Numerical Zoom for a Multi-Scale Problem

http://www.ann.jussieu.fr/pironneau

Olivier Pironneau[1]

[1]University of Paris VI, Laboratoire J.-L. Lions, **Olivier.Pironneau@upmc.fr**

with J.-B. Apoung Kamga, A. Lozinski

The Site of Bure

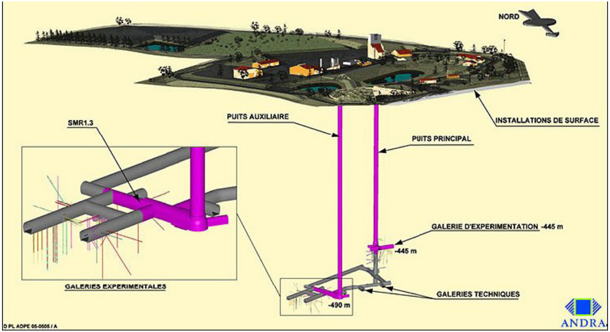


Figure: Schematic view of the Bure project (East of France)

Nuclear waste is cooled, processed, then buried safely for 1M years
Simulation requires a super computer, or does it really?

The COUPLEX I Test Case

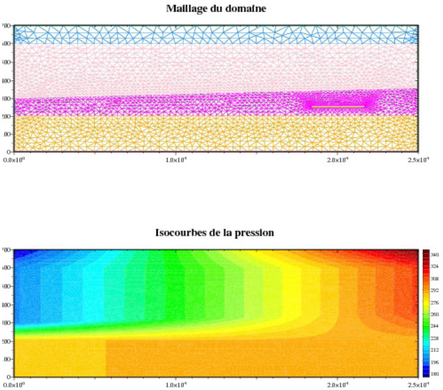


Figure: A 2D multilayered geometry 20km long, 500m high with permeability variations $\frac{K^+}{K^-} = O(10^9)$. Hydrostatic pressure by a FEM.

$$\nabla \cdot (K \nabla H) = 0, \quad H \text{ or } \frac{\partial H}{\partial n} \text{ given on } \Gamma$$

COUPLEX I : Concentration of Radio-Nucleides

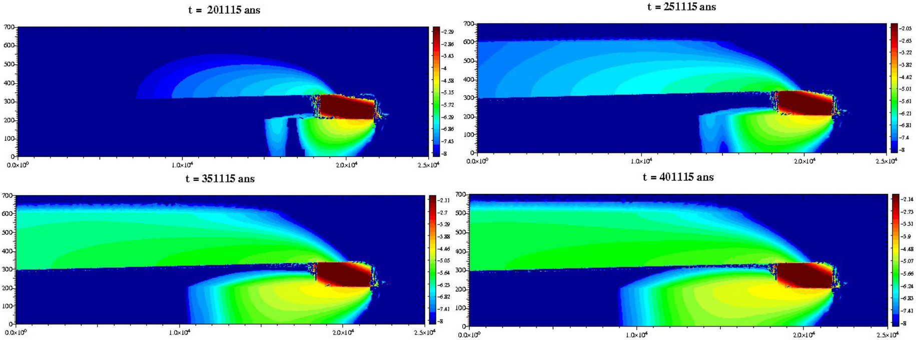
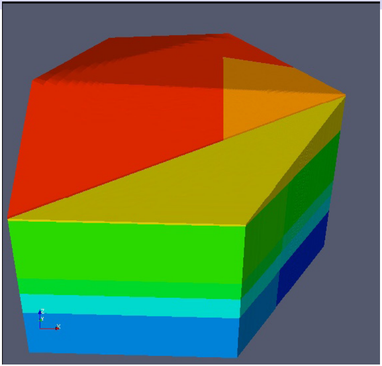


Figure: Concentration at 4 times with Discontinuous Galerkin FEM (Apoung-Despré).

$$r\partial_t c + \lambda c + u\nabla c - \nabla \cdot (K\nabla c) = q(t)\delta(x - x_R)$$

Couplex II: Geological figures

Layer	Permeability
Tithonien	3.10^{-5}
Kimmeridgien I	3.10^{-4}
Kimmeridgien II	10^{-12}
Oxfordien I	2.10^{-7}
Oxfordien II	8.10^{-9}
Oxfordien III	4.10^{-12}
Callovo-Oxfordien	10^{-13}
Dogger	2.510^{-6}



Layer decomposition: $K^+\frac{\partial H}{\partial n}^+ = K^-\frac{\partial H}{\partial n}^-$ implies that $\frac{\partial H}{\partial n}^+ = O(\frac{K^-}{K^+})$.
So $\frac{\partial H}{\partial n}|_{KI-KII} \approx 0$ is a B.C. that decouples the top from the bottom.
Later $H^-|_{KII} = H^+$ is used as B.C for the bottom.
Note that the Callovo-Oxfordian+Oxfordian III have $H|_\Gamma$ given from top and bottom separate calculations.

