Méthodes qualitatives pour les problèmes de diffraction inverses en électromagnétisme

Main topic: Transmission Eigenfrequencies for Dielectrics and Their Use in The Identification Problem

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Joint work with

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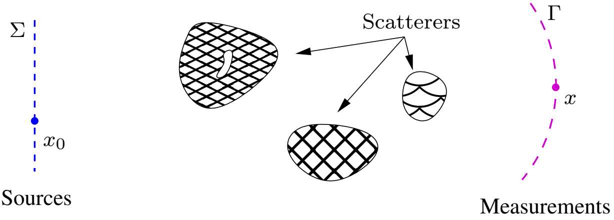
Motivation

Radar, Sonar, Medical Imaging, Non destructive testing, ···



Inverse problem: Determine the geometry (imaging) and some physical properties (identification) of inclusions from the knowledge of diffracted fields (associated with several incident waves).

- nonlinear problem
- unstable with respect to measurement error (ill-posed problem)
- uniqueness is not guaranteed with only several measurements?

None of the existing numerical methods can efficiently treat the problem in its general setting.

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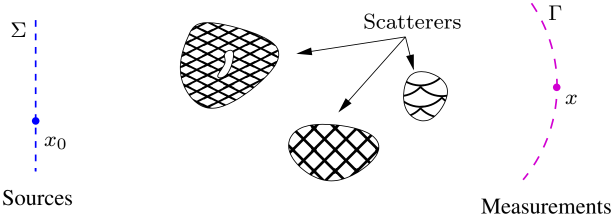

Inverse problem: Determine the geometry (imaging) and some physical properties (identification) of inclusions from the knowledge of diffracted fields (associated with several incident waves).

Commonly used approaches: (linearization of the inverse problem)

- The Kirchhoff approximation (high frequencies)
- The Born approximation (weak scatterer, low frequencies, small inclusions)

$\Rightarrow$ Provide a satisfying solution for a large variety of practical problems

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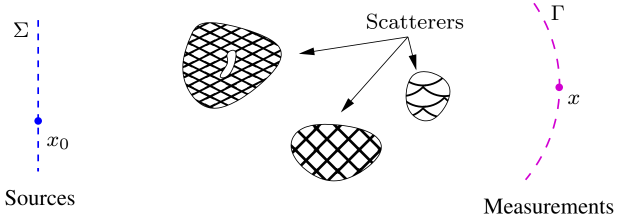

In our case: We consider problems for which the linearization is not possible (strongly non linear problems)

- inclusions with high contrast, frequency in the resonant regime
- important multiple scattering effects (closely spaced objects)

$\Rightarrow$ We use multistatic data at fixed or multiple frequencies.

Goal: Get reliable qualitative information with few a priori information.

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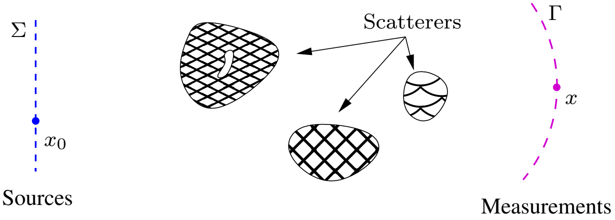


Typical targeted applications:

- Imaging of urban infrastructures, landmines, . . . (GPR)
- Microwave imaging of malignancies or blood content tissues
- Nondestructive testing of complex media

Physical (nonhomogeneous, anisotropic) properties are not known (a priori).

Outline

- **Part I.** Beginning of the story...

  Sampling Methods for Imaging problems

  - Imaging of anisotropic inclusions using electromagnetic waves
  - Interior transmission problem

- **Part II.** The story continues...

  Qualitative identification procedures

  - Transmission eigenvalues (or eigenfrequencies)!
  - Application to the identification problem and to nondestructive testings

Sampling Methods for Imaging problems

Mathematical Model: Forward problem

- Constant permeability $\mu = \mu_0$.
- ϵ and σ can be matrices. $\epsilon = \epsilon_0$ and $\sigma = 0$ in $\mathbb{R}^3 \setminus D$.
- Inclusion D is characterized by the index N

$$\boxed{N(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\epsilon_0 \omega}, \quad N(x) = I \text{ in } \mathbb{R}^3 \setminus D}$$

