Radiative Transfer in Fluids: Mathematical Analysis and Numerical Simulations

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What is Radiative Transfer?

Radiative Transfer studies the interaction of electromagnetic radiation viewed as a gas of photons with a background medium (fluid or plasma, e.g. stellar or planetary atmospheres)

How the various frequencies in the radiation coming from the Sun interact with the atmosphere of the Earth is important in the understanding of

•the blue color of a cloudless sky (J.W. Strutt Rayleigh 1871)
•the greenhouse effect (J. Fourier 1824)

The Radiative Intensity

Radiative intensity

$$I_\nu(t,\vec{x},\vec{\omega});= ch\nu \underbrace{f(t,\vec{x},\vec{\omega},\nu)}_{\text{photon \# density}}\;,\quad \vec{x}\in\mathbf{R}^3\,,\ |\vec{\omega}|=1$$

Example: Planck's function for a black body at temperature T

$$B_\nu(T):=\frac{2h\nu^3}{c^2(e^{h\nu/kT}-1)}$$

Stefan-Boltzmann law

$$\pi\int_0^\infty B_\nu(T)d\nu=\Sigma_R T^4\,,\qquad \Sigma_R:=\tfrac{2\pi^5k^4}{15c^2h^3}$$

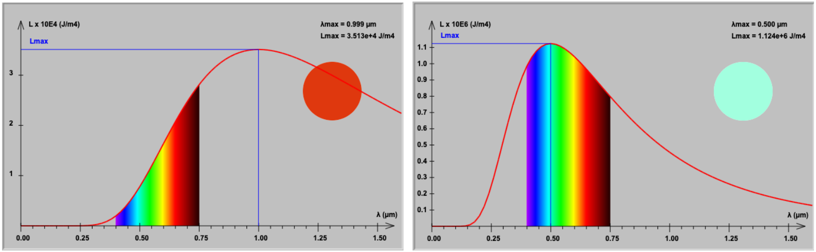


Figure: Planck's function in terms of the wavelength at temperatures $2900K$ (left) and $5800K$ (right). For a black body at $288K$ (Earth's mean temperature), most of the emitted radiation is infrared.

Source: ressources.univ-lemans.fr

Kinetic equation for the radiative intensity

$$\left(\tfrac{1}{c}\partial_t + \vec{\omega}\cdot\nabla_{\vec{x}}\right)I_\nu + \underbrace{\rho\bar{\kappa}_\nu I_\nu}_{\text{absorption}} = \underbrace{\rho\bar{\kappa}_\nu(1-a_\nu)B_\nu(T)}_{\text{LTE reemission}} + \underbrace{\rho\bar{\kappa}_\nu a_\nu J_\nu}_{\text{scattering}}$$

mean radiative intensity $\quad J_\nu(t,\vec{x}) := \frac{1}{4\pi}\int_{\mathbf{S}^2} I_\nu(t,\vec{x},\vec{\omega})d\vec{\omega}$

where

- $\rho =$ background fluid density
- $\bar{\kappa}_\nu =$ absorption coefficient
- $[0,1] \ni a_\nu =$ scattering albedo

Local Thermodynamic Equilibrium (LTE) relaxation to the Planck function, analogous to a BGK model in the kinetic theory of gases

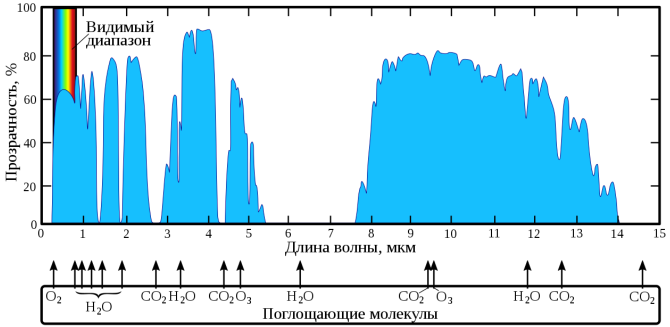


Figure: Transmittance of Earth's atmosphere in terms of the wavelength.
Source: commons.wikimedia.org/wiki/File:Atmosfaerisk spredning-ru.svg

