Asymptotic-Preserving schemes for highly anisotropic elliptic problems

Alexei Lozinski

Institut de Mathématiques de Toulouse
Université Paul Sabatier

e-mail : alexei.lozinski@math.univ-toulouse.fr

Joint work with

Pierre Degond	(Toulouse)
Fabrice Deluzet	(Toulouse)
Jacek Narski	(Toulouse)
Claudia Negulescu	(Marseille)

Financial support by :
Fondation RTRA "Sciences et Technologies Aéronautiques et Spatiales"
projet "Plasmax" (IMT et ONERA-Toulouse)

Outline

- Problem : A diffusion equation with extremely strong anisotropy in the diffusion matrix.
- Goal : Design an algorithm which solves the problem regardless of the anisotropy strength.
- Tool : Asymptotic Preserving (AP) approach.

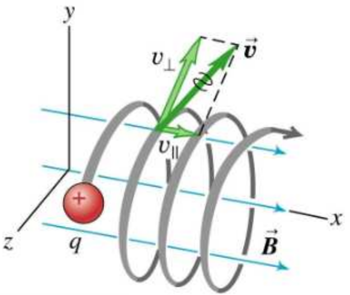
A typical boundary value problem

$$
\begin{cases}
-\dfrac{\partial^2 \phi^\varepsilon}{\partial x^2} - \dfrac{1}{\varepsilon}\dfrac{\partial^2 \phi^\varepsilon}{\partial y^2} = f & \text{in } (0,1)\times(0,1), \\
\phi^\varepsilon = 0 & \text{for } x = 0 \text{ and } x = 1, \\
\dfrac{\partial \phi^\varepsilon}{\partial y} = 0 & \text{for } y = 0 \text{ and } y = 1,
\end{cases}
$$

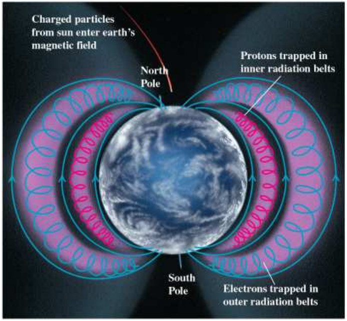
with $\varepsilon = 10^{-10}$, for example.

Motivations can be found in

- Flows in porous media,
- Semiconductor modeling,
- Image processing,
- Plasma physics :

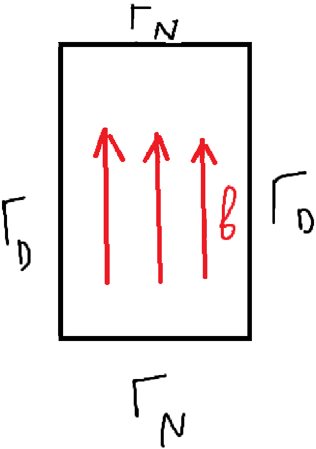


Plasma in a strong magnetic field : modeling resulting in an anisotropic equation for the electron density ...

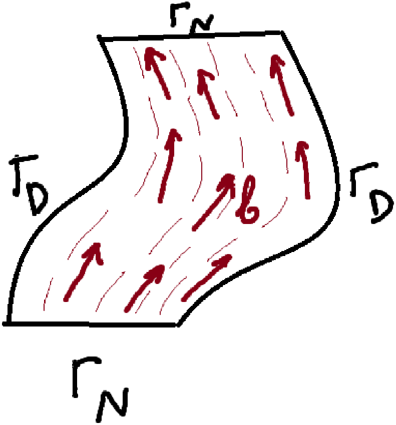


Ionospheric plasma : anisotropic equation for the electric potential ...

Anisotropic diffusion : notations



The simple geometry



The general geometry

Anisotropic diffusion : notations

- The direction of the anisotropy is defined by a given smooth vector field b, $|b| = 1$. The field lines of b cannot be closed curves.
- $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ and outward normal n.
- We suppose $b \cdot n = 0$ on Γ_D so that Ω is a tube made out of field lines.
- We'll also need the notations $\Gamma_D = \Gamma_{in} \cup \Gamma_{out}$ depending on $b \cdot n$.
- Notations for parallel and perpendicular derivatives :

$$\nabla_{||}\phi := (b \cdot \nabla\phi)b \, , \nabla_{\perp}\phi := (Id - b \otimes b)\nabla\phi \, ,$$

so that

$$\nabla\phi = \nabla_{||}\phi + \nabla_{\perp}\phi$$

- The functions space for the solutions :

$$\mathcal{V} = \{\phi \in H^1(\Omega) \mid \phi_{\Gamma_D} = 0\}$$

Anisotropic diffusion equation

- The boundary value problem with $\varepsilon > 0$

$$\begin{cases} -\dfrac{1}{\varepsilon}\nabla_{\parallel} \cdot \left(A_{\parallel}\nabla_{\parallel}\phi^{\varepsilon}\right) - \nabla_{\perp} \cdot (A_{\perp}\nabla_{\perp}\phi^{\varepsilon}) = f & \text{in } \Omega, \\ \dfrac{1}{\varepsilon} n \cdot \left(A_{\parallel}\nabla_{\parallel}\phi^{\varepsilon}\right) + n \cdot (A_{\perp}\nabla_{\perp}\phi^{\varepsilon}) = 0 & \text{on } \Gamma_N, \\ \phi^{\varepsilon} = 0 & \text{on } \Gamma_D. \end{cases}$$

- The limit problem ($\varepsilon \to 0$)

$$\begin{cases} -\nabla_{\parallel} \cdot \left(A_{\parallel}\nabla_{\parallel}\phi^{0}\right) = 0 & \text{in } \Omega, \\ n \cdot \left(A_{\parallel}\nabla_{\parallel}\phi^{0}\right) = 0 & \text{on } \Gamma_N, \\ \phi^{0} = 0 & \text{on } \Gamma_D. \end{cases}$$

Anisotropic diffusion equation

- The boundary value problem with $\varepsilon > 0$

$$\begin{cases} -\frac{1}{\varepsilon}\nabla_{\|}\cdot\left(A_{\|}\nabla_{\|}\phi^{\varepsilon}\right) - \nabla_{\perp}\cdot\left(A_{\perp}\nabla_{\perp}\phi^{\varepsilon}\right) = f & \text{in } \Omega, \\ \frac{1}{\varepsilon} n\cdot\left(A_{\|}\nabla_{\|}\phi^{\varepsilon}\right) + n\cdot\left(A_{\perp}\nabla_{\perp}\phi^{\varepsilon}\right) = 0 & \text{on } \Gamma_N, \\ \phi^{\varepsilon} = 0 & \text{on } \Gamma_D. \end{cases}$$

- The limit problem ($\varepsilon \to 0$)

$$\begin{cases} -\nabla_{\|}\cdot\left(A_{\|}\nabla_{\|}\phi^{0}\right) = 0 & \text{in } \Omega, \\ n\cdot\left(A_{\|}\nabla_{\|}\phi^{0}\right) = 0 & \text{on } \Gamma_N, \\ \phi^{0} = 0 & \text{on } \Gamma_D. \end{cases}$$

The limit problem is **ill posed** !