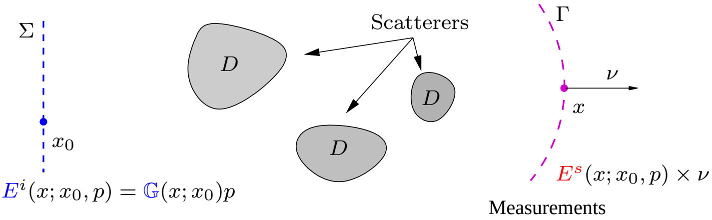
Wave number: $k^2 = \epsilon_0\, \mu_0\, \omega^2$

$$\begin{cases} (i) & \operatorname{curl}\operatorname{curl} E - k^2\, N\, E = p\delta(x - x_0) \quad \text{in } \mathbb{R}^3, \\ (ii) & E^s = E - E^i \in H_{loc}(\operatorname{curl}, \mathbb{R}^3) \\ (iii) & \lim\limits_{r\to\infty} (\operatorname{curl} E^s \times x - ik\, r\, E^s) = 0 \end{cases}$$

Incident electromagnetic field E^i:

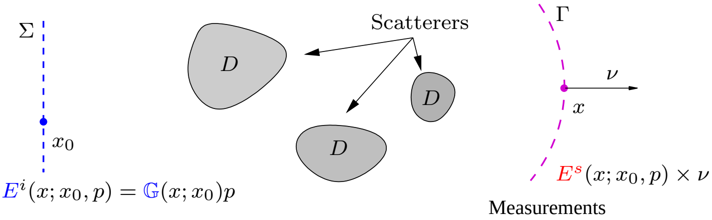
$$\operatorname{curl}\operatorname{curl} E^i - k^2 E^i = p\delta(x - x_0) \text{ in } \mathbb{R}^3$$

Mathematical Model: Inverse problem



$\mathbb{G}(x;x_0)$: the Green tensor of the background medium.

Mathematical Model: Inverse problem



$\mathbb{G}(x; x_0)$: the Green tensor of the background medium.

Inverse problem: Determine D and N from the knowledge of $E^s(x; x_0, p) \times \nu$, for all $x_0 \in \Sigma$, $p \in T_{x_0}$, and all $x \in \Gamma$.

Theorem (*Cakoni-Colton*, 2003): Assume that the complement of D is connected, and $(N - I)$ is positive definite on D, then D is uniquely determined.

However, N cannot be uniquely determined, even if measurements are available for a range of frequencies $k \in [k_1, k_2]$.

Retrieve the geometry: Sampling Methods

Linear Sampling Method, Factorization, Probe Method, MUSIC, Reciprocity Gap . . .

A quick development in recent years after the introduction of the Linear Sampling Method by *D. Colton* and *A. Kirsch* in 1996...

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- Algorithms are not based on a linearization assumption
- Do not require a forward solver and a priori knowledge on the physical parameters
- Quick methods (trivially parallel)!

Limitations:

- Use multi-static data (at a fixed frequency)
- Precision strongly dependent on the number of measurements
- Only the geometrical information is provided

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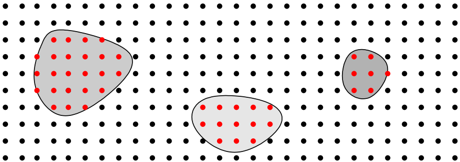
Tutorial Monographs:

- *Cakoni-Colton* ('06) LSM for the scalar problem
- *Grinberg-Kirsch* ('08) The Factorization method

Principle of a “sampling method”

Associate with a *sampling point* z of the probed domain a criterion $\mathcal{G}(z)$ that indicates whether z is in the interior or the exterior of the scatterer.

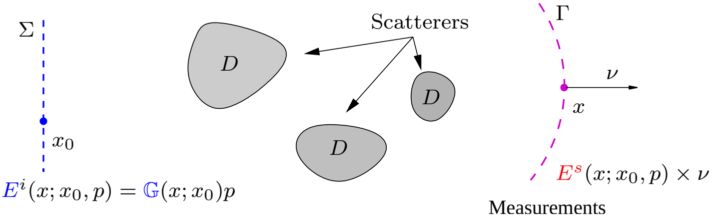
Inversion scheme: consider a collection of points $(z_i)_{i=1,\cdots,N}$ meshing the probed region and evaluate $\mathcal{G}(z_i)$ for each z_i.



The computation of $\mathcal{G}$ needs to be quick!

Proposed methods differ according to the choice and the way of computing of $\mathcal{G}$.

The Linear Sampling Method (LSM)



$\mathbb{G}(x; x_0)$: the Green tensor of the background medium.