Coupling Radiation to the Fluid Energy Balance

Energy balance in an incompressible fluid with radiation

$$\partial_t \underbrace{\left(\rho(\tfrac{1}{2}|\vec{u}|^2 + c_V T) + \frac{4\pi}{c}\int_0^\infty J_\nu d\nu\right)}_{\text{kinetic+internal+radiative energy}} + \nabla_{\vec{x}}\cdot((p - \rho\vec{g}\cdot\vec{x})\vec{u})$$

$$+\nabla_{\vec{x}}\cdot\underbrace{\left(\rho\vec{u}(\tfrac{1}{2}|\vec{u}|^2 + c_V T) + \int_0^\infty\int_{\mathbf{S}^2}\vec{\omega}I_\nu d\vec{\omega}d\nu\right)}_{\text{kinetic+internal+radiative energy flux}}$$

$$= \nabla_{\vec{x}}(\rho c_P \kappa_T \nabla_{\vec{x}} T) + \nabla_{\vec{x}}(\mu_F(\nabla_{\vec{x}} u + (\nabla_{\vec{x}} u)^T) \cdot \vec{u})$$

where

- c_V, c_P = specific heats
- κ_T = heat diffusivity
- μ_F = viscosity
- $\vec{g}$ = gravitational field

Heat Equation

Substracting the kinetic energy balance leads to the heat equation

$$\rho c_V(\partial_t + \vec{u}\cdot\nabla_{\vec{x}})T = 4\pi\int_0^\infty \rho\bar{\kappa}_\nu(1-a_\nu)(J_\nu - B_\nu(T))d\nu$$
$$+\nabla_{\vec{x}}(\rho c_P \kappa_T \nabla_{\vec{x}} T) + \underbrace{\tfrac{1}{2}\mu_F|\nabla_{\vec{x}}u + (\nabla_{\vec{x}}u)^T|^2}_{\text{viscous heating}}$$

Simplifying assumptions Henceforth assume

•that $|\vec{u}| \ll 1$ and neglect viscous heating

•the radiative intensity is quasi-static

$$\tfrac{1}{c}|\partial_t I_\nu(t,\vec{x},\vec{\omega})| \ll 1$$

•the radiative intensity is slowly varying in the horizontal variables

$$\vec{x} = (x,y,z) \text{ and } |\partial_x I_\nu(t,\vec{x},\vec{\omega})| + |\partial_y I_\nu(t,\vec{x},\vec{\omega})| \ll 1$$

$\Longrightarrow$ **stratified** radiative transfer

With $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$, set

$$\mu = \omega_z = \cos\theta \quad \text{and} \quad (\omega_x, \omega_y) = \sin\theta(\cos\alpha, \sin\alpha)$$

and

$$\mathcal{I}_\nu(t, \vec{x}, \mu) := \tfrac{1}{2\pi}\int_0^{2\pi} I_\nu(t, \vec{x}, \vec{\omega})d\alpha \implies J_\nu(t, \vec{x}) = \tfrac{1}{2}\int_{-1}^{1} \mathcal{I}_\nu(t, \vec{x}, \mu)d\mu$$

The coupled radiative transfer+temperature equation becomes

$$\begin{cases}(\partial_t + \vec{u}\cdot\nabla_{\vec{x}})T - \frac{c_P}{c_V}\kappa_T\Delta_{\vec{x}}T = 4\pi\int_0^\infty \frac{\bar{\kappa}_\nu}{c_V}(1-a_\nu)(J_\nu - B_\nu(T))d\nu \\ \mu\partial_z\mathcal{I}_\nu + \rho\bar{\kappa}_\nu\mathcal{I}_\nu = \rho\bar{\kappa}_\nu a_\nu J_\nu + \rho\bar{\kappa}_\nu(1-a_\nu)B_\nu(T)\end{cases}$$

Assume that $(x, y) \in \mathcal{O}$ (bounded domain in $\mathbf{R}^2$) and $z \in (0, Z)$, and set $\Omega := \mathcal{O} \times (0, Z)$.