Infinity of solutions in $\mathcal{G} = \{\phi \in \mathcal{V} \mid \nabla_{\|}\phi = 0\}$

The problem with small $\varepsilon > 0$ is **ill conditionned** !

Weak formulation

- The problem with $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int_\Omega \left(A_\parallel \nabla_\parallel \phi^\varepsilon\right)\cdot\nabla_\parallel\psi + \int_\Omega (A_\perp \nabla_\perp \phi^\varepsilon)\cdot\nabla_\perp\psi = \int_\Omega f\psi \quad \forall\psi\in\mathcal{V}$$

- The limit problem ($\varepsilon \to 0$) :
  taking test functions in $\mathcal{G} = \{\phi \in \mathcal{V} \mid \nabla_\parallel\phi = 0\}$ yields

$$\phi^0 \in \mathcal{G} : \qquad \int_\Omega A_\perp \nabla_\perp \phi^0 \cdot \nabla_\perp \psi = \int_\Omega f\psi \quad \forall\psi\in\mathcal{G}$$

Weak formulation

- The problem with $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int_\Omega \left(A_\| \nabla_\| \phi^\varepsilon\right) \cdot \nabla_\| \psi + \int_\Omega \left(A_\perp \nabla_\perp \phi^\varepsilon\right) \cdot \nabla_\perp \psi = \int_\Omega f\psi \quad \forall \psi \in \mathcal{V}$$

- The limit problem ($\varepsilon \to 0$) :
  taking test functions in $\mathcal{G} = \{\phi \in \mathcal{V} \mid \nabla_\| \phi = 0\}$ yields

$$\phi^0 \in \mathcal{G} : \qquad \int_\Omega A_\perp \nabla_\perp \phi^0 \cdot \nabla_\perp \psi = \int_\Omega f\psi \quad \forall \psi \in \mathcal{G}$$

This shows

- The limit model admits a well posed formulation
- The perturbed problem ($\varepsilon << 1$) would be still ill conditioned under a straight-forward discretization

AP – Assymptotic Preserving schemes

The concept introduced by S. Jin, JCP'99

A typical straightforward scheme :

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "h\to 0", "\text{OK}"'] \arrow[d, "\varepsilon\to 0"', "\text{not OK}"] & P^\varepsilon \arrow[d] \\ P_h^0 \arrow[r] & P^0 \end{tikzcd}$$

AP – Assymptotic Preserving schemes

The concept introduced by S. Jin, JCP'99

A typical straightforward scheme :

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "h\to 0", "\text{OK}"'] \arrow[d, "\varepsilon\to 0"', "\text{not OK}"] & P^\varepsilon \arrow[d] \\ P_h^0 \arrow[r] & P^0 \end{tikzcd}$$

An AP scheme :

$$\begin{tikzcd} \tilde{P}_h^\varepsilon \arrow[r, "h\to 0", "\text{OK}"'] \arrow[d, "\varepsilon\to 0"', "\text{OK}"] & \tilde{P}^\varepsilon \arrow[d] \\ \tilde{P}_h^0 \arrow[r] & P^0 \end{tikzcd}$$

Here $\tilde{P}^\varepsilon$ is an AP reformulation of P^ε :

- $\tilde{P}^\varepsilon$ is equivalent to P^ε, but
- a straight-forward discretization of $\tilde{P}^\varepsilon$ is robust with respect to the limit $\varepsilon \to 0$

Hilbert space decomposition

P. Degond, F. Deluzet, and C. Negulescu, SIAM Multiscale Model. Simul. (2009/10)

- Decomposing the space $\mathcal{V}$ into a direct sum $\mathcal{G}$ plussomething

$$\mathcal{V} = \mathcal{G} \oplus^{\perp} \mathcal{A}$$

- This entails the decomposition of the solution

$$\phi^{\varepsilon} = p^{\varepsilon} + q^{\varepsilon}$$
$$p^{\varepsilon} \in \mathcal{G} \quad , \quad q^{\varepsilon} \in \mathcal{A}$$

- $\mathcal{G}$ — functions with parallel **G**radient $= 0$
- $\mathcal{A}$ — functions with **A**varage $= 0$
- $\mathcal{G} \perp \mathcal{A}$ in the sence of $L^2(\Omega)$
- p^{ε} — mean value of ϕ^{ε} along field lines (weigted average if $b \neq const$
- q^{ε} — fluctuations with zero mean value

Asymptotic Preserving reformulation

$$\begin{cases} \displaystyle\int_\Omega (A_\perp \nabla_\perp p^\varepsilon) \cdot \nabla_\perp \eta + \int_\Omega (A_\perp \nabla_\perp q^\varepsilon) \cdot \nabla_\perp \eta = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\ \displaystyle\int_\Omega \left(A_\parallel \nabla_\parallel q^\varepsilon\right) \cdot \nabla_\parallel \xi & \\ \displaystyle\qquad + \varepsilon \int_\Omega (A_\perp \nabla_\perp p^\varepsilon) \cdot \nabla_\perp \xi + \varepsilon \int_\Omega (A_\perp \nabla_\perp q^\varepsilon) \cdot \nabla_\perp \xi = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{A} \end{cases}$$

Asymptotic Preserving reformulation

$$
\begin{cases}
\int_\Omega (A_\perp \nabla_\perp p^\varepsilon) \cdot \nabla_\perp \eta + \int_\Omega (A_\perp \nabla_\perp q^\varepsilon) \cdot \nabla_\perp \eta = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\
\int_\Omega \left(A_\parallel \nabla_\parallel q^\varepsilon \right) \cdot \nabla_\parallel \xi & \\
\quad + \varepsilon \int_\Omega (A_\perp \nabla_\perp p^\varepsilon) \cdot \nabla_\perp \xi + \varepsilon \int_\Omega (A_\perp \nabla_\perp q^\varepsilon) \cdot \nabla_\perp \xi = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{A}
\end{cases}
$$

Question : How to discretize $\mathcal{A}$ and $\mathcal{G}$?