COUPLEX II Hydrostatic Pressure

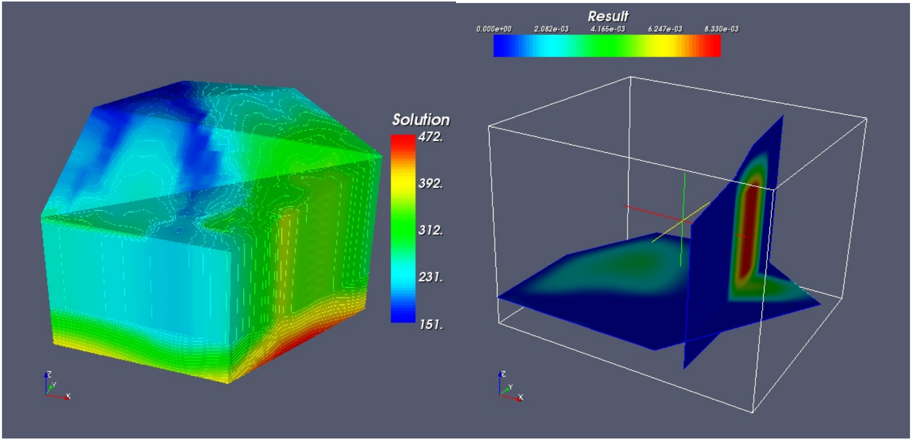


Figure: Final result and comparison with a global solution on a supercomputer (Apoung)

The Clay Layer with the repository

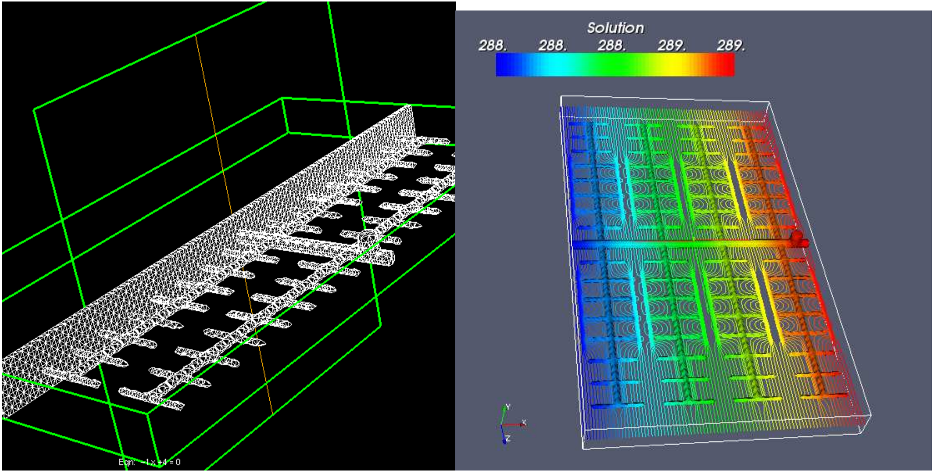


Figure: A computation within the clay layer only with Dirichlet B.C. from the surrounding layers (Apoung-Delpino). Left: a geometrical zoom

First Numerical Zoom

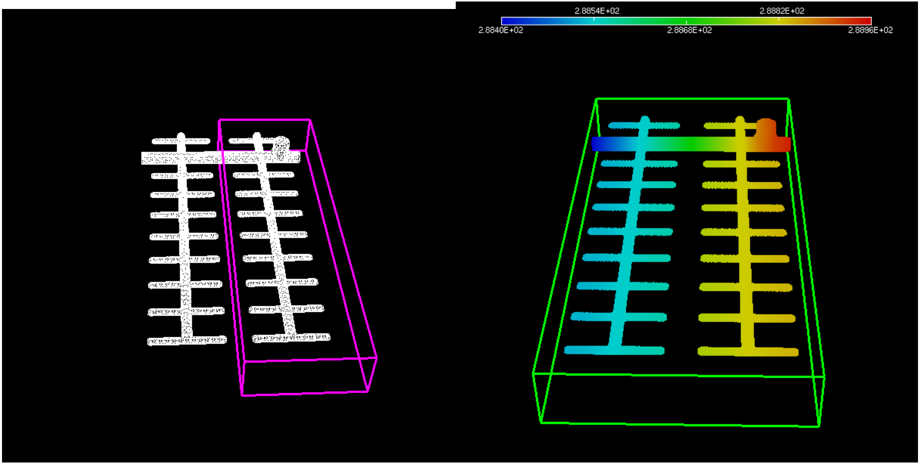


Figure: Mesh and Sol of Darcy's in a portion of the entire site.