Near field equation of LSM: seek $g_z \in \tilde{H}_{\text{curl}}^{-\frac{1}{2}}(\Sigma)$ such that for all $x \in \Gamma$,

$$(F\, g_z)(x) := \int_{\Sigma} \nu(x) \times E^s(x; x_0, g_z(x_0))\, ds(x_0) \simeq \nu(x) \times \mathbb{G}(x; z)\, q$$

Fg scattered wave associated with: $(\mathcal{H}g)(x) := \int_{\Sigma} \mathbb{G}\,(x, y) g(y)\, ds(y)$

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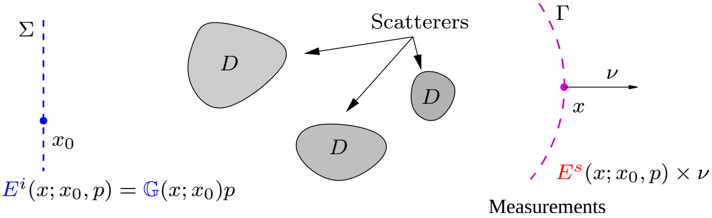

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Criterion: $\mathcal{G}(z) = \|g_z\|$ has large values if $z \notin D$ as compared with $z \in D$.

Interior Transmission Problem

$$\boxed{(F\, g_z)(x) \simeq \nu(x) \times \mathbb{G}(x; z)\, q \quad x \in \Gamma,}$$

for $z \in D$, is equivalent to finding E and E_0 solutions to

$$\begin{cases} \operatorname{curl}\operatorname{curl} E - k^2\, N\, E = 0 & \text{in } D, \\ \operatorname{curl}\operatorname{curl} E_0 - k^2\, E_0 = 0 & \text{in } D. \end{cases} \tag{1}$$

$$\begin{cases} (E - E_0) \times \nu = (\mathbb{G}(\cdot; z)\, q) \times \nu & \text{on } \partial D, \\ (\operatorname{curl} E - \operatorname{curl} E_0) \times \nu = \operatorname{curl}(\mathbb{G}(\cdot; z)\, q) \times \nu & \text{on } \partial D. \end{cases} \tag{2}$$

Problem (2)-(1) is called the **Interior Transmission Problem**.

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Space of solutions: $(E, E_0) \in L^2(D)^3$ satisfy (1) in the distributional sense; $E - E_0 \in H(\operatorname{curl}, D)$; $\operatorname{curl}(E - E_0) \in H(\operatorname{curl}, D)$ and satisfy (2)

Interior Transmission Problem

Theorem (H. 04): Assume that $(N-1)^{-1}$ is bounded and that $\mathcal{R}e(N-1)^{-1}$ is a positive definite matrix field on D then ITP is of Fredholm type.

Definition: Transmission eigenvalues (eigenfrequencies) are the values of k for which uniqueness of ITP solutions fails.

Observations:

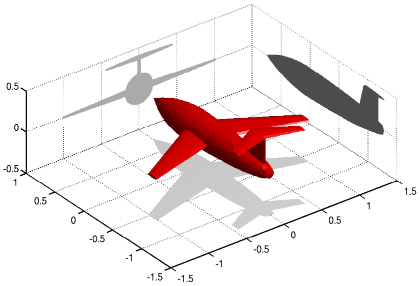
- Uniqueness for ITP $\longrightarrow$ Injectivity of F ($\longleftrightarrow$ 1 is not an eigenvalue of the scattering matrix)
- The LSM fails for k being a transmission eigenvalue.

Theorem (continued):

- If $\mathcal{I}m(N) > 0$ in D then the solution is unique.
- If not, then the set of k for which uniqueness fails is at most discrete.

3-D Numerical Results (farfield data, perfect scatterer)

Collino-Fares-H. '05



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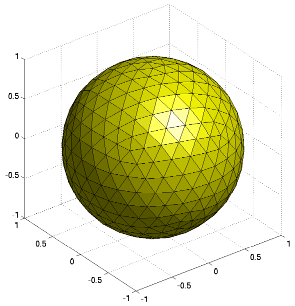
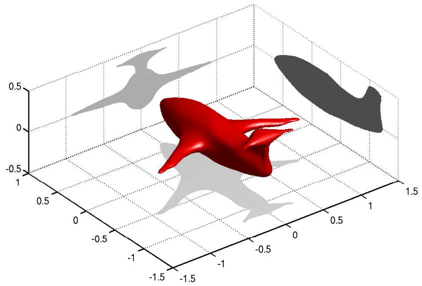


Aperture (sources-measurements)



Reconstruction

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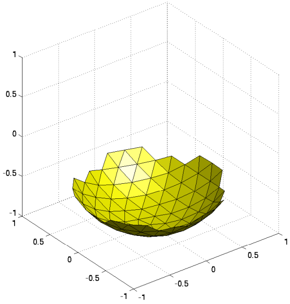