The answer is simple in the “simple geometry”

- The function space $\mathcal{G}$ is

$$\mathcal{G} = \{u(x,y) = u(x),\ \ u(\cdot) \in H_0^1(0, L_x)\}$$

- Provided we use a tensor-product finite-element space $\mathcal{V}_h$ like $\mathbb{Q}_k$, we discretize $\mathcal{G}$ in a straight-forward manner :

$$\mathcal{G}_h = \{u_h \in \mathcal{V}_h \text{ s.t. } u_h(x,y) = u_h(x)\}$$

- We still have the decomposition :

$$\mathcal{V}_h = \mathcal{G}_h \oplus^{\perp} \mathcal{A}_h$$

with

$$\mathcal{A}_h = \{u_h \in \mathcal{V}_h \text{ s.t. } \int_0^{L_y} u_h(x,y)dy = 0,\ \ \forall x\}$$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

The $\mathcal{A}$ space

$$\begin{cases} a_\perp(p^\varepsilon, \eta) + a_\perp(q^\varepsilon, \eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\ a_\parallel(q^\varepsilon, \xi) & \\ \qquad + \varepsilon a_\perp(p^\varepsilon, \xi) + \varepsilon a_\perp(q^\varepsilon, \xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{A} \end{cases}$$

With the notations

$$a_\perp(u,v) = \int_\Omega (A_\perp \nabla_\perp u) \cdot \nabla_\perp v, \quad a_\parallel(u,v) = \int_\Omega \left(A_\parallel \nabla_\parallel u\right) \cdot \nabla_\parallel v$$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

The $\mathcal{A}$ space

$$\begin{cases} a_\perp(p^\varepsilon,\eta) + a_\perp(q^\varepsilon,\eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\ a_\parallel(q^\varepsilon,\xi) + \int_\Omega l^\varepsilon \xi & \\ \qquad + \varepsilon a_\perp(p^\varepsilon,\xi) + \varepsilon a_\perp(q^\varepsilon,\xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{V} \\ \int_\Omega q^\varepsilon \chi = 0 & \forall \chi \in \mathcal{G} \end{cases}$$

With the notations

$$a_\perp(u,v) = \int_\Omega (A_\perp \nabla_\perp u)\cdot \nabla_\perp v, \quad a_\parallel(u,v) = \int_\Omega \left(A_\parallel \nabla_\parallel u\right)\cdot \nabla_\parallel v$$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

The $\mathcal{A}$ space

$$\begin{cases} a_\perp(p^\varepsilon,\eta) + a_\perp(q^\varepsilon,\eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\ a_{||}(q^\varepsilon,\xi) + \int_\Omega l^\varepsilon \xi & \\ \qquad + \varepsilon a_\perp(p^\varepsilon,\xi) + \varepsilon a_\perp(q^\varepsilon,\xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{V} \\ \int_\Omega q^\varepsilon \chi = 0 & \forall \chi \in \mathcal{G} \end{cases}$$

The $\mathcal{G}$ space

$$p \in \mathcal{G} \Leftrightarrow \begin{cases} \nabla_{||} p = 0 \\ p \in \mathcal{V} \end{cases} \Leftrightarrow \begin{cases} \int_\Omega A_{||} \nabla_{||} p \cdot \nabla_{||} \lambda \, dx = a_{||}(p,\lambda) = 0, \quad \forall \lambda \in \mathcal{L} \\ p \in \mathcal{V} \end{cases}$$

with $\mathcal{L} := \{\lambda \in L^2(\Omega) \; / \; \nabla_{||}\lambda \in L^2(\Omega) \, , \;\; \lambda_{|\Gamma_{in}} = 0\}$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

Full system – AP5 scheme

$$
\begin{cases}
a_\perp(p^\varepsilon,\eta) + a_\perp(q^\varepsilon,\eta) = \int_\Omega f\eta \quad \forall \eta \in \mathcal{G} \\
\\
a_\parallel(q^\varepsilon,\xi) \\
\qquad + \varepsilon a_\perp(p^\varepsilon,\xi) + \varepsilon a_\perp(q^\varepsilon,\xi) = \varepsilon \int_\Omega f\xi \quad \forall \xi \in \mathcal{A}
\end{cases}
$$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

Full system – AP5 scheme

$$
\begin{cases}
a_\perp(p^\varepsilon,\eta) + a_\perp(q^\varepsilon,\eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{G} \\
a_\parallel(q^\varepsilon,\xi) + \int_\Omega l^\varepsilon \xi & \\
\qquad + \varepsilon a_\perp(p^\varepsilon,\xi) + \varepsilon a_\perp(q^\varepsilon,\xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{V} \\
\int_\Omega q^\varepsilon \chi = 0 & \forall \chi \in \mathcal{G}
\end{cases}
$$

What to do in the general geometry :

Lagrange multipliers for $\mathcal{A}$ and $\mathcal{G}$; P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2010)

Full system – AP5 scheme

$$
\begin{cases}
a_\perp(p^\varepsilon,\eta) + a_\perp(q^\varepsilon,\eta) + a_{||}(\lambda^\varepsilon,\eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{V} \\
a_{||}(p^\varepsilon,\kappa) = 0 & \forall \kappa \in \mathcal{L} \\
a_{||}(q^\varepsilon,\xi) + \int_\Omega l^\varepsilon \xi & \\
\qquad + \varepsilon a_\perp(p^\varepsilon,\xi) + \varepsilon a_\perp(q^\varepsilon,\xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{V} \\
\int_\Omega q^\varepsilon \chi + a_{||}(\chi,\mu^\varepsilon) = 0 & \forall \chi \in \mathcal{V} \\
a_{||}(\tau, l^\varepsilon) = 0 & \forall \tau \in \mathcal{L}
\end{cases}
$$

Here we search for $p^\varepsilon,\ q^\varepsilon,\ l^\varepsilon \in \mathcal{V}$ and $\lambda^\varepsilon,\ \mu^\varepsilon \in \mathcal{L}$.

Why is this AP ?