Second Zoom

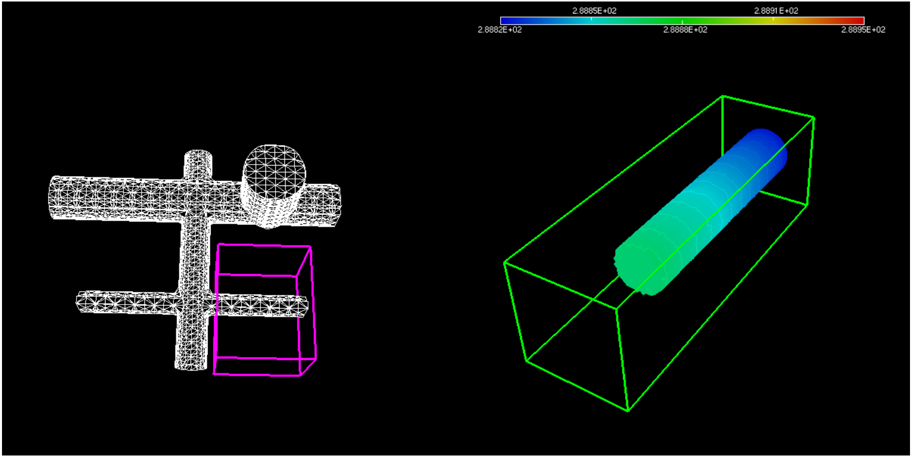
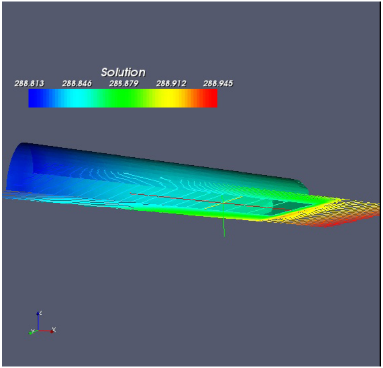
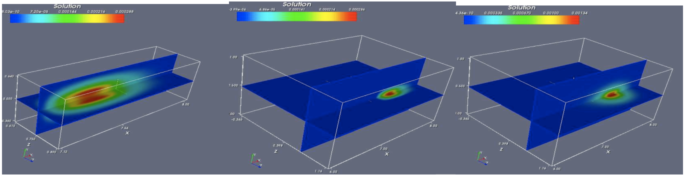


Figure: Mesh and Sol around a single gallery capable of evaluating the impact of a lining around the gallery.

Last Zoom and upscale comp. of the concentration





What are the errors in the end?

Why Numerical Zoom

Graphical zoom are always needed and is easy when the problem is solved on the fine mesh

- Numerical zoom are needed when it is very expensive or impossible to solve the full problem
- For instance if the problem has multiple scales
- Improved precision may be found necessary a posteriori

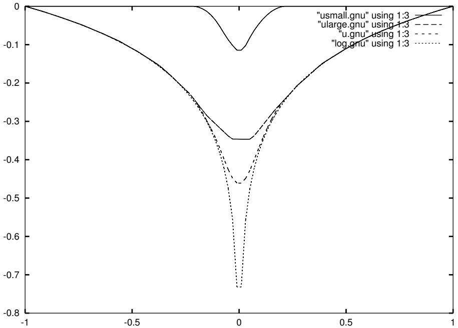
- Numerical zoom methods exist:
  - Steger's Chimera method,
  - J.L. Lions's Hilbert space decomposition (HSD),
  - Glowinski-He-Rappaz-Wagner's Subspace correction methods (SCM), etc.
- We need error estimates.

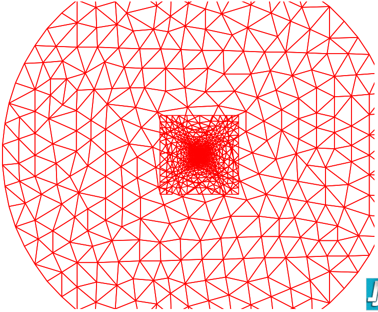
Hilbert Space Decomposition Method (JL. Lions)

Model problem (for instance $K = I,\ f = 1 + \delta(|x - x_0|)$):

$$-\nabla(K\nabla v) = f \text{ in } \Omega, \quad u|_\Gamma = 0$$

$$v = U + u, \quad U \in H_0^1(\Omega), \quad u|_\Lambda \in H_0^1(\Lambda)$$
$$\beta(U^{n+1} - U^n) - \Delta(U^{n+1} + u^n) = f \text{ in } \Omega$$
$$\beta(u^{n+1} - u^n) - \Delta(U^n + u^{n+1}) = f \text{ in } \Lambda$$





Discretization and Proof of Uniqueness (Brezzi)

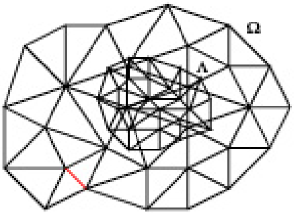
Find $U_H \in V_{0H} \approx H_0^1(\Omega)$, $u_h \in V_{0h} \approx H_0^1(\Lambda)$

$$a(U_H + u_h, W_H + w_h) = (f, W_H + w_h) \quad \forall W_H \in V_{0H} \ \forall w_h \in V_{0h}$$

Theorem The solution is unique if there are no vertices belong to both triangulations.

