_{(\mathbb{R}^3)^4}\int_0^{\pi}\int_0^{2\pi}\left[\Psi(v',w')+\Psi(v'_*,w'_*)-\Psi(v_*,w_*)-\Psi(v,w)\right]$$
$$B(\theta)F(v,w,t)F(v_*,w_*,t)dvdwdv_*dw_*d\theta d\varphi$$

$$v'=\frac{1}{2}(v+v_*)+\frac{1}{2}|v-v_*|\sigma,\quad v'_*=\frac{1}{2}(v+v_*)-\frac{1}{2}|v-v_*|\sigma$$
$$w'=\frac{1}{2}(w+w_*)+\frac{1}{2}|w-w_*|\omega,\quad w'_*=\frac{1}{2}(w+w_*)-\frac{1}{2}|w-w_*|\omega$$

Scattering and Boltzmann Eq.

and the sphere is parametrized as

$$\sigma = \cos(\theta)\frac{v - v_*}{|v - v_*|} + \sin(\theta)[l\cos(\varphi) + l_1\sin(\varphi)]$$

$$\omega = \cos(\theta)\frac{w - w_*}{|w - w_*|} + \sin(\theta)[l\cos(\varphi) + l_2\sin(\varphi)],$$

where $l = \frac{(v-v_*)\wedge(w-w_*)}{|(v-v_*)\wedge(w-w_*)|}$,

and $l_1,\ l_2$ are chosen so that

$$\left(\frac{v - v_*}{|v - v_*|}, l, l_1\right), \qquad \left(\frac{w - w_*}{|w - w_*|}, l, l_2\right)$$

are two direct orthonormal bases

Coupling method

- **Kinetic scattering** with the same jump condition as before

$$\partial_t n + v.\nabla n + n(t,x,v) = \int n(t,x,V(v;h))\mathrm{det} D_v V d\mu(h)$$

is non-expansive for $\mathbf{d_1}$, $c(x,v,y,w) = \alpha|x-y| + |v-w|$

- **Theorem (Otto)** The porous media equation is non-expansive for $\mathbf{d_2}$

$$\frac{\partial n}{\partial t} - \Delta A(n) = 0$$

Proof uses the Brenier map and Monge-Ampere equation (see also Bolley-Carrillo)

No proof known by 'coupling'

- **Golse-Paul :**

• There is a 'quantic MK distance'.
• It can be used for the limit of quantum particles

Organisation of the talk

Examples of stuctured equations

The Kermack-McKendrick renewal equation (1927)

$$\begin{cases} \frac{d}{dt}S(t) = B - \mu_S S(t) - I(t)S(t) \\ I(t) := \int_0^\infty \beta(s) n_I(t,s) ds \\ \frac{\partial}{\partial t} n_I(t,s) + \frac{\partial}{\partial s} n_I(t,s) + (\mu_I + \gamma(s)) n_I(t,s) = 0 \\ n_I(t, s=0) = I(t)S(t) \end{cases}$$

Theorem (Magal, McCluskey, Webb, 2010). Define

$$\mathcal{E}(t) = \int_0^\infty \psi(s) \bar{n}_I(s) \Big[\frac{n_I(t,s)}{\bar{n}_I(s)} - \ln \frac{n_I(t,s)}{\bar{n}_I(s)} \Big] ds + \bar{S} \ln S(t) - S(t).$$

Then, we have

$$\frac{d}{dt}\mathcal{E}(t) \le -D(t) \le 0, \qquad D(t) = \frac{\mu_S}{S(t)} (\bar{S} - S(t))^2.$$

The renewal equation

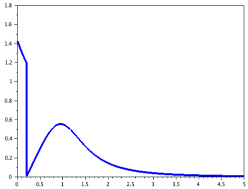
$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad t \geq 0,\ x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \\ n(t=0, x) = n^0(x) \end{cases}$$

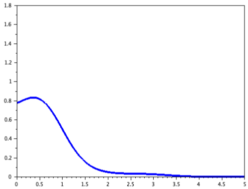
$$b,\ d \in L^\infty_+(0,\infty).$$

- Very useful (demography, cell cycle, anomalous diffusions)
- Very standard (Feller)
- Nonlinear versions are complex

The renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad t \geq 0,\ x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y, I(t)) n(t,y) dy \\ I(t) = \int q(x) n(t,x) dx \end{cases}$$





Growth-fragmentation/TCP

Another useful equation

$$\begin{cases} \dfrac{\partial n(t,x)}{\partial t} + \dfrac{\partial [g(x)n(t,x)]}{\partial x} + b(x)n(t,x) = 2\displaystyle\int_x^\infty b(y)\kappa(x,y)n(t,y)dy \\ n(t,x=0)=0, \qquad g(0)>0, \\ n(t=0,x)=n^0(x). \end{cases}$$

- $b(y) =$ is the division rate of cells/polymers/messages of sizes y
- $\kappa(x,y) = 0 \qquad$ for $x > y$
- $\int_0^y \kappa(x,y)dx = 1, \qquad \int_0^y x\,\kappa(x,y)dx = y/2$
- $\frac{d}{dt}\int_0^\infty n(t,x)dx = \int_0^\infty b(x)n(t,x)dx$
- $\frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx.$

Many other structured equations

Age and size structured

$$\begin{cases} \frac{\partial n(t,x,z)}{\partial t} + \frac{\partial n(t,x,z)}{\partial x} + \frac{\partial [g(z)\, n]}{\partial z} + d(x,z)\, n(t,x,z) = 0, \quad x > 0,\ z > 0 \\ n(t,x,z=0) = 0, \\ n(t,x=0,z) = \int_{x=0}^{\infty} \int_{z'=z}^{\infty} d(x',z')\kappa(z,z')n(t,x',z')dx'dz \end{cases}$$

