Solutions explosives, contraintes sur l'état et comportement asymptotique pour des équations de Hamilton-Jacobi visqueuses

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Plan of the talk

Let Ω be a bounded smooth subset in $\mathbb{R}^N$. If $1 < q \leq 2$, there exists a maximal solution of

$$-\Delta u + |\nabla u|^q + \lambda u = f(x) \qquad \text{in } \Omega\,, \tag{1}$$

which satisfies the boundary condition

$$u(x) \to +\infty \quad \text{as } x \to \partial\Omega \quad (\text{i.e. } \mathrm{dist}(x, \partial\Omega) \to 0)\,. \tag{2}$$

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- Link with state constraint problems
- Qualitative properties and blow-up profile
- Applications to the ergodic problem with Dirichlet conditions
- How the blow-up solutions (and their profile) play a role in the large time behaviour for viscous HJ

$$u_t - \Delta u + |\nabla u|^q = f\,, \qquad u_{|_{\partial\Omega \times (0,T)}} = 0$$

1. Boundary blow-up solutions and state constraint

Let Ω be a bounded smooth subset in $\mathbb{R}^N$, and $f \in L^\infty$. When $1 < q \leq 2$, there exists a (unique) solution u of the problem

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Given a Brownian motion B_t and the SDE

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

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find an optimal feedback control $a \in C(\Omega)$ such that X_t does never leave the domain Ω.

NB: This is clearly a singular problem: one has to use unbounded controls (singular at $\partial\Omega$).

Given the cost functional

$$J(x,a) = E\int_0^{\infty} \left\{ f(X_t) + \gamma_q \, |a(X_t)|^{q'} \right\} e^{-\lambda\, t} dt$$

where $q' = \frac{q}{q-1}$, one defines the value function

$$u(x) = \inf_{a \in \mathcal{A}} J(x,a)$$

where $\mathcal{A} = \{a \in C(\Omega) : X_t \in \Omega\,, \forall t > 0 \, a.s.\}$.

By Dynamic programming principle one expects u to be the maximal solution of

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Actually

- if $1 < q \leq 2$ then $u(x) \to \infty$ on the boundary.
- if $q > 2$ the maximal solution is bounded (indeed, continuous in $\overline{\Omega}$), and the situation is very different (see e.g. [Barles-Da Lio], [Barles], [Capuzzo Dolcetta-Leoni-Porretta]).

Results from [Lasry-PL. Lions]:

1. (case $\lambda > 0$) Let $1 < q \leq 2$. Then the value function u is the unique solution (in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$) of

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They also studied the case $\lambda = 0$ and the ergodic limit $\lambda \to 0$:

2. (case $\lambda = 0$) There exists a unique constant c_0 such that the problem

$$\begin{cases} -\Delta u + |\nabla u|^q + c_0 = f(x) \quad \text{in } \Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

admits a solution. Moreover the solution u is unique (up to an additive constant).

2. Qualitative properties and blow-up profile

The asymptotic of u when $x \to \partial\Omega$ was determined in [LL]:

$$\begin{cases} u(x) \sim C_q d(x)^{-\frac{2-q}{q-1}} & \text{if } 1 < q < 2, \\ u(x) \sim -\log(d(x)) & \text{if } q = 2, \end{cases} \qquad [d(x) := \mathrm{dist}\,(x, \partial\Omega)]$$

where $C_q = \frac{(q-1)^{-\frac{2-q}{q-1}}}{2-q}$.

- The blow-up of u is independent on the dimension N (essentially, 1-D behaviour).
- The blow-up is faster as $q \to 1$ and decreases up to a logarithmic rate when $q = 2$.

Remark : $u \in L^1$ if and only if $\frac{2-q}{q-1} < 1$ which means $\frac{3}{2} < q \leq 2$.

Recent results also characterize the asymptotics of ∇u (and then the behaviour of the optimal control and the dynamics).

In [Porretta-Véron '06] we prove that

$$\lim_{x \to \partial\Omega} \quad d(x)^{\frac{1}{q-1}} \nabla u(x) = \tilde{c}_q \, \nu(x)$$

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$$\frac{\partial u}{\partial \nu} \sim \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \quad \text{and} \quad \frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right) .$$

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This is the expected scaling from the asymptotics of u: set $\alpha = \frac{2-q}{q-1}$

$$\begin{cases} \text{if} \quad 1 < q < 2 \,, \quad u \sim C_q d(x)^{-\alpha} \ \to \nabla u \sim -C_q \alpha \, d(x)^{-(\alpha+1)} \nabla d(x) \\ \text{if} \quad q = 2 \,, \quad u \sim -\log(d(x)) \ \to \nabla u \sim -\frac{1}{d(x)} \nabla d(x) \end{cases}$$

(note: $\alpha + 1 = \frac{1}{q-1}$, $\tilde{c}_q = C_q \alpha$ and $\nabla d(x) = -\nu$)

Rmk: The first order asymptotic of ∇u is crucial to deal with the nonlinear term $|\nabla u|^q$ as:

$$|\nabla u|^q = |\nabla u|^{q-2}\nabla u \nabla u \simeq \frac{c_q}{d(x)}\nabla u \cdot \nu(x) \tag{3}$$

Next one can use this information to get further qualitative properties.

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We set $S = C_q\, d(x)^{-\alpha}$ and we look at the equation for the error term $z = u - S$ which looks like [using (3)...]

$$-\Delta z + z - \tfrac{\alpha+2}{d(x)}\nabla z \nabla d(x) + O(d^\alpha\,|\nabla z|^2) = f(x) + g(x),$$

$$g = \Delta S - S - |\nabla S|^q$$

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This equation has a singular but well-oriented transport term. We estimate $|\nabla z|$ and get a full description near the boundary:

- precise behaviour of normal and tangent components
- curvature effects in the optimal control profile

Theorem (Leonori–Porretta Siam '07)

Being ν and τ the normal and tangent vectors, we have, as $d(x) \to 0$,

$$\frac{\partial u}{\partial \nu} = \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \left[1 + \frac{(N-1)}{2} \kappa(\overline{x})\, d(x) + o(d(x)) \right], \quad \forall 1 < q \leq 2,$$

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Key point: Bernstein's type gradient estimates for singular equations

$$-\Delta z + \lambda z + H(x, \nabla z) = F(x) \qquad \text{in } \Omega$$

like our model $H(x, \nabla z) \sim -\frac{\mu}{d(x)} \nabla z \nabla d(x) + \gamma\, d^\alpha |\nabla z|^2$.

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where $\psi \in L^\infty(\Omega)$.

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Note in particular:

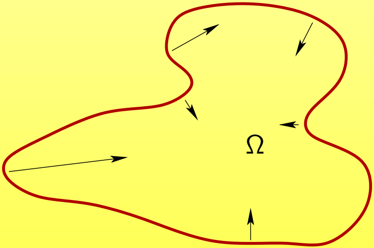
(i) The control tangentially is zero on $\partial\Omega$ if $q \neq 2$, bounded if $q = 2$.

(ii) On the hypersurfaces parallel to $\partial\Omega$, the control is maximum where the domain has a maximal mean curvature

The "constrained dynamics"

Near the boundary, the dynamics looks like

$$\begin{cases} dX_t = \left[\frac{q'}{d(X_t)} + \frac{q'(N-1)}{2}\,\kappa(\overline{X_t})\right] \nabla d(X_t) dt + \sqrt{2}