Proof

If $u_h = U_H$ on Λ then they are linear on Λ because $\Delta u_h = \Delta U_H$ and each is a distribution on the edges. The only singularity, if any, are at the intersection of both set of edges (which are points), but being in H^{-1} it cannot be singular at isolated points. So $\Delta u_h = \Delta U_H|_\Lambda = 0$



Subspace Correction Method (SCM)

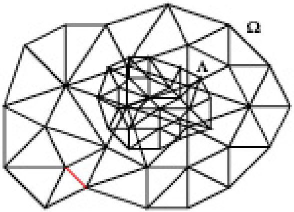
Find $U_H \in V_{0H} \approx H_0^1(\Omega)$, $u_h \in V_{0h} \approx H_0^1(\Lambda)$

$$a(U_H + u_h, W_H + w_h) = (f, W_H + w_h) \quad \forall W_H \in V_{0H} \quad \forall w_h \in V_{0h}$$

Theorem (Lozinski et al)

If u_H is computed with FEM of degree r and u_h with FEM of degree s, then with $q = \max\{r, s\} + 1$,

$$\|u_H + u_h - u\|_1 \leq c(H^r \|u\|_{H^q(\Omega\backslash\Lambda)} + h^s \|u\|_{H^q(\Lambda)})$$



Iterative process? Inexact quadrature?

Proof of the Theorem for 2 Zooms (I)

Let $u_{Hh\hbar} \in V_H + V_h + V_\hbar$ be a solution of:

$$a(u_{Hh\hbar}, v_{Hh\hbar}) = (f, v_{Hh\hbar}) \;\; \forall v_{Hh\hbar} \in V_H + V_h + V_\hbar$$

For some $u^1 \in H^1_0(\Omega)$, $u^2 \in H^1_0(\Lambda)$, $u^3 \in H^1_0(O)$, we choose $w_H = \pi_H u^1$, $w_h = \pi_h u^2$, $w_\hbar = \pi_\hbar u^3$, where the π are interpolants. Let $w_{Hh\hbar} = w_H + w_h + w_\hbar$ and $v_{Hh\hbar} = u_{Hh\hbar} - w_{Hh\hbar}$. Then

$$a(u, v_{Hh\hbar}) = a(u_{Hh\hbar}, v_{Hh\hbar}) \;\text{ and so }\; a(v_{Hh\hbar}, v_{Hh\hbar}) = a(u - w_{Hh\hbar}, v_{Hh\hbar})$$

Therefore $\|v_{Hh\hbar}\| \leq \|u - w_{Hh\hbar}\|$ and so

$$\|u - u_{Hh\hbar}\| \leq \|u - w_{Hh\hbar}\| + \|v_{Hh\hbar}\| \leq 2\|u - w_{Hh\hbar}\|$$

Finally, if $u^1 + u^2 + u^3 = u$,

$$\begin{aligned} \|u - w_{Hh\hbar}\| &\leq \|u^1 - w_H\| + \|u^2 - w_h\| + \|u^3 - w_\hbar\| \\ &\leq C(H\|u^1\|_2 + h\|u^2\|_2 + \hbar\|u^3\|_2) \end{aligned}$$

Proof of the Theorem for 2 Zooms (II)

Take u^1an extension in Λ of $u|_{\Omega\setminus\Lambda}$. Then

- $\|u^1\|_2 \leq \|u\|_{2,\Omega\setminus\Lambda}$ and $v^1 := u - u^1 \in H_0^1(\Lambda) \Rightarrow \|v^1\|_2 = \|u - u^1\|_{2,\Lambda}$.

Next, by taking u^2 to be an extension in O of $v^1|_{\Omega\setminus O}$ we secure $u^2 - v^1 \in H_0^1(O)$ and $\|u^2\|_2 = \|v^1\|_{2,\Lambda\setminus O}$.

Now $u^3 := u - u^1 - u^2 = v^1 - u^2 \in H_0^1(O)$ and so $\|u^3\|_2 = \|u^3\|_{2,O}$. This proves that

$$\|u - u_{Hh\hbar}\| \leq C(H\|u\|_{2,\Omega\setminus\Lambda} + h\|u\|_{2,\Lambda\setminus O} + \hbar\|u\|_{2,O})$$

Hilbert Space Decomposition with Inexact Quadrature

$$a_h(u_1+u_2,w_1+w_2)=a_h(u_1,w_1)+a_h(u_2,w_2)+a_h(u_1,w_2)+a_h(u_2,w_1)$$

2 grids: $\{T_k^1\}$ $\{T_k^2\}$ $a_h(u,v)=\sum_k\sum_{j=1..3}\frac{|T_k^1|}{3}\frac{\nabla u\cdot\nabla v}{I_{\Omega^1}+I_{\Omega^2}}|_{\xi^1_{jk}}$ + id with T_k^2

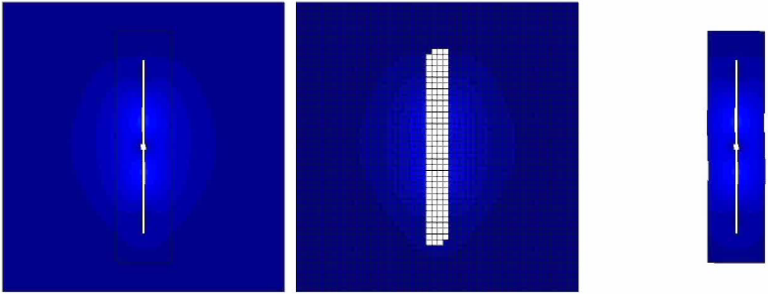
The gradients are computed on their native grids at vertices ξ.