That's becase setting $\varepsilon = 0$ yields

- $q^\varepsilon \in \mathcal{A}$ and
$$\int_\Omega A_\parallel \nabla_\parallel q^\varepsilon \nabla_\parallel \xi \quad = 0 \quad \forall \xi \in \mathcal{A}$$
hence $\nabla_\parallel q^\varepsilon = 0$, hence $q^\varepsilon = 0$.
- The first two equation now give $\phi^\varepsilon = p^\varepsilon$ with

$$\begin{cases} \int_\Omega (A_\perp \nabla_\perp p^\varepsilon)\cdot \nabla_\perp \eta + \int_\Omega A_{||} \nabla_{||} \lambda^\varepsilon \nabla_{||} \eta = \int_\Omega f \eta & \forall \eta \in \mathcal{V} \\ \int_\Omega A_{||} \nabla_{||} p^\varepsilon \nabla_{||} \kappa = 0 & \forall \kappa \in \mathcal{L} \end{cases}$$

Discretization

- Take any finite element space $V_h \subset H^1(\Omega)$ such that

$$v_h|_{\Gamma_D} = 0 \quad \forall v_h \in V_h$$

  We have tried $\mathbb{P}_1$, $\mathbb{P}_2$; $\mathbb{Q}_1$ and $\mathbb{Q}_2$ finite elements on a rectangle
- Take L_h as the subspace of V_h such that

$$\lambda_h|_{\Gamma_{in}} = 0 \quad \forall \lambda_h \in L_h$$

Discretization

- Take any finite element space $V_h \subset H^1(\Omega)$ such that

$$v_h|_{\Gamma_D} = 0 \quad \forall v_h \in V_h$$

  We have tried $\mathbb{P}_1$, $\mathbb{P}_2$; $\mathbb{Q}_1$ and $\mathbb{Q}_2$ finite elements on a rectangle

- Take L_h as the subspace of V_h such that

$$\lambda_h|_{\Gamma_{in}} = 0 \quad \forall \lambda_h \in L_h$$

- It is important to require $L_h \subset V_h$

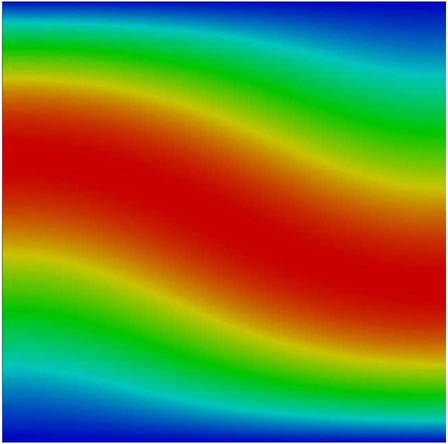
$$\lambda_h|_{\Gamma_D} = 0 \quad \forall \lambda_h \in L_h$$

  although it is not necessarily true on the continuous level.

☞ On the discrete level, we do not have the uniqueness of Lagrange multipliers, if $L_h \not\subset V_h$.

Numerical results using AP5 with $\mathbb{Q}_2$ finite elements on a regular grid on a square

Limit solution



Analytical solution

$$\phi^\varepsilon = \sin\left(\pi y + \alpha(y^2 - y)\cos(\pi x)\right) + \varepsilon\cos(2\pi x)\sin(\pi y)$$

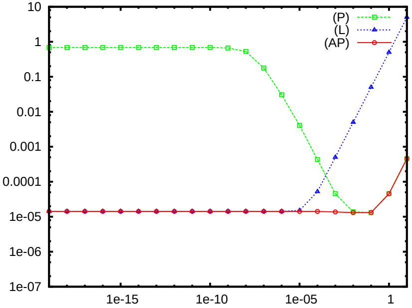
Field b

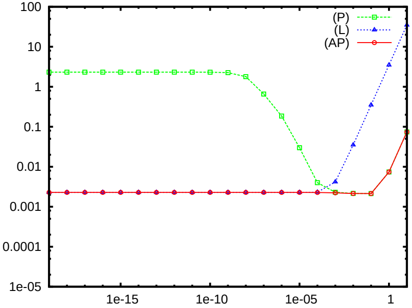
$$b = \frac{B}{|B|}$$

$$B = \begin{pmatrix} \alpha(2y-1)\cos(\pi x) + \pi \\ \pi\alpha(y^2 - y)\sin(\pi x) \end{pmatrix}$$

Error vs. ε

Mesh size : 50×50

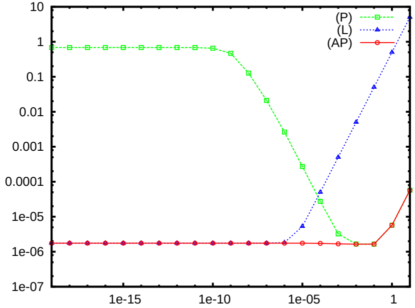


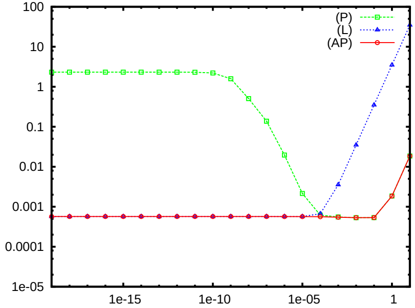


L^2 error on the left, H^1 error on the right

Error vs. ε

Mesh size : 100×100

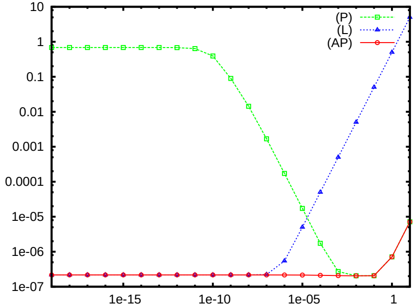


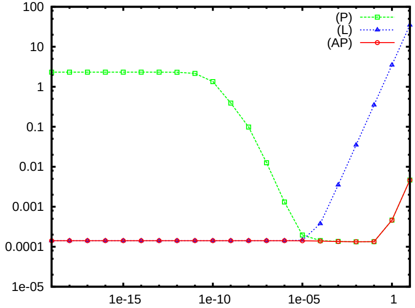


L^2 error on the left, H^1 error on the right

Error vs. ε

Mesh size : 200×200





L^2 error on the left, H^1 error on the right

3D test case

Analytical solution

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\phi^{\varepsilon} = \sin(\pi y)\sin(\pi z) + \varepsilon\cos(2\pi x)\sin(\pi y)\sin(\pi z)$$