Assume that $\rho =$ Const., that $\vec{u}$ satisfies $\nabla_{\vec{x}} \cdot \vec{u} = 0$ is a solution of the Navier-Stokes equations. Denoting $\kappa_\nu = \rho \bar{\kappa}_\nu$

$$\begin{cases} \vec{u} \cdot \nabla_{\vec{x}} T - \frac{c_P}{c_V} \kappa_T \Delta_{\vec{x}} T = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu)(J_\nu - B_\nu(T)) d\nu \\ \mu \partial_z \mathcal{I}_\nu + \kappa_\nu \mathcal{I}_\nu = \kappa_\nu a_\nu J_\nu + \kappa_\nu (1 - a_\nu) B_\nu(T) \\ \\ \mathcal{I}_\nu(0, \mu) = \mu Q_\nu^+, \quad \mathcal{I}_\nu(Z, -\mu) = \mu Q_\nu^-, \quad 0 < \mu < 1 \\ \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0, \quad \vec{u}\Big|_{\partial\Omega} = 0 \end{cases}$$

Algorithm: Iteration on the Sources

(1) Data: $Q_\nu^\pm$ and

$$S_\nu(\vec{x}) := \tfrac{1}{2}\int_0^1 \mu(e^{-\kappa_\nu z/\mu} Q_\nu^+(x,y) + e^{-\kappa_\nu (Z-z)/|\mu|} Q_\nu^-(x,y))d\mu$$

(2) Choose $T^0(\vec{x})$ and $J_\nu^0(\vec{x}) = S_\nu(\vec{x})$ for all $\vec{x} \in \Omega$

(3) For all $(x,y) \in \mathcal{O}$ do

(a) for all $0 < z < Z$ and $\nu > 0$, compute $J^n(\vec{x})$ by

$$J_\nu^n(\vec{x}) = S_\nu(\vec{x}) + \int_0^Z \tfrac{\kappa_\nu}{2} E_1(\kappa_\nu|z-\zeta|)(a_\nu J_\nu^{n-1} + (1-a_\nu)B_\nu(T^{n-1})))(x,y,\zeta)d\zeta$$

(b) compute $T^n(\vec{x})$ by solving

$$\begin{cases} \vec{u}\cdot\nabla_{\vec{x}} T^n - \tfrac{c_P}{c_V}\kappa_T \Delta_{\vec{x}} T^n = \tfrac{4\pi}{\rho c_V}\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu^n - B_\nu(T^n))d\nu \\ \tfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0 \end{cases}$$

(4) end for; (5) return T

Notation for the exponential integral

$$E_1(\theta) := \int_\theta^\infty \frac{e^{-y}}{y} dy\,, \qquad C_1(k) := \int_0^{kZ/2} E_1(\theta) d\theta$$

Thm A Assume $0 \leq a_\nu \leq a_M < 1$ and $0 < \kappa_m \leq \kappa_\nu \leq \kappa_M$, and $0 \leq Q_\nu^\pm \leq B_\nu(T_M)$, and set $T^0(\vec{x}) = 0$.
(a) One has

$$0 \leq S_\nu = J^0 \leq J^1 \leq \ldots \leq J_\nu^n \leq J_\nu^{n+1} \leq B_\nu(T_M)$$
$$0 = T^0 \leq T^1 \leq \ldots \leq T^n \leq T^{n+1} \leq T_M$$

(b) One has $(J_\nu^n, T^n) \to (J_\nu, T)$ solution of the radiative transfer+heat equation system in the limit as $n \to \infty$

Pbms Convergence rate? Uniqueness of the solution?