Proposition *When vertices of $\mathcal{T}^i$ are strictly inside the T^j the discrete Solution is unique and* $\|u_h^1+u_h^2-u\|_1\le\frac{c}{C}h(\|u^1\|_2+\|u^2\|_2)$

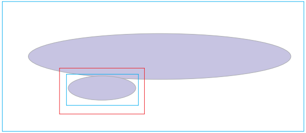
		$u-(u_1+u_2)$		
$N1$	L^2 error	rate	∇L^2 error	rate
10	$1.696E-02$	–	$2.394E-01$	–
20	$5.044E-03$	1.75	$1.204E-01$	0.99
40	$1.129E-03$	2.16	$5.596E-02$	1.10

Table: Numerical L^2 and H^1 errors, and convergence rate. Results are sensitive to rotation and translation of inner mesh

It is an Old Problem: Chimera



From W. A. Wall



Chimera: Computes

$$-\Delta U = 1, \text{ in } \Omega\backslash B_{x_0}(r) \qquad U|_{\partial B_{x_0}(r)} = u, \; U|_{\partial\Omega} = 0$$
$$-\Delta u = 1 + \delta(x_0) \text{ in } B_{x_0}(r+\rho), \qquad u|_{\partial B_{x_0}(r+\rho)} = U$$

which is also Schwarz' domain decomposition method

Harmonic Patch Iterator for Speed-up (Lozinski)

Proximity of vertices could lead to drastically slow convergence $\Rightarrow$

for $n = 1 ... N$ **do**

Find $\lambda_H^n \in V_H^0 = \{v_H \in V_{0H} : \mathrm{supp}\, v_H \subset \Lambda\}$ such that

$$a(\lambda_H^n, \mu) = \langle f|v\rangle - a(u_h^{n-1}, \mu), \quad \forall \mu \in V_{0H}$$

Find $u_H^n \in V_{0H}$ such that

$$a(u_H^n, v) = \langle f|v\rangle - a(u_h^{n-1}, v) - a(\lambda_H^n, v), \quad \forall v \in V_{0H}$$

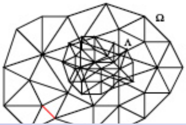
Find $u_h^n \in V_{0h}$ such that

$$a(u_h^n, v) = \langle f|v\rangle - a(u_H^{n-1}, v), \quad \forall v \in V_{0h}$$

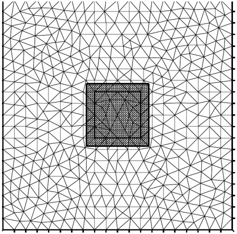
Set $u_{Hh}^n = u_H^n + u_h^n$

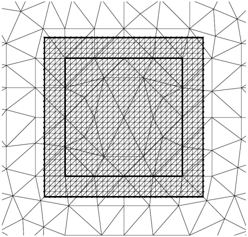
end for

Note: with $\tilde{u}_h^{n-1} = u_h^{n-1} + \lambda_H^n$ is it Schwarz?



Harmonic Patches





1/10		1/20		1/40	
H^1	L^2	H^1	L^2	H^1	L^2
1.00	8.50E−1	9.98E−1	8.48E−1	9.98E−1	8.49E−1
1.03E−2	6.18E−3	2.18E−3	1.25E−3	6.08E−4	4.05E−4
1.01E−2	5.22E−3	2.36E−3	1.27E−3	6.42E−4	4.36E−4
8.93E−3	4.79E−3	2.10E−3	1.11E−3	5.56E−4	3.01E−4
9.40E−3	5.09E−3	2.16E−3	1.17E−3	5.91E−4	3.72E−4
8.72E−3	4.89E−3	2.09E−3	1.09E−3	5.51E−4	2.87E−4
11		4		3	
0.8236		0.9339		0.9698	

Discrete one way Schwarz

If the Λ_h is a submesh of Ω_H then the same algorithm is:

1: **for** $n = 1...N$ **do**
2: Find $u_H^n - g_H \in V_{0H}$ such that

$$a(u_H^n, v) = \langle f|v\rangle - a_h(w_h^{n-1}, v) + a_\Lambda(u_H^{n-1}, v), \quad \forall v \in V_{0H}$$

3: Find $w_h^n \in V_h$ such that (r_h is a trace interpolation operator)

$$a(w_h^n, v) = \langle f|v\rangle, \quad \forall v \in V_{0h}, \quad w_h^n|_{\partial\Lambda} = r_h u_H^n|_{\partial\Lambda}$$

4: **end for**
5: Set

$$u_{Hh}^n = \begin{cases} w_h^n, & \text{in } \Lambda \\ u_H^n, & \text{outside } \Lambda \end{cases}$$

Implementation in 2D with freefem++ (F. Hecht)