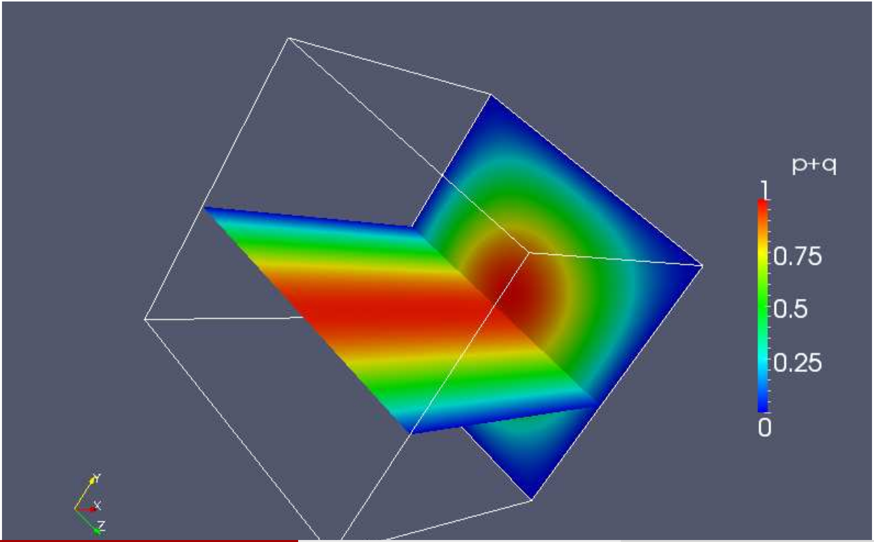
$$p^{\varepsilon} = \sin(\pi y)\sin(\pi z)$$

$$q^{\varepsilon} = \varepsilon\cos(2\pi x)\sin(\pi y)\sin(\pi z)$$

$$f = 2\pi^2(2+\varepsilon)(\cos(2\pi x)+1)\sin(\pi y)\sin(\pi z)$$

3D test case

Mesh size : $40 \times 40 \times 40$, $\varepsilon = 10^{-7}$



Looking out for simpler reformulations (AP3)

- The precise form of the decomposition $\mathcal{V} = \mathcal{G} \oplus \mathcal{A}$ is not really important.
- Why not to replace $\mathcal{A}$ by $\mathcal{L} := \{\lambda \in L^2(\Omega) \ / \ \nabla_{||}\lambda \in L^2(\Omega)\,, \ \ \lambda_{|\Gamma_{in}} = 0\}$?
- We have then

$$\mathcal{V} \approx \mathcal{G} \oplus \mathcal{A}$$

(strictly speaking, this decomposition is valid only on sufficiently smooth functions)

Looking out for simpler reformulations (AP3)

- The precise form of the decomposition $\mathcal{V} = \mathcal{G} \oplus \mathcal{A}$ is not really important.
- Why not to replace $\mathcal{A}$ by $\mathcal{L} := \{\lambda \in L^2(\Omega) \ / \ \nabla_{||}\lambda \in L^2(\Omega)\,, \ \ \lambda_{|\Gamma_{in}} = 0\}$?
- We have then
$$\mathcal{V} \approx \mathcal{G} \oplus \mathcal{A}$$
(strictly speaking, this decomposition is valid only on sufficiently smooth functions)
- We search now for $p^\varepsilon \in \mathcal{V}$, $q^\varepsilon \in \mathcal{L}$, $\lambda^\varepsilon \in \mathcal{L}$ such that

$$\begin{cases} a_\perp(p^\varepsilon, \eta) + a_\perp(q^\varepsilon, \eta) + a_{||}(\lambda^\varepsilon, \eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{V} \\ a_{||}(p^\varepsilon, \kappa) = 0 & \forall \kappa \in \mathcal{L} \\ a_{||}(q^\varepsilon, \xi) + \varepsilon a_\perp(p^\varepsilon, \xi) + \varepsilon a_\perp(q^\varepsilon, \xi) = \varepsilon \int_\Omega f\xi & \forall \xi \in \mathcal{L} \end{cases}$$

and set $\phi^\varepsilon = p^\varepsilon + q^\varepsilon$ as before.

One can do even better

The idea of Jacek Narski

- We keep the decomposition $\mathcal{V} \approx \mathcal{G} \oplus \mathcal{L}$ (zero inflow conditions)
- But we set

$$\phi^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$$

instead of $\phi^\varepsilon = p^\varepsilon + q^\varepsilon$

- Just substitute it into the governing equation
- We search now for $p^\varepsilon \in \mathcal{V}$ and $q^\varepsilon \in \mathcal{L}$ such that

$$\begin{cases} a_\perp(p^\varepsilon + \varepsilon q^\varepsilon, \eta) + a_{||}(q^\varepsilon, \eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{V} \\ a_{||}(p^\varepsilon, \kappa) = 0 & \forall \kappa \in \mathcal{L} \end{cases}$$

One can do even better

The idea of Jacek Narski

- We keep the decomposition $\mathcal{V} \approx \mathcal{G} \oplus \mathcal{L}$ (zero inflow conditions)
- But we set

$$\phi^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$$

  instead of $\phi^\varepsilon = p^\varepsilon + q^\varepsilon$
- Just substitute it into the governing equation
- We search now for $p^\varepsilon \in \mathcal{V}$ and $q^\varepsilon \in \mathcal{L}$ such that

$$\begin{cases} a_\perp(p^\varepsilon + \varepsilon q^\varepsilon, \eta) + a_{||}(q^\varepsilon, \eta) = \int_\Omega f\eta & \forall \eta \in \mathcal{V} \\ a_{||}(p^\varepsilon, \kappa) = 0 & \forall \kappa \in \mathcal{L} \end{cases}$$

☞ In the limit $\varepsilon = 0$ we have $\phi^\varepsilon = p^\varepsilon$ by definition and q^ε acts like the Lagrange multiplier for the constraint $p^\varepsilon \in \mathcal{G}$.

Our best reformulation (AP2)

P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2011)

- Reintroduce $\phi^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$ into the scheme and forget about p^ε.
- We search now for $\phi^\varepsilon \in \mathcal{V}$ and $q^\varepsilon \in \mathcal{L}$ such that

$$\begin{cases} a_\perp(\phi^\varepsilon, \eta) + a_{||}(q^\varepsilon, \eta) & = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V} \\ a_{||}(\phi^\varepsilon, \kappa) - \varepsilon a_{||}(q^\varepsilon, \kappa) & = 0, \quad \forall \kappa \in \mathcal{L} \end{cases}$$

Our best reformulation (AP2)

P. Degond, F. Deluzet, AL, J. Narski and C. Negulescu (2011)

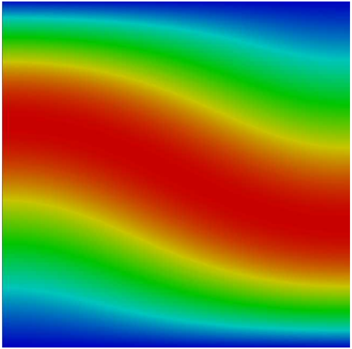
- Reintroduce $\phi^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$ into the scheme and forget about p^ε.
- We search now for $\phi^\varepsilon \in \mathcal{V}$ and $q^\varepsilon \in \mathcal{L}$ such that

$$\begin{cases} a_\perp(\phi^\varepsilon, \eta) + a_{||}(q^\varepsilon, \eta) & = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V} \\ a_{||}(\phi^\varepsilon, \kappa) - \varepsilon a_{||}(q^\varepsilon, \kappa) & = 0, \qquad \forall \kappa \in \mathcal{L} \end{cases}$$