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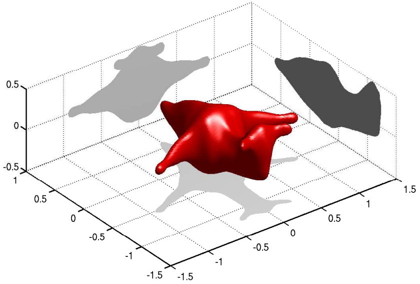

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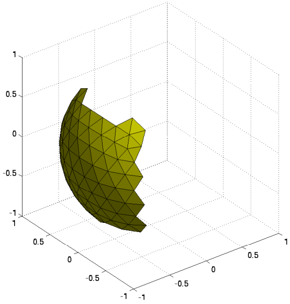

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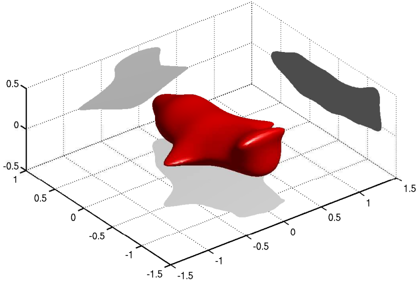

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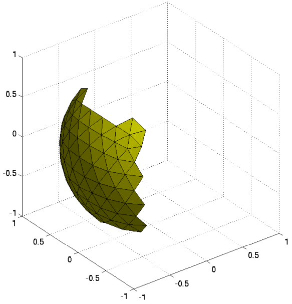

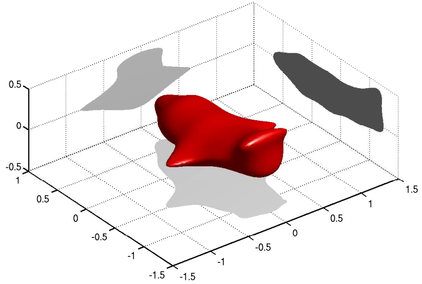

Aperture (sources-measurements)

Reconstruction

Numerical examples for dielectrics with experimental data from the Fresnel institute : Corocco et al '09

A qualitative approach for the identification problem

The identification problem: an original approach
Cakoni-Colton-H. (2008-2009)

Example of geometry reconstruction using LSM (M'B Fares)

Goal: Obtain information on N, assuming that D (or an approximation of it) is known.

The idea is to use the transmission eigenvalues...

Interior Transmission Problem: equivalent formulation

Assume that $(N-1)^{-1}$ is bounded a.e. in D.

Define: $\mathbf{u} = E - E_0$.

$\mathbf{u} \in \mathcal{U}(D) := \{\mathbf{u} \in H(\mathrm{curl}, D);\ \mathrm{curl}\mathbf{u} \in H(\mathrm{curl}, D)\}$,

satisfies

$$\begin{aligned} &\mathrm{curl}\,\mathrm{curl}(N-1)^{-1}\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathrm{curl}\,\mathrm{curl}(N-1)^{-1}\mathbf{u} \\ &\quad -k^2 N(N-1)^{-1}\mathrm{curl}\,\mathrm{curl}\mathbf{u} + k^4 N(N-1)^{-1}\mathbf{u} \;=\; 0 \text{ in } D, \end{aligned}$$

and

$$\begin{cases} \mathbf{u} \times \nu = (\mathbb{G}(\cdot; z)\, q) \times \nu, & \text{on } \partial D, \\ (\mathrm{curl}\,\mathbf{u}) \times \nu = \mathrm{curl}(\mathbb{G}(\cdot; z)\, q) \times \nu & \text{on } \partial D. \end{cases}$$

Reciprocally:

$$E = (N-1)^{-1}(\mathrm{curl}\,\mathrm{curl}\mathbf{u} - k^2\mathbf{u})/k^2$$
$$E_0 = (N-1)^{-1}(\mathrm{curl}\,\mathrm{curl}\mathbf{u} - k^2 N\mathbf{u})/k^2$$

Solvability of the Interior Transmission Problem

The key point is to observe that the variational formulation can be written in the form

$$\boxed{\mathcal{A}_k(\mathbf{u},\mathbf{u}') - k^2\mathcal{B}(\mathbf{u},\mathbf{u}') = 0} \quad \text{for all } \mathbf{u}' \in \mathcal{U}_0(D), \tag{3}$$

with $\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\mathrm{curl}, D);\ \mathrm{curl}\mathbf{u} \in H_0(\mathrm{curl}, D)\}$ and

$$\begin{array}{lcl} \mathcal{A}_k(\mathbf{u},\mathbf{u}') & = & \big((N-1)^{-1}(\mathrm{curlcurl}\mathbf{u} - k^2\mathbf{u}),\ (\mathrm{curlcurl}\mathbf{u}' - k^2\mathbf{u}')\big)_D \\ & + & k^4\,(\mathbf{u},\ \mathbf{u}')_D \\ \mathcal{B}(\mathbf{u},\mathbf{u}') & = & (\mathrm{curl}\mathbf{u},\ \mathrm{curl}\mathbf{u}')_D \end{array}$$

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Definition: Transmission eigenvalues (eigenfrequencies) are the values of k for which uniqueness of ITP solutions fails; i.e. $\exists\ \mathbf{u} \in \mathcal{U}_0(D)$ non trivial solution of (3)

$\longrightarrow$ Quadratic eigenvalue problem.