Age and space structured

$$\begin{cases} \frac{\partial n(t,x,z)}{\partial t} + \frac{\partial n(t,x,z)}{\partial x} + d(x)n(t,x,z) = 0, \quad x > 0,\ z \in \mathbb{R}^d \\ n(t,x=0,z) = \int_0^{\infty}\int_{\mathbb{R}^d} d(x)n(t,x,z+\varepsilon\eta)k(\eta)dxd\eta \end{cases}$$

- Doumic, M. ; Hoffmann, M. ; Krell, N. ; Robert, L. et al (2015)
- Berry, H. ; Lepoutre, T. ; González, A. ; Acta Appl. Math. (2016)
- Calvez, V. ; Gabriel, P. ; Mateos G. ; Asymptot. Anal. (2019)
- Franck M. ; Goudon T. ; KRM (2018)

Which general tools ?

- Generalized relative entropy

- Modified Monge-Kantorovich distance

Organisation of the talk

1. Why structured equations ?

2. Renewal and growth fragmentation eqs

3. Structured equations

3bis. Generalised relative entropy

4. Monge-Kantorovich distance

Generalized relative entropy

All (linear) equations preserving positivity satisfy the GRE

$$\mathcal{L}\varphi + \lambda_0\varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^*\psi + \lambda_0\psi = 0, \quad \psi > 0$$

Let

$$\frac{\partial n(t)}{\partial t} + \mathcal{L}n(t,x) = 0.$$

GRE Principle (Michel, Mischler, P., 2004). For $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t}\frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt}\int \psi(x)\varphi(x)H\big(u(t,x)\big)\,dx = -D_H(t) \leq 0$$

Also true for any triplet $(n_1(t,x), n_2(t,x), \psi(t,x))$.

Usual relative entropy for conservative systems correspond to $\psi \equiv 1$

Generalized relative entropy

All (linear) equations preserving positivity satisfy the GRE

$$\mathcal{L}\varphi + \lambda_0\varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^*\psi + \lambda_0\psi = 0, \quad \psi > 0$$

Let

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GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t}\frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt}\int \psi(x)\varphi(x)H\big(u(t,x)\big)\,dx = -D_H(t) \le 0$$

- $\int \psi(x)n(t,x)dx = e^{\lambda_0 t}\int \psi(x)n^0(x)dx$ Conservation law
- $\|u(t,\cdot\|_\infty$ decreases
- $\|\frac{d}{dt}\ln n(t,x)\|_\infty$ decreases (2nd Collatz-Wielandt formula)
- Explains Kermack-McKendrick $H(u) = u - \ln u$.

Generalized relative entropy

Spectral gap. Aim is to find $\lambda_1 > 0$ such that

$$\|e^{-\lambda_0 t} n(t,x) - \rho^0 \varphi\| \leq C e^{-\lambda_1 t}$$

- Poincaré inequality : when $\int \psi \varphi u(x) = 0$

$$\lambda_1 \int \psi \varphi H(u) dx \leq D_H(u)$$

- Doeblin's method
- Method of integral equation
- Bacry-Emery's method

(Ryzhik, BP, Doumic, Gabriel, Mischler, Cañizo, Yoldas, Laurencot)

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Structured equation

Theorem. (N. Fournier, BP)
The Renewal and Growth-Fragmentation equations (and many others) are non-expansive for the cost type$^{(1)}$

$$c(x-y) = \min(|x-y|, a), \qquad a \text{ related to Lipschitz bounds of coef.}$$

History.
Fournier and Locherbach (neuron networks)
Chafai, Malrieu, Paroux, Guillin, Zitt... (TCP connections)

Structured equation

Renewal equation as an example.

$$\begin{cases} \frac{\partial n(x,t)}{\partial t} + \frac{\partial [g(x)n]}{\partial x} + d(x)n = b(x)N(t), & t \geq 0,\ x \geq 0, \\ n(x=0,t) = 0, \qquad N(t) = \int_0^\infty d(x)n(x,t)dx. \end{cases}$$

$$g' \leq 0, \qquad g(0) \geq 0, \qquad \int_0^\infty b = 1$$

$$0 < a < 1 \qquad \text{and} \qquad a \leq \inf_{|x-y|<1} \frac{|x-y| \max(d(x), d(y))}{|d(x) - d(y)|}$$

Example : $d(x) = \alpha + \beta x^p$, $p \geq 1$.

$$c(x-y) = \min(|x-y|, a)$$

$$d_{MK}(n_1(t), n_2(t)) \leq d_{MK}(n_1^0, n_2^0)$$

Structured equation

$$\frac{\partial v}{\partial t} + \frac{\partial g(x)v}{\partial x} + \frac{\partial g(y)v}{\partial y} + \max(d(x), d(y))v$$
$$= b(x)\delta(x-y)\int \min(d(x'), d(y'))\, v(dx', dy', t)$$
$$+ b(x)\int \left(d(x') - d(y)\right)_+ v(dx', y, t)$$
$$+ b(y)\int \left(d(y') - d(x)\right)_+ v(x, dy', t).$$

with an initial data v^0 whose marginals are n_1^0 and n_2^0.

$$c(x,y) \geq \int c(z,y)b(z)dz \frac{\left(d(x)-d(y)\right)_+}{\max(d(x),d(y))}$$
$$+ \int c(x,z)b(z)dz \frac{\left(d(y)-d(x)\right)_+}{\max(d(x),d(y))}$$

Conclusion

- The coupling method is a simple and general tool for several conservative PDEs and a few nonlinear PDEs
- The dual problem is also used
- Mainly used to study limits of particle systes/long time behavior
- Connected to other doubling of variables ?

THANK YOU