☞ Setting $\varepsilon = 0$ yields the correct limit problem :

$$\begin{cases} a_\perp(\phi^0, \eta) + a_{||}(q^0, \eta) & = \int_\Omega f\eta, \quad \forall \eta \in \mathcal{V} \\ a_{||}(\phi^0, \kappa) & = 0, \qquad \forall \kappa \in \mathcal{L} \end{cases}$$

Numerical results using AP2 with $\mathbb{Q}_2$ finite elements on a regular grid on a square



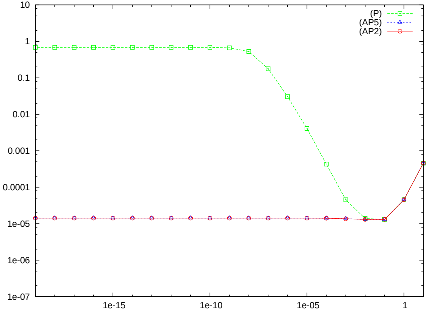
The limit solution

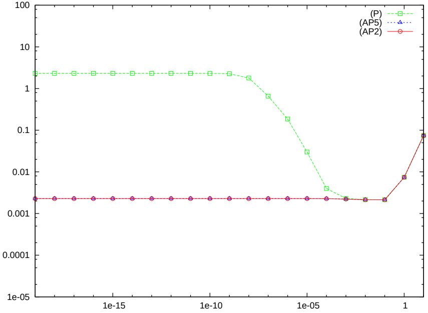
h	L^2-error	H^1-error
0.1	5.7×10^{-3}	1.86×10^{-1}
0.05	7.3×10^{-4}	4.7×10^{-2}
0.025	9.1×10^{-5}	1.18×10^{-2}
0.0125	1.14×10^{-5}	2.96×10^{-3}
0.00625	1.43×10^{-6}	7.4×10^{-4}
0.003125	1.78×10^{-7}	1.85×10^{-4}
0.0015625	2.23×10^{-8}	4.6×10^{-5}

The error for ϕ^ε in L^2 and H^1 norms for different mesh sizes for $\varepsilon = 10^{-3}$ for the AP2 scheme.

Error vs. ε

Mesh size : 50×50

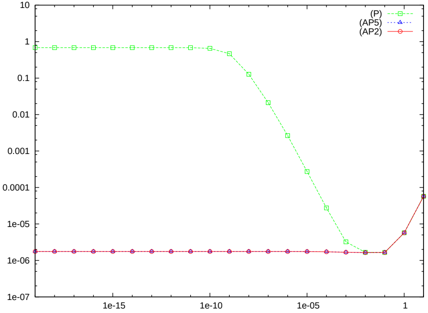


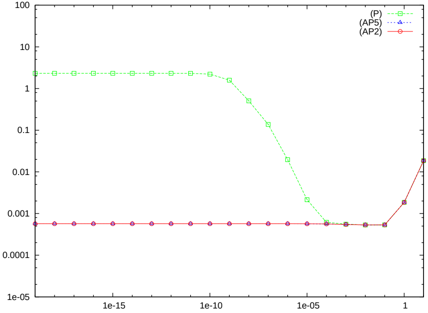


L^2 error on the left, H^1 error on the right

Error vs. ε

Mesh size : 100×100

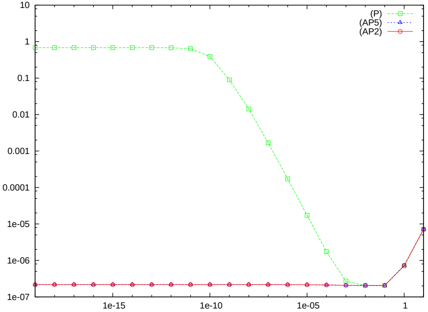


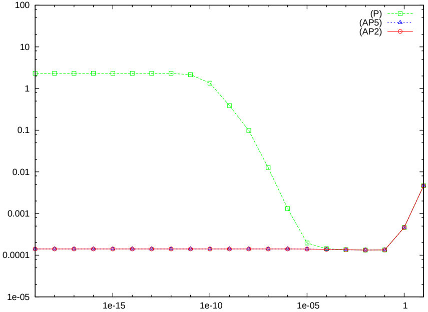


L^2 error on the left, H^1 error on the right

Error vs. ε

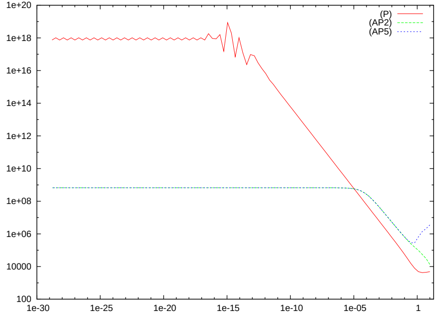
Mesh size : 200×200



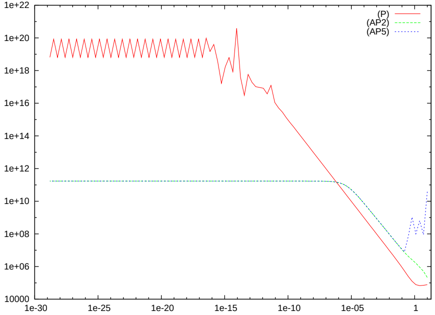


L^2 error on the left, H^1 error on the right

Condition numbers



mesh size : 100×100 points



mesh size : 400×400 points

Condition number estimate provided by the MUMPS solver for the (AP2), (AP5) and (P=straightforward) schemes.