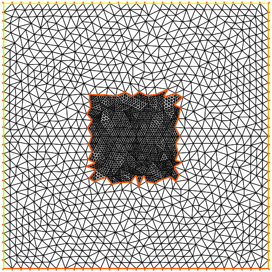
// embedded meshes with keyword splitmesh

```
int n=10, m=4;
real x0=0.33,y0=0.33,x1=0.66,y1=0.66;
mesh TH=square(n,n);
mesh Th = splitmesh(TH,(x>x0 && x<x1 && y>y0 && y<y1)*m);

mesh THth = splitmesh(TH,1+(x>x0 && x<x1 && y>y0 &&
y<y1)*(m-1));
solve aH(U,V) = int2d(TH)(K*(dx(U)*dx(V)+dy(U)*dy(V)))
+ int2d(Th)(K*(dx(u)*dx(V)+dy(u)*dy(V)))
- int2d(THh)(K*(dx(Uold)*dx(V)+dy(Uold)*dy(V)))
- int2d(TH)(f*V) + on(dOmega, U=g);
```

$K = 1$ except in a Disk 0.1 in the center where $K = 100$:

$$u = y - \frac{1}{2}, \text{ in the disk } = -\frac{1+K}{4} - \frac{(1-K)\delta^2}{4(x^2+y^2)} \text{ elsewhere} \qquad (1)$$



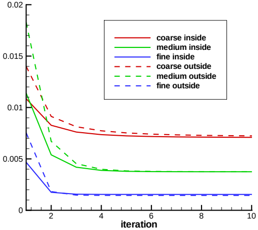


Figure: The initial mesh Ω_H is is divided 4 times in the zoom. Convergence history for 3 different initial meshes of the unit square: a coarse, medium (documented in the text) and fine mesh. 3 curves correspond to the errors on the mesh H and 3 for the mesh h.

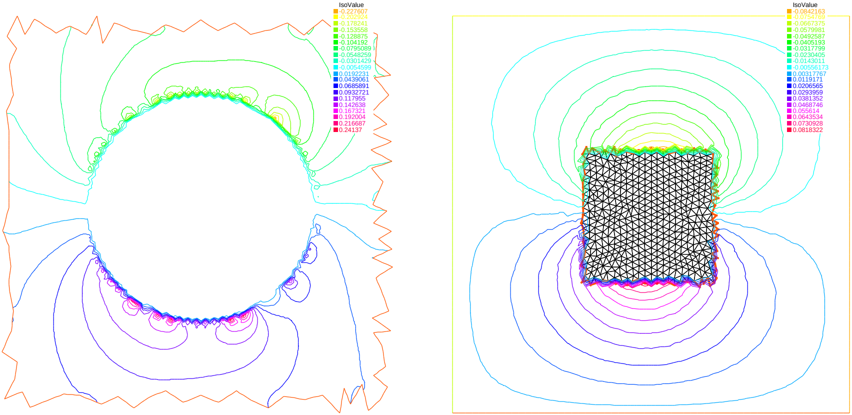
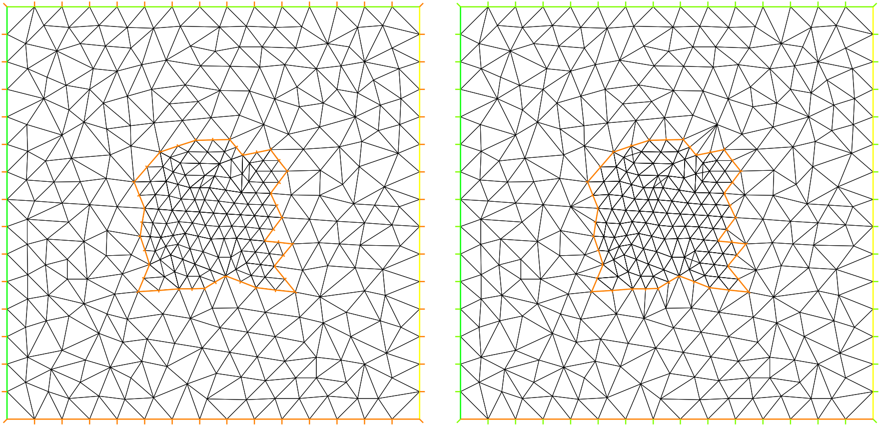


Figure: Error at each point for the converge solution in Λ (left) and outside (right) Λ on the fine mesh of Fig. The color scales from -0.23 to 0.24 on the left and from -0.08 to 0.08 on the right.

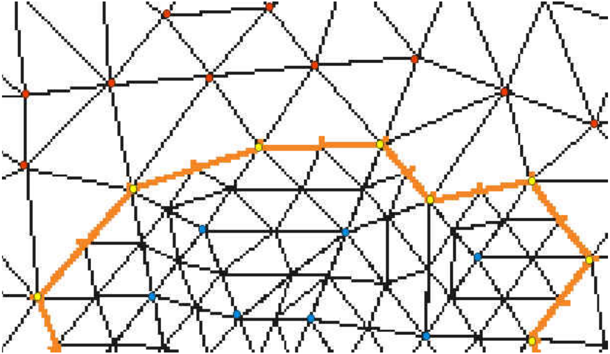
Embedded Meshes: Relation with Schwarz' DDM



Left: Divide the Triangles which have a vertex in $(.33, .66)^2 \Rightarrow$ not a valid mesh. Right: a valid mesh is obtained by joining the hanging vertices to their opposite vertex.