Thm B Under the same assumptions as Thm A, and if

$$\sup_{\nu>0}((1-a_\nu)C_1(\kappa_\nu)) + \sup_{\nu>0}(a_\nu C_1(\kappa_\nu)) = \gamma < 1$$

(a) The algorithm above converges exponentially fast

$$\int_\Omega \int_0^\infty (|J_\nu - J_\nu^n| + \kappa_\nu(1-a_\nu)|B_\nu(T)-B_\nu(T^n)|)d\nu d\vec{x}$$
$$\leq \tfrac{\gamma^n|\Omega|}{1-\gamma}(1+\tfrac{1}{\kappa_m(1-a_M)}) \int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_M)d\nu$$

(b) There exists at most one solution such that

$$0 \leq T \in L^\infty(\Omega) \quad \text{and} \quad I_\nu \geq 0 \text{ a.e. on } \Omega \times \mathbf{S}^2 \times (0,\infty)$$

Consider

$$\mathcal{B}(T) := \int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T) d\nu \,, \qquad \text{increasing on } (0,+\infty)$$

Lemma 1 For all $R \in L^{6/5}(\Omega)$ there exists a weak solution of

$$(H) \qquad -\lambda \Delta_{\vec{x}} T + \vec{u} \cdot \nabla_{\vec{x}} T + \mathcal{B}(T_+) = R \,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0$$

(1) If $R \geq 0$ a.e. on Ω and $|\{\vec{x} \in \Omega \text{ s.t. } R(\vec{x}) > 0\}| > 0$, the solution T of (H) is unique and $T \geq 0$ a.e. on Ω;

(2) If $R' \in L^{6/5}(\Omega)$ and $R' \geq R$ a.e. on Ω the weak solution T' of (H) with r.h.s. R' satisfies $T \leq T'$ a.e. on Ω.

For all $\epsilon > 0$, apply the Leray-Lions theorem [BSMF 1965] to the (nonlinear) functional $\mathcal{A}_\epsilon : H^1(\Omega) \to H^1(\Omega)'$ defined by the formula

$$\langle \mathcal{A}_\epsilon T, \phi \rangle := \int_\Omega (\epsilon T\phi + \nabla_{\vec{x}} T \cdot (\lambda \nabla_{\vec{x}}\phi + \phi u) + \mathcal{B}(T_+)\phi) d\vec{x}$$

Since B_ν is increasing for each $\nu > 0$, the function $\mathcal{B}$ is increasing

$$\langle \mathcal{A}_\epsilon T - \mathcal{A}_\epsilon T', T - T' \rangle = \epsilon \| T - T' \|^2_{L^2} + \| \nabla_{\vec{x}}(T - T')\|^2_{L^2} + \int_\Omega \underbrace{(T - T')(\mathcal{B}(T) - \mathcal{B}(T'))}_{\geq 0} d\vec{x}$$

Let $\epsilon \to 0$, by Rellich and Banach-Alaoglu, find T_{ϵ_n} s.t. for $p \in [1, 6)$

$$\|T_\epsilon\|_{L^2} = O\left(\frac{1}{\sqrt{\epsilon}}\right), \quad T_{\epsilon_n,+} \to T_+ \text{ in } L^p(\Omega), \quad \nabla_{\vec{x}} T_{\epsilon_n} \rightharpoonup \nabla_{\vec{x}} T \text{ in } L^2(\Omega)$$

and pass to the limit in $\mathcal{A}_{\epsilon_n}$ as $\epsilon_n \to 0$ $\square$

Lemma 2 The function

$$(0,+\infty)\ni k\mapsto C_1(k)=\int_0^{kZ/2}E_1(\theta)d\theta\in(0,1)\text{ is increasing}$$

and

$$\int_0^Z\int_0^Z\tfrac{1}{2}kE_1(k|z-\zeta|)|f(\zeta)|d\zeta dz\le C_1(k)\int_0^Z|f(\zeta)|d\zeta$$

Proof Since E_1 is decreasing, by symmetric rearrangement

$$\sup_{0<\zeta<Z}\int_0^Z kE_1(k|\zeta-z|)dz=\sup_{0<\zeta<Z}\int_{\mathbf{R}}kE_1(k|\zeta-z|)\mathbf{1}_{[0,Z]}(z)dz$$

$$\le\int_{\mathbf{R}}kE_1(k|z|)\underbrace{(\mathbf{1}_{[0,Z]})^*}_{=\mathbf{1}_{[-\frac{Z}{2},\frac{Z}{2}]}}(z)dz=\int_{-kZ/2}^{kZ/2}E_1(\theta)d\theta=2C_1(k)$$