Computational cost

Comparison between the Asymptotic Preserving schemes AP2, AP5 and the Singular Perturbation model (P) for $h = 0.01$ (100 mesh points in each direction) and fixed $\varepsilon = 10^{-6}$:

- matrix size
- number of nonzero elements
- computation time

method	# rows	# non zero	CPU time
AP2	20×10^3	621×10^3	5.227 s
AP5	50×10^3	1563×10^3	13.212 s
P	10×10^3	157×10^3	3.707 s

Extension to the case of a non constant ε

- The extension to the general case $\varepsilon = \varepsilon(x)$ is very simple.
- Instead of writing $u = p + \varepsilon q$, we introduce q as

$$\nabla_{\parallel} q = \frac{1}{\varepsilon} \nabla_{\parallel} u$$

with again $q \in \mathcal{L}$, i.e. $q = 0$ on Γ_{in}.
- This leads to almost the same reformulated system as before :

$$\begin{cases} \int_\Omega (A_\perp \nabla_\perp u^\varepsilon) \cdot \nabla_\perp v \, dx + \int_\Omega A_{||} \nabla_{||} q^\varepsilon \cdot \nabla_{||} v \, dx = \int_\Omega fv \, dx, & \forall v \in \mathcal{V} \\ \int_\Omega A_{||} \nabla_{||} u^\varepsilon \cdot \nabla_{||} w \, dx - \int_\Omega \varepsilon A_{||} \nabla_{||} q^\varepsilon \cdot \nabla_{||} w \, dx = 0, & \forall w \in \mathcal{L} \, . \end{cases}$$

The benchmark with a variable ε

- We take a benchmark where ε is of order 1 in a part of Ω ond of order ε_{min} in the remaining part :

$$\varepsilon(x,y) = \frac{1}{2}\left[1 + \tanh\left(a(x_0 - x)\right) + \varepsilon_{min}\left(1 - \tanh\left(a(x_0 - x)\right)\right)\right],$$

with

$$a = 50, \quad x_0 = 0.25$$

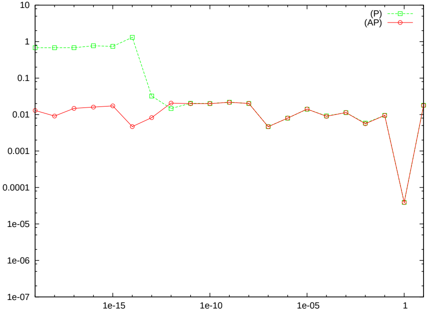
- The exact solution :

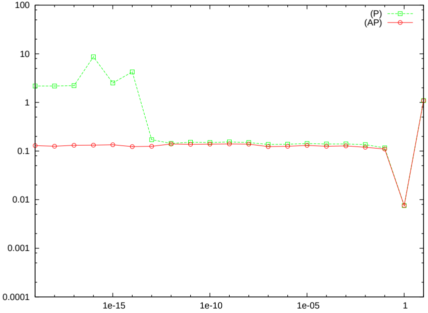
$$u^\varepsilon = \sin\left(\pi y + \alpha(y^2 - y)\cos(\pi x)\right) + \varepsilon\cos\left(2\pi x\right)\sin\left(\pi y\right)$$

- The shape of Ω and the field b are as in the previous numerical experiments

Error vs. $\varepsilon_{\min}$

Mesh size : 50×50

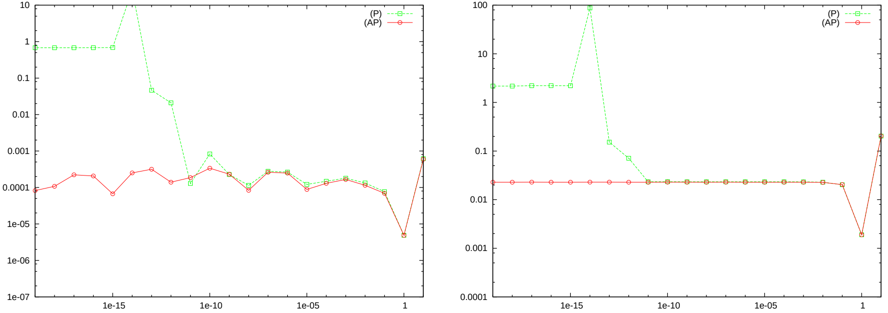




L^2 error on the left, H^1 error on the right

Error vs. ε_{min}

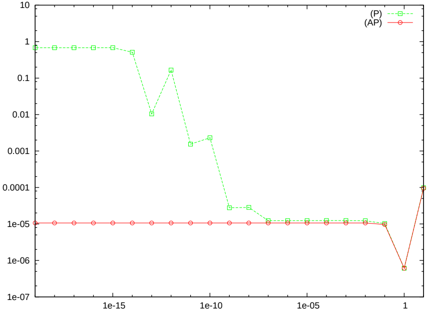
Mesh size : 100×100

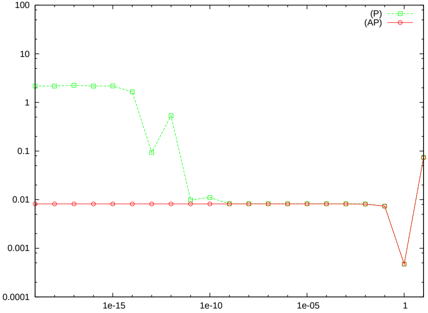


L^2 error on the left, H^1 error on the right

Error vs. $\varepsilon_{\min}$

Mesh size : 200×200





L^2 error on the left, H^1 error on the right

What can be proved for AP5

Provided $f \in L^2(\Omega)$ and for every $\varepsilon > 0$, the AP5 formulation admits a unique solution $(p^\varepsilon, q^\varepsilon) \in \mathcal{G} \times \mathcal{A}$, where $\phi^\varepsilon := p^\varepsilon + q^\varepsilon$ is the unique solution in $\mathcal{V}$ to the original problem.
These solutions satisfy the bounds

$$||\phi^\varepsilon||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)}, \quad ||p^\varepsilon||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)},$$
$$||\nabla_{||} q^\varepsilon||_{L^2(\Omega)} \leq C\varepsilon||f||_{L^2(\Omega)},$$

with an ε-independent constant $C > 0$. We thus have

$$\phi^\varepsilon \to \phi^0, \quad p^\varepsilon \to \phi^0 \text{ and } \quad q^\varepsilon \to 0 \quad \text{in} \quad H^1(\Omega) \text{ as } \varepsilon \to 0,$$

where $\phi^0 \in \mathcal{G}$ is the unique solution of the Limit problem.

What can be proved for AP2

Provided $f \in L^2(\Omega)$ and for every $\varepsilon > 0$, the AP2 formulation admits a unique solution $(\phi^\varepsilon, q^\varepsilon) \in \mathcal{V} \times \mathcal{L}$, where ϕ^ε is the unique solution in $\mathcal{V}$ to the original problem.
These solutions satisfy the bounds

$$||\phi^\varepsilon||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)}, \quad ||\nabla_{||} q^\varepsilon||_{L^2(\Omega)} \leq C||f||_{L^2(\Omega)},$$

with an ε-independent constant $C > 0$. We thus have

$$\phi^\varepsilon \to \phi^0, \text{ and } \quad q^\varepsilon \rightharpoonup \lambda^0 \quad \text{in} \quad H^1(\Omega) \text{ as } \varepsilon \to 0\,,$$

where $\phi^0 \in \mathcal{G}$ is the unique solution of the Limit problem and λ^0 is the corresponding Lagrange multiplier.