It can be shown that it is also a linear **non self-adjoint** eigenvalue problem.

Existence of transmission eigenvalues

Remark: In the case of perfect conductors these special frequencies correspond to Maxwell's eigenfrequencies of the cavity D,
i.e the $k > 0$ such that:

$$\begin{cases} \exists\, \mathbf{u} \in H_0(\mathrm{curl}, D); \mathbf{u} \neq 0; \\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u} = 0 \quad \text{in } D. \end{cases}$$

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Existence of transmission eigenvalues:

1. Colton-Kress (91') : Spherically, stratified medium: existence of infinitely many.
2. Païvaïrinta-Sylvester ('08) : (for the scalar problem), Kirsch ('09) Cakoni-H. ('09) (generalization to the Maxwell problem): There exist eigenvalues for N sufficiently large (or sufficiently small).
3. Cakoni-H.-Gintides ('10) : existence of infinitely many without restrictions on N

Existence of transmission eigenvalues

$$\boxed{\mathcal{A}_k(\mathbf{u},\mathbf{u}') - k^2\mathcal{B}(\mathbf{u},\mathbf{u}') = 0} \quad \text{for all } \mathbf{u}' \in \mathcal{U}_0(D),$$

Idea: Consider the eigenvalues of A_k with respect to the compact and non injective operator B, i.e. the $\lambda_j(k^2)$ such that $\ker(A_k - \lambda_j(k^2)B) \neq \{0\}$.

Theorem: The eigenvalues of $(\lambda_j(k^2))_{j\geq 1}$ are given by

$$\lambda_j(k^2) = \min_{W\subset\mathcal{U}_j} \big(\max_{u\in W\setminus\{0\}} (A_k u, u)/(Bu, u) \big)$$

where U_j denotes the set of all j-dimensional subspaces W of $\mathcal{U}_0(D)$ such that $W \cap \ker(B) = \{0\}$.

The transmission eigenvalues are $k^2 = \tau$, solutions to

$$\lambda_j(\tau) = \tau \text{ for some } j \geq 1.$$

Existence of transmission eigenvalues : basic theorem

Theorem: Assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

- $A_{\tau_0} - \tau_0 B$ is positive on $\mathcal{U}_0$,
- $A_{\tau_1} - \tau_1 B$ is non positive on a m-dimensional subspace W_m of U.

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, m$, has at least one solution in $[\tau_0, \tau_1]$.

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Assume that $\mathcal{I}m(N) = 0$ a.e. in D and $\|N(x)\|_2 \geq \delta > 1$ for $x \in D$.

- $A_\tau - \tau B$ is positive for all $\tau < \tau_0 = \lambda(D)/\sup_D \|N\|_2$,

  where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

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- Existence of τ_1 and W_m?

Existence of transmission eigenvalues

Idea: use existence of eigenvalues for spherical geometries.

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Assume $\|N\|_\infty \geq n^* > 1$

A_τ^* same as A_τ with N replaced with n_*

$$((A_\tau - \tau B)\mathbf{u}, \mathbf{u})_{\mathcal{U}_0} \leq ((A_\tau^* - \tau B)\mathbf{u}, \mathbf{u})_{\mathcal{U}_0}$$

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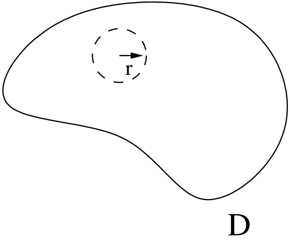
$\tau = k_1^*(r)^2$ first tr. eigen. associated with n_* for the sphere S of radius r.

$\mathbf{u}_S = \mathbf{u}_1^*$ inside S, extended by 0 outside...

$$((A_\tau - \tau B)\mathbf{u}_S, \mathbf{u}_S)_{\mathcal{U}_0} \leq 0$$



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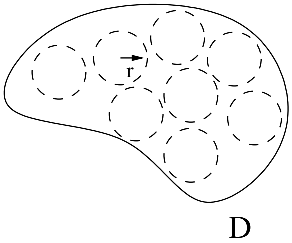

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Idea: use existence of eigenvalues for spherical geometries.