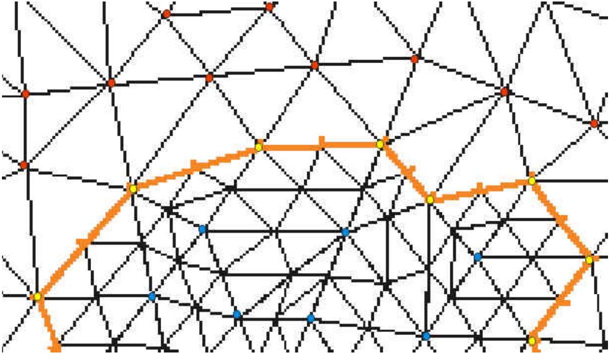
Relation with DDM by Schur Complement

vertices = coarse mesh of $\Omega_H \backslash \Lambda$
vertices = coarse mesh of Λ_H
vertices = coarse mesh vertices at the interface
vertices = extra vertices on the fine mesh of $\partial\Lambda_h$
Black vertices = extra vertices on the fine mesh of Λ_h



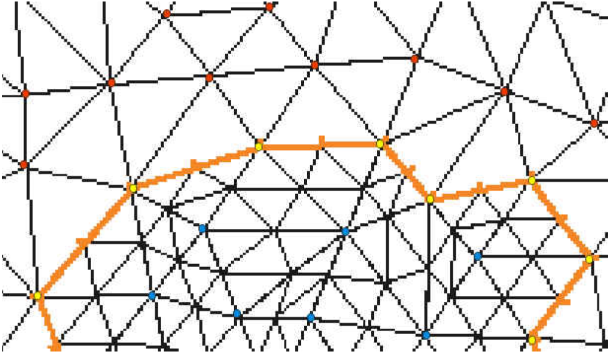
FEM with hanging nodes + Discrete Schwarz

$$\begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^n \\ u_2^n \\ u_3^{n-1} \end{pmatrix} = \tilde{f}$$



One way Schwarz Zoom

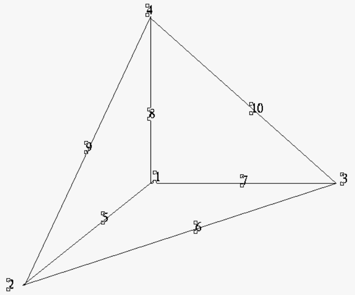
$$a(w_h^n, v) = \langle f|v\rangle, \quad \forall v \in V_{0h}, \quad w_h^n|_{\partial\Lambda} = r_h u_H^n|_{\partial\Lambda}$$
$$a(\mathbf{u}_H^n, v) = \langle f|v\rangle - a_h(w_h^{n-1}, v) + a_\Lambda(u_H^{n-1}, v), \quad \forall v \in V_{0H}$$



Implementation in 3D

The new elements have vertices

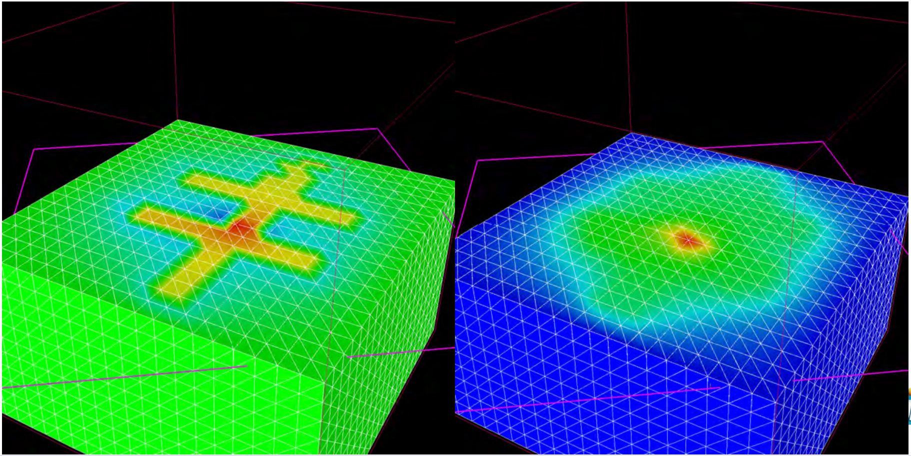
```
(1,5,7,8), (2,6,5,9), (3,7,6,10), (8,9,10,4),
(5,8,9,10), (5,9,6,10), (5,6,7,10), (5,7,8,10)
```



Vertex and Middle Numbers of an initial tetrahedron

A 3D case

There are 3 cylindrical galleries making a “Lorraine Cross" : figure **??.** A source term f is added at x_0 near the center of the main gallery (it is an exponential function $\exp(-0.1|x - x_0|^2)$; the Dirichlet conditions are set to zero, so the problem simulates a leak at x_0.