□

Notation for $f \equiv f(\nu) \in L^1((0,\infty))$ and $g \equiv g(\mu) \in L^1([-1,1])$

$$\langle f \rangle := \int_0^\infty f(\nu) d\nu \qquad \tilde{g} := \tfrac{1}{2}\int_{-1}^{1} g(\mu) d\mu\,, \qquad \langle\!\langle h \rangle\!\rangle := \langle \tilde{h} \rangle$$

Lemma 3 Let (I_ν, T) and (I'_ν, T') satisfy RT+heat equations; then

$$\partial_z \langle\!\langle \mu(\mathcal{I}_\nu - \mathcal{I}'_\nu)_+ \rangle\!\rangle + \vec{u} \cdot \nabla_{\vec{x}}(T - T')_+ - \lambda \Delta_{\vec{x}}(T - T')\mathbf{1}_{T>T'} = -D_1 - D_2$$

where $\mathcal{I}_\nu$ is the average of I_ν in (ω_x, ω_y) and

$$\begin{cases} D_1 = \langle\!\langle \kappa_\nu(1-a_\nu)((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{T>T'}) \rangle\!\rangle \geq 0 \\ D_2 = \langle\!\langle \kappa_\nu a_\nu((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} \rangle\!\rangle \geq 0 \end{cases}$$

NB in Lemma 3 we have set $\lambda := \frac{\rho C_P \kappa_T}{4\pi}$ and replaced $\vec{u}$ with $\frac{\rho C_V}{4\pi}\vec{u}$

Proof of the Monotonicity Lemma

•Multiplying both sides of the RT equation by $\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu}$ and integrating the resulting expressions in μ and ν, and both sides of the heat equation by $\mathbf{1}_{T > T'}$ leads to

$$\partial_z \langle\!\langle \mu(\mathcal{I}_\nu - \mathcal{I}'_\nu)_+ \rangle\!\rangle + \vec{u} \cdot \nabla_{\vec{x}} (T - T')_+ - \lambda \Delta_{\vec{x}} (T - T') \mathbf{1}_{T > T'} \\ = -D_1 - D_2$$

•Since $\widetilde{(\mathcal{I}_\nu - \mathcal{I}'_\nu)} = (J_\nu - J'_\nu)$ and $\mathbf{1}_{J_\nu > J'_\nu}$ is independent of μ

$$D_2 = \langle\!\langle \kappa_\nu a_\nu ((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{J_\nu > J'_\nu}) \rangle\!\rangle$$

Since $z \mapsto \mathbf{1}_{z>0}$ is nondecreasing, one has

$$(a - b)(\mathbf{1}_{a>0} - \mathbf{1}_{b>0}) \geq 0$$

so that

$$((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{J_\nu > J'_\nu}) \geq 0 \implies D_2 \geq 0$$

•Since $T \mapsto B_\nu(T)$ is increasing on $(0,+\infty)$ for all $\nu > 0$, one has

$$\mathbf{1}_{T>T'} = \mathbf{1}_{B_\nu(T)>B_\nu(T')} \text{ is independent of } \mu, \nu$$

Since $z \mapsto \mathbf{1}_{z>0}$ is nondecreasing,

$$\begin{aligned}&((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{T>T'})\\ =&((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{B_\nu(T)>B_\nu(T')}) \geq 0\end{aligned}$$