Assume $\|N\|_\infty \geq n^* > 1$

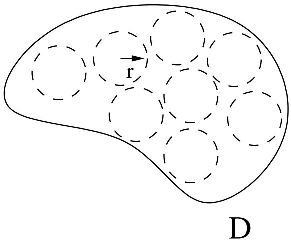
A_τ^* same as A_τ with N replaced with n_*

$$((A_\tau - \tau B)\mathbf{u}, \mathbf{u})_{\mathcal{U}_0} \leq ((A_\tau^* - \tau B)\mathbf{u}, \mathbf{u})_{\mathcal{U}_0}$$

$\tau = k_1^*(r)^2$ first tr. eigen. associated with n_* for the sphere S of radius r.

$\mathbf{u} \in \text{Span}\{\mathbf{u}_{S_j} \; ; \; j = 1, \ldots, N(r)\}$

$$((A_\tau - \tau B)\mathbf{u}, \mathbf{u})_{\mathcal{U}_0} \leq 0$$



$\Longrightarrow$ Existence of $N(r)$ tr. eigenvalues !

But: $N(r) \to \infty$ as $r \to 0$.

First application: a priori lower bound on material coefficients

- If $\|N(x)\|_2 \geq \delta > 1$ for all $x \in D$ then for any positive transmission eigenvalue k we have

$$\sup_D \|N(x)\| \geq \lambda(D)/k^2$$

  where $\lambda(D)$ denotes the first Dirichlet eigenvalue of $-\Delta$ on D.

- $N \to k_1(N, D)$ "is decreasing":

$$n_* \leq N \leq n^* \Longleftrightarrow k_1(n^*, D) \leq k_1(N, D) \leq k_1(n_*, D)$$

- $D \to k_1(N, D)$ "is decreasing".

How to compute the transmission eigenvalues?

Without knowing N!

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$$(F\, g_z)(x) \simeq \nu(x) \times \mathbb{G}(x; z)\, q (=: E_e(\cdot; z, q))$$

with bounded "Herglotz norm" is no longer guaranteed if k is a transmission eigenvalue.

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Algorithm: We fix a point z in D and evaluate

$$k \mapsto \|g_\alpha(\cdot; z, q)\|$$

where

$$(\alpha + F^* F)\, g_\alpha(\cdot; z, q) = F^* \left(E_e(\cdot; z, q)\right)$$

We expect to observe peaks at the irregular frequencies...

How to compute the transmission eigenvalues

Justification: Let F_δ be the operator corresponding to noisy measurements

$$\|F_\delta - F\| \leq \delta$$

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$$(\alpha(\delta) + F_\delta^* F_\delta)\, g_\delta(\cdot; z, q) = F_\delta^* \left(E_e(\cdot; z, q)\right)$$

Assumptions:

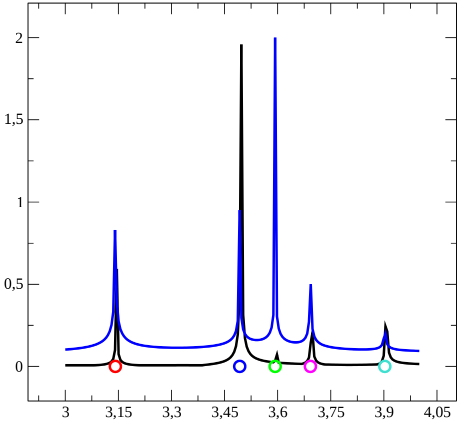
- F_δ is a family of uniformly compact operators with dense range
- $\sqrt{\alpha(\delta)}/\delta \to 0$ as $\delta \to 0$

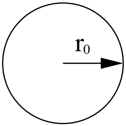
Theorem: Let k be a transmission eigenvalue, then for a.e. $z \in D$

$$\lim_{\delta \to 0} \|\mathcal{H} g_\delta(\cdot; z, q)\|_{L^2(D)} = \infty$$

Main "formal idea of the proof": $E_e(\cdot; z, q)$ cannot be "orthogonal" to transmission eigenvectors for all z in an open set of D.

Validation in the spherical case





Sphere with radius $r_0 = 1$ and $n = 4$.