Numerical analysis

Something that our limit problem (L) and the scheme (AP2) resemble very much :

- An abstract saddle-point problem :
  find $(u, q) \in X \times M$ such that

$$\begin{aligned} a(u, v) + b(q, v) &= < f, v > && \forall v \in X \\ b(w, u) &= 0 && \forall w \in M \end{aligned}$$

- Approximation by penalty (cf. the book by Girault&Raviart, section 4.3) :
  find $(u^\varepsilon, q^\varepsilon) \in X \times M$ such that

$$\begin{aligned} a(u^\varepsilon, v) + b(q^\varepsilon, v) &= < f, v > && \forall v \in X \\ b(w, u^\varepsilon) - \varepsilon c(v^\varepsilon, w) &= 0 && \forall w \in M \end{aligned}$$

- We have existence, uniqueness, convergence $(u^\varepsilon, q^\varepsilon) \to (u, q)$ under the assumtion

$$\forall q \in M \quad : \quad \sup_{v \in X} \frac{b(q, v)}{||v||_X} \geq \beta ||q||_M$$

Does it apply to the analysis of (AP2) ?

- No
- We could define

$$X = \mathcal{V}, \quad M = \mathcal{L}$$

$$a(u,v) = a_\perp(u,v), \quad b(q,v) = a_{||}(q,v), \quad c(q,w) = a_{||}(q,w)$$

so that (AP2) seems like the penalized saddle-point problem above.

- But the inf-sup does not hold :

$$\inf_{q\in\mathcal{L}} \sup_{v\in\mathcal{V}} \frac{a_{||}(q,v)}{||v||_V} = \inf_{q\in\mathcal{L}} \sup_{v\in\mathcal{V}} \frac{\int \nabla_{||} q \cdot \nabla_{||} v}{\left(\int |\nabla_{||} v|^2 + \int |\nabla_\perp v|^2\right)^{1/2}} = 0$$

The proper setting of the limit problem

- Consider the problem : find $(u, q) \in \mathcal{V} \times \mathcal{L}$ with usual $\mathcal{V}$ and $\mathcal{L}$ such that

$$\begin{aligned} a_{\perp}(u, v) + a_{||}(q, v) &= (f, v) && \forall v \in \mathcal{V} \\ a_{||}(u, w) - \varepsilon a_{||}(q, w) &= 0 && \forall w \in \mathcal{L} \end{aligned}$$

- Define the norm

$$|q|^* = \sup_{v \in \mathcal{V}} \frac{a_{||}(v, q)}{||v||_V}$$

Searching $(u, q) \in \mathcal{V} \times \mathcal{L}^*$ is the proper setting for the limit problem $\varepsilon = 0$.

- This choice does not work when $\varepsilon > 0$. Indeed, the term $a_{||}(q, w)$ makes no sense if we suppose only $(q, w) \in \mathcal{L}^*$.
- A new norm on $\mathcal{L}$

$$|q|_{\varepsilon} = |q|^* + \sqrt{\varepsilon}|q|_{||}$$

This is indeed a norm on our old $\mathcal{L}$ since the new norm is equivalent to the old one

$$|q|_{\varepsilon} \leq (1 + \sqrt{\varepsilon})|q|_{||} \text{ and } |q|_{||} \leq \frac{1}{\sqrt{\varepsilon}}|q|_{\varepsilon}$$

Well-posedness

Introducing the coupled bilinear norm on $X = \mathcal{V} \times \mathcal{L}$

$$c((u,q),(v,w)) = a_\perp(u,v) + a_{||}(q,v) + a_{||}(u,w) - \varepsilon a_{||}(q,w)$$

we see immediately that it is continuous with the ε-independent constant

$$\begin{aligned} c((u,q),(v,w)) &\leq |u|_\perp |v|_\perp + |q|_{||}|v|_{||} + |u|_{||}|w|_{||} + \varepsilon|q|_{||}|w|_{||} \\ &\leq 2|u,q|_X |v,w|_X \end{aligned}$$

with

$$|u,q|_X = (|u|_V^2 + |q|_\varepsilon^2)^{1/2}$$

We also have the infsup

$$\inf_{(u,q)\in X} \sup_{(v,w)\in X} \frac{c((u,q),(v,w))}{|u,q|_X |v,w|_X} \geq \beta$$

with an ε-independent constant β.

Discretization

- Let us now add the subscript h everywhere :

$$\begin{aligned} a_\perp(u_h, v_h) + a_{||}(q_h, v_h) &= (f, v_h) && \forall v_h \in V_h \\ a_{||}(u_h, w_h) - \varepsilon a_{||}(q_h, w_h) &= 0 && \forall w_h \in L_h \end{aligned}$$

- The bilinear form c is still continuous on $X_h = V_h \times L_h$. But we also need the infsup

$$\inf_{(u_h,q_h)\in X_h} \sup_{(v_h,w_h)\in X_h} \frac{c((u_h, q_h), (v_h, w_h))}{|u_h, q_h|_X |v_h, w_h|_X} \geq \beta \tag{1}$$

Well-posedness of the discrete problem

We give first a general result providing the sufficient conditions for the discrete inf-sup to hold.

Lemma

Suppose that $L_h \subset V_h$ and the discrete inf-sup for the form $a_{||}$ holds with respect to the star norm

$$\inf_{q_h \in L_h} \sup_{v_h \in V_h} \frac{a_{||}(q_h, v_h)}{|q_h|^* |v_h|_V} \geq \alpha_1$$

with some $\alpha_1 > 0$. Moreover, for any $u_h \in V_h$ denote $u_h^0 \in L_h$ such that $a_{||}(u_h^0, w_h) = a_{||}(u_h, w_h)$ for all $w_h \in L_h$ and suppose

$$|u_h|_\perp^2 + |u_h^0|_{||}^2 \geq \alpha_2 |u_h|_V^2$$

with some $\alpha_2 > 0$. Then the discrete inf-sup property (1) holds with a constant $\beta > 0$ that depends only on α_1 and α_2.

Conclusions

- The AP approach provides a powerful tool to construct the robust numerical methods for anisotropic problems, including the situations where the anisotropy strength varies wildly throughout the domain
- The decomposition of the type "$p^\varepsilon + \varepsilon q^\varepsilon$" will be very helpful in more complicated problems in plasma simulation, as for instance, Euler-Lorentz equations

$$\partial_t n + \nabla \cdot (nu) = 0$$
$$m(\partial_t (nu) + \nabla \cdot (nu \otimes u)) + \nabla p(n) = en(E + u \times B)$$

An abundant literature exists already on the AP schemes for this kind of problems (P. Degond, M.-H. Vignal, S . Brull, A. Mouton ...)