This implies that

$$D_1 \geq 0$$

NUMERICAL SIMULATIONS

The Absorption Coefficient κ_ν

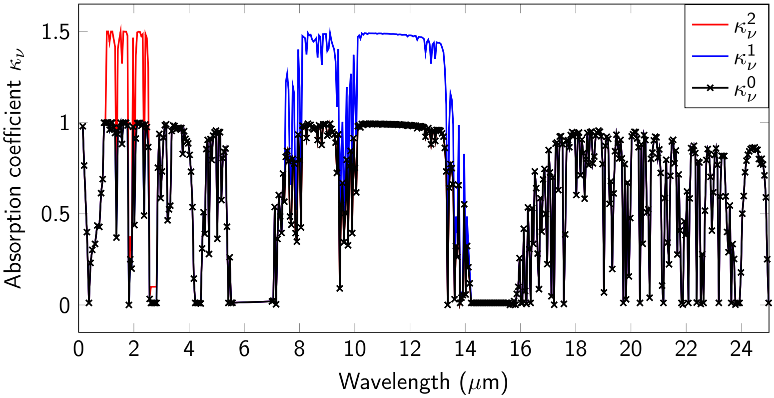


Figure: Absorption κ_ν^0 read from Gemini measurements. Set enhanced absorption: $\kappa_\nu^1 > \kappa_\nu^0$ in the infrared range $2-3\mu m$ and $\kappa_\nu^2 > \kappa_\nu^0$ in the range $8-14\mu m$. The $\times$ marks are the 487 grid points for the ν-integrals. Enhanced high values are truncated at $\kappa = 1.5$.

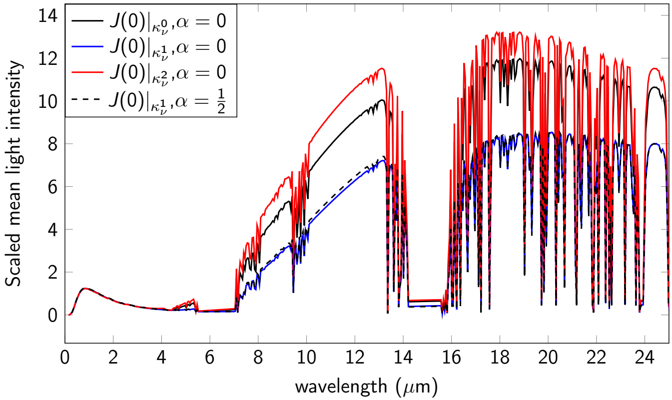


Figure: Computed mean radiation intensities $J_\nu(0)$ at the ground level for κ_ν^0, κ_ν^1, κ_ν^2 without scattering and for κ_ν^0 with isotropic scattering at altitude 6-9km and Rayleigh scattering above 9km both with $\alpha = \frac{1}{2}$.

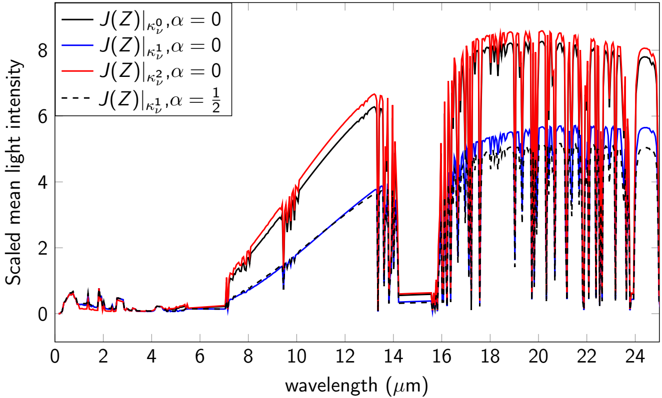


Figure: Computed mean radiation intensities $J_\nu(Z)$ at the top of the troposphere for κ_ν^0, κ_ν^1, κ_ν^2 without scattering and for κ_ν^0 with isotropic scattering at altitude 6-9km and Rayleigh scattering above 9km both with $\alpha = \frac{1}{2}$.

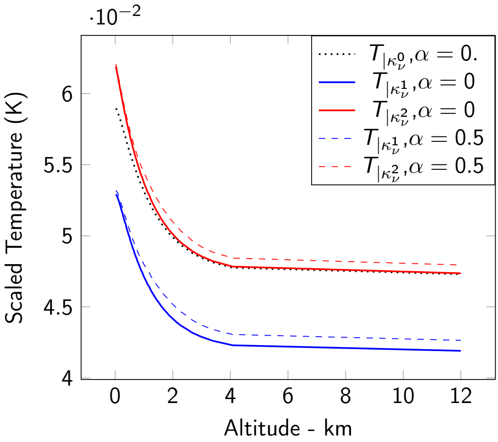
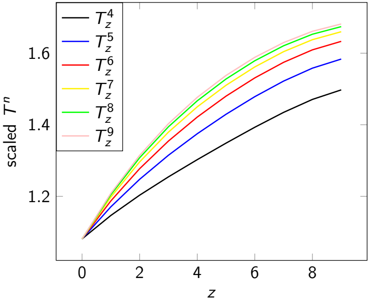


Figure: Temperatures $z \to T(z)$ in Kelvin divided by 4798 computed with κ_ν^0, κ_ν^1 and κ_ν^2 without scattering ($\alpha = 0$) and with a scattering $\alpha = \frac{1}{2}$.