Blue line: $\|g(\cdot, z)\|$ in terms of the wave number k

Dark line: Computed eigenvalues using numerical solutions to ITP

Circles: Computed eigenvalues using speration of variables

Possible applications to nondestructive testing

Idea: Use eigenfrequencies to obtain a qualitative information on the presence of faults inside D

Advantage: We do not need to know N or to solve any direct problem

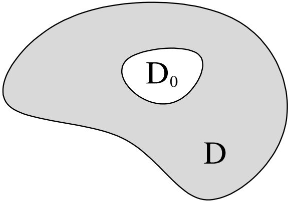
Limitation: Only a qualitative information can be obtained

First investigations: test the presence of cavities inside D (application: cavities inside composite materials)

$$N = 1 \text{ in } D_0 \subset D$$

$\Longrightarrow$ difficulty in the mathematical analysis:

$$N - 1 \text{ is not invertible a.e. in } D!$$



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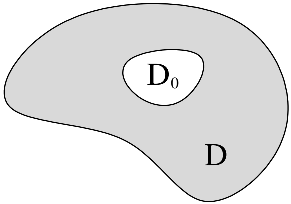

with $\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\mathrm{curl}, D);\ \mathrm{curl}\mathbf{u} \in H_0(\mathrm{curl}, D)\}$ and

$$\mathcal{U}(D, D_0, k^2) := \{u \in \mathcal{U}_0(D) \text{ such that } \mathrm{curlcurl}\mathbf{u} - k^2\mathbf{u} = 0 \text{ in } D_0\}$$

Assume that k^2 is not a *Dirichlet and Neumann* eigenvalue of $-\Delta$ in D_0 and that N and $(N-1)^{-1}$ are positive definite in $D \setminus D_0$.

Then, solving ITP is equivalent to solve for $\mathbf{u} \in \mathcal{U}(D, D_0, k^2)$ such that

$$\boxed{\mathcal{A}_k(\mathbf{u}, \mathbf{u}') - k^2\mathcal{B}(\mathbf{u}, \mathbf{u}') = 0} \quad \text{for all } \mathbf{u}' \in \mathcal{U}(D, D_0, k^2),$$

$$\begin{aligned}
\mathcal{A}_k(\mathbf{u}, \mathbf{u}') &:= \left((N-1)^{-1}(\mathrm{curlcurl}\mathbf{u} - k^2\mathbf{u}),\ (\mathrm{curlcurl}\mathbf{u}' - k^2\mathbf{u}')\right)_{D\setminus D_0} \\
&+ k^4\,(\mathbf{u},\ \mathbf{u}')_D \\
\mathcal{B}(\mathbf{u}, \mathbf{u}') &:= (\mathrm{curl}\mathbf{u},\ \mathrm{curl}\mathbf{u}')_D
\end{aligned}$$

ITP with presence of cavities: existence of transmission eigenvalues

Difficulty: The variational space

$$\mathcal{U}(D, D_0, k^2) := \{u \in \mathcal{U}_0(D) \text{ such that } \mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u} = 0 \text{ in } D_0\}$$

depends on k...

Theorem: $k \longmapsto P_k$, where $P_k : \mathcal{U}_0(D) \to \mathcal{U}(D, D_0, k^2) \subset \mathcal{U}_0(D)$ denotes the orthogonal projection operator, is continuous.

$\Longrightarrow$ Similar analysis as in the case without cavities carry over by considering

$$I + k^2 \, P_k^* A_k^{-1/2} B_k A_k^{-1/2} P_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$$

$\Longrightarrow$ Existence of infinitely many transmission eigenvalues.

Countability of transmission eigenvalues

Countability of transmission eigenvalues cannot be obtained this way since $k \longmapsto P_k$ cannot be analytic.

We substitute to P_k a projection like, but analytic operator:
$\tilde{P}_k : \mathcal{U}_0(D) \to \mathcal{U}(D, D_0, k^2) \subset \mathcal{U}_0(D)$

$$\tilde{P}_k \mathbf{u} := \mathbf{u} - \chi \theta_k \mathbf{u}$$

χ is a C^∞ cutoff function $= 1$ in D_0 and 0 outside D.
$\theta_k \mathbf{u} := \int_{D_0} \Im \mathbb{G}(x, y)(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2 \mathbf{u})(y)\, dy$

Theorem: If k^2 is a transmission eigenvalue then there exists non trivial solution $\mathbf{u} \in \mathcal{U}_0(D)$ to

$$\boxed{\mathcal{A}_k(\tilde{P}_k \mathbf{u}, \overline{\tilde{P}_k \overline{\mathbf{u}'}}) + \alpha(\theta_k \mathbf{u}, \overline{\theta_k \overline{\mathbf{u}'}})_{\mathcal{U}} - k^2 \mathcal{B}(\tilde{P}_k \mathbf{u}, \overline{\tilde{P}_k \overline{\mathbf{u}'}}) = 0}$$

for all $\mathbf{u}' \in \mathcal{U}_0(D)$.