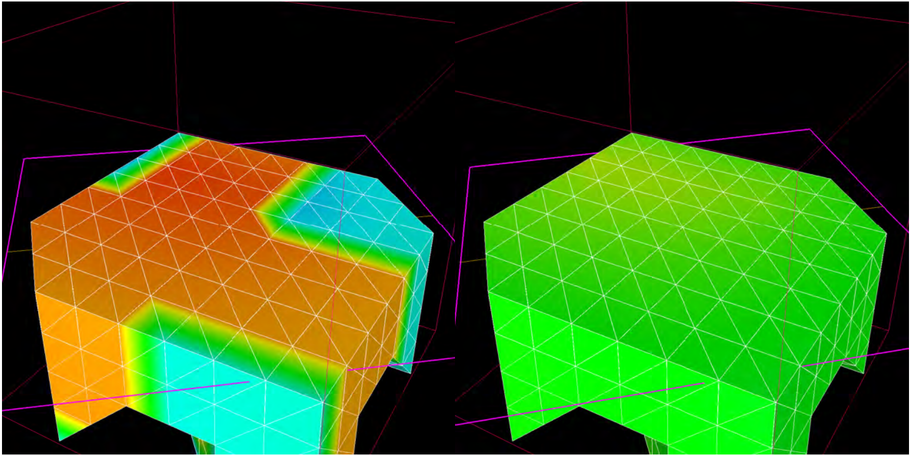


Figure: Hydrostatic pressure in the zoom. On the right $H(x, y, z)$ is changed into $-H(x, y, z)$ to show the galleries.

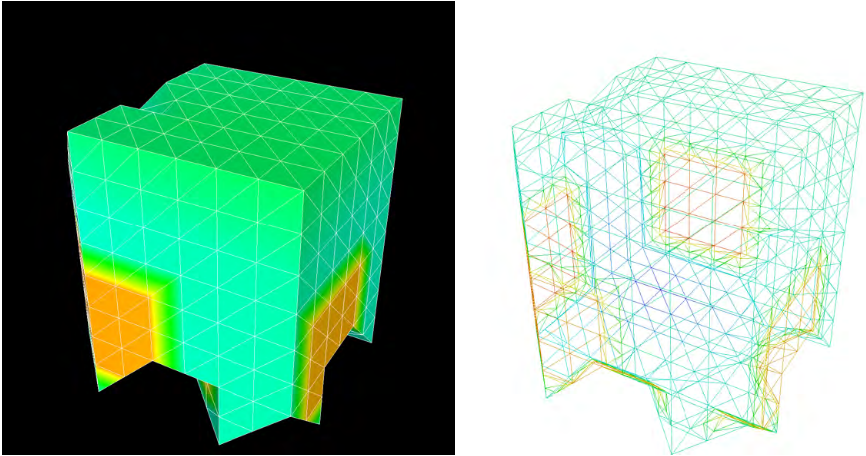
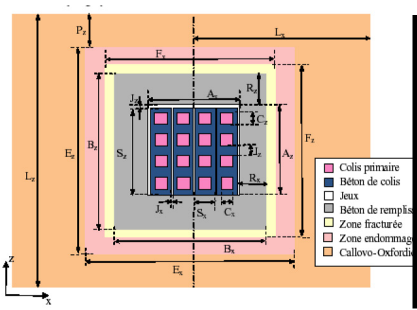
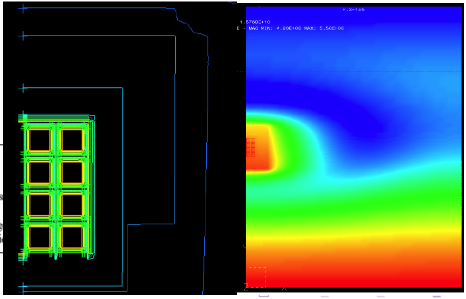


Figure: Two views of the complete zoom geometry and the Hydrostatic pressure on the boundary with the convention that H is $-H$ in the galleries.

Conclusion

- Numerical zooms are inevitable
- Precision: given by GHLR.
- With embedded meshes:
  - similar to DDM
  - convergence similar to full overlapping Schwarz
- Advice to code developer: since DDM is built in due to computer architecture why not add the zoom facility also!





Bibliography on www.ann.jussieu.fr/pironneau

- Apoung-Kamga J.B. and J.L., Pironneau : O. Numerical zoom. DDM16 conf. proc, New-York Jan 2005. D. Keyes ed.
- Brezzi,F., Lions, J.L., Pironneau, O. : Analysis of a Chimera Method. C.R.A.S., **332**, 655-660, (2001).
- P. Frey: `medit`, `http://www.ann.jussieu.fr/~frey`
- R. Glowinski, J. He, A. Lozinski, J. Rappaz, and J. Wagner. Finite element approximation of multi-scale elliptic problems using patches of elements. Numer. Math., 101(4):663–687, 2005.
- Hecht F.., O. Pironneau: `http://www.freefem.org`
- Lions, J.L., Pironneau, O. : Domain decomposition methods for CAD. C.R.A.S., **328** 73-80, (1999).
- J. HE, A. Lozinsk and J. Rappazi: Accelerating the method of finite element patches using harmonic functions. C.R.A.S. 2007.
- Steger J.L. : The Chimera method of flow simulation. Workshop on applied CFD, Univ. of Tennessee Space Institute, (1991).
- Wagner J. : FEM with Patches and Appl. Thesis 3478, EPFL, 2006.