Case 2: Heating of a Pool (2D, $\vec{u} = 0$)



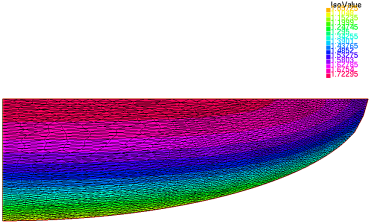


Figure: Left: Convergence of the sequence T^n. Right: Color map of $T(x, z)$ at iteration 10.

Double iteration loop; inner loop of 3 iterations to resolve the T^4 nonlinearity in the heat equation (coming from the Stefan-Boltzmann law if κ and a are both independent of ν)

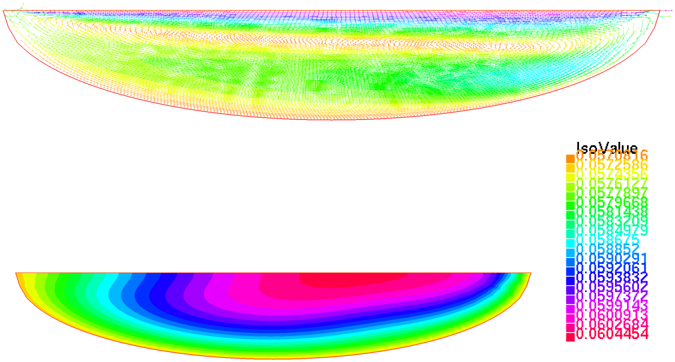


Figure: Velocity field and temperature. Wind velocity $(10, 0)^T$, velocity field solution of Navier-Stokes. Temperature given at the bottom. Solution of the heat equation computed by a time-marching algorithm based on the formula of characteristics for the drift + a quasi-Newton method for a variational formulation of temperature diffusion.

Case 4: Lake Geneva (3D, potential flow)

Set $\vec{u} = \nabla_{\vec{x}} p$, where $\Delta_{\vec{x}} p = 0$ with Dirichlet conditions for p on the red part of the boundary (inlet and outlet of the Rhône) and $\partial_n p = 0$ elsewhere. FEM P1 method with 33810 tetrahedra, and 1287 triangles on the surface.

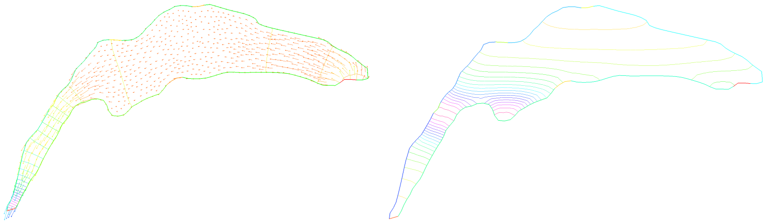


Figure: Left: velocity vectors and pressure isolines at the surface of the lake. Right: isolines of the surface temperature.

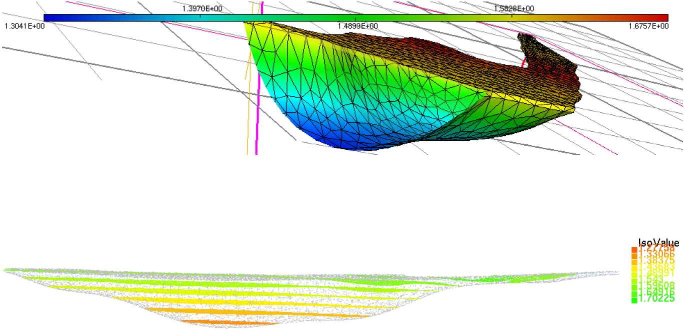


Figure: Top: perspective view of a 3D color map of the temperature on the side of the lake past a middle vertical plane. Bottom: perspective view showing some temperature level surfaces inside the lake.