ITP with presence of cavities

- The set of transmission eigenvalues is discrete and $+\infty$ is the only point of accumulation.
- If $D_0 \subset D_0'$ and $N_1 \geq N_2 > 1$ then $k_1(D, D_0', N_1) > k_1(D, D_0, N_2)$.
- Transmission eigenvalues satisfy

$$k^2 \geq k_*^2(D, D_0) = \frac{\lambda(D, D_0)\mu(D, D_0)}{\lambda(D, D_0) + \sup |N| \, \mu(D, D_0)}$$

$$\lambda(D, D_0) = \inf_{k \geq 0} \left(\inf_{\psi \in \mathcal{U}_0(D, D_0, k)^*} \|\nabla \psi\|^2_{L^2(D)} / \|\psi\|^2_{L^2(D \setminus \overline{D}_0)} \right).$$

$$\mu(D, D_0) = \inf_{k \geq 0} \left(\inf_{\psi \in \mathcal{U}_0(D, D_0, k)^*} \|\nabla \psi\|^2_{L^2(D)} / \|\psi\|^2_{L^2(D_0)} \right)$$

- If k^2 is a transmission eigenvalue then for a.e. $z \in D$, $\lim_{\delta \to 0} \|\mathcal{H} g_\delta\|_{L^2(D)} = \infty$.

Numerical validation (spherical case)

Sphere of radius 1, index $n = 4$

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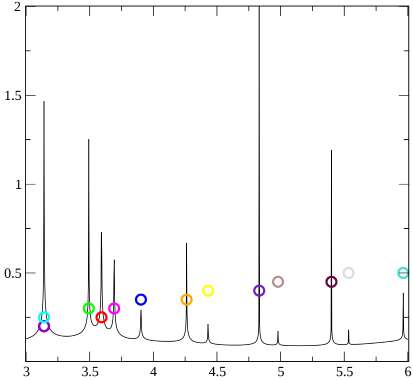

without cavities

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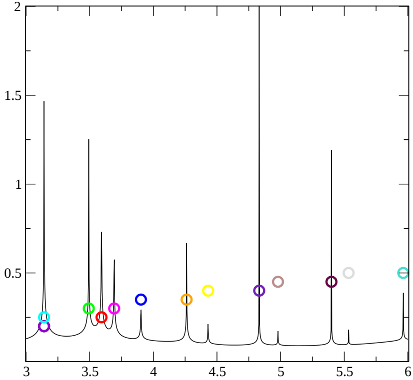
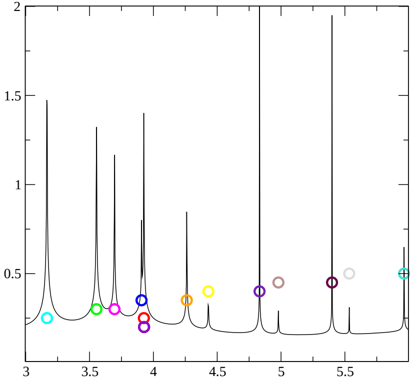


without cavities



cavity, $r = 0.2$

Numerical validation (spherical case)

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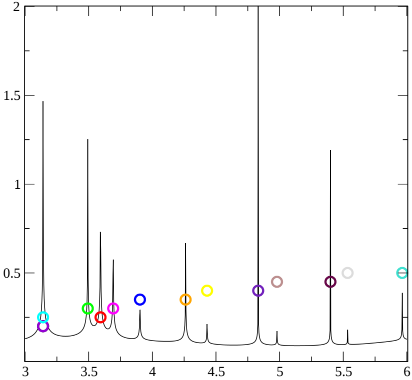
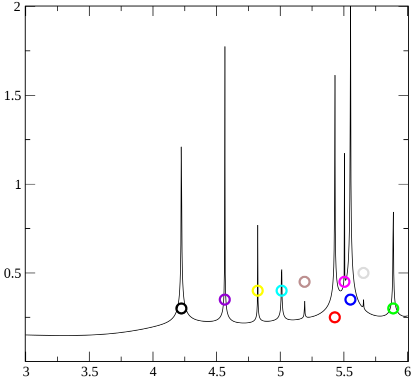


without cavities



cavity, $r = 0.5$

Some open problems and perspectives

- Improve a-priori estimates on physical properties in terms of transmission eigenfrequencies.
- Extend applications to non-destructive testing of other types of faults and materials (dispersive).
- The case of $N - I$ changing sign (photo-acoustic tomography).
- Stable identification from limited aperture data.

- What about the structure of transmission eigenfunctions... and can we say more about physical interpretations of transmission eigenfrequencies?
- Inverse spectral problem?

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End!

Thank you!