Discretization of Lake Geneva: F. Hecht (New Developments in FreeFem++ J. Numer. Math. 2012)

Case 5: Planetary Atmosphere Heated by the Sun

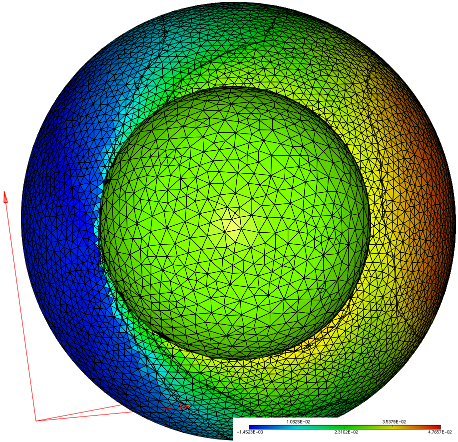


Figure: Temperature in the atmosphere of a planet heated by the Sun. Thermal diffusion propagates heat in unlit regions, with or without the presence of a counterclockwise rotating wind. (Aspect ratio enlarged.)

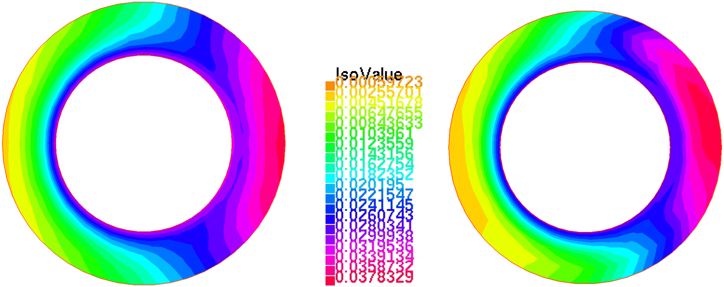


Figure: Temperature in the atmosphere of a planet heated by the Sun on the right with (right) and without (left) almost counterclockwise rotating wind (the axis of rotation is not perpendicular to the figure).

Wind velocity=rotating Poiseuille flow, axis$\neq$direction of the Sun
RT boundary condition: with Q deduced from $Q_{Sun} = 1300 W/m^2$

$$\mathcal{I}_\nu(Z, -\mu) = 0 \text{ and } \mathcal{I}_\nu(0, \mu) = Q\mu B_\nu(T_{Sun}), \quad 0 < \mu < 1$$

Conclusions

•Existence for the radiative transfer equation with frequency dependence coupled with incompressible Navier-Stokes-Fourier system
•Monotonicity structure of RT discovered by Mercier (SIMA1987) adapted to this setting leading to uniqueness and exponential convergence of an algorithm based on iteration on the sources
•Fast numerical simulations in the case of stratified RT (integral equation for the angle averaged radiative intensity) coupled with the fluid equations solved with FreeFEM https://freefem.org
•Extension to the case of Rayleigh's scattering kernel

$$p(\vec{\omega},\vec{\omega}') = \tfrac{3}{16\pi}(1+(\vec{\omega}\cdot\vec{\omega}')^2)$$

•Increasing κ_ν in frequency ranges corresponding to greenhouse gases in the RT equation **alone** seems insufficient to explain the enhancement of greenhouse effect

References

•Textbooks on RT

S. Chandrasekhar "Radiative Transfer" Clarendon Press 1950
G.C. Pomraning "Radiation Hydrodynamics" Pergamon Press 1973
A. Fowler "Mathematical Geoscience" chap. 2 Springer 2011

•Papers

F.G., O. Pironneau:
https://arxiv.org/abs/2107.13857
https://hal.sorbonne-universite.fr/hal-03419670v1

Comments on the temperature computation in case 1 in
C.Bardos, F.G., O. Pironneau:
https://arxiv.org/abs/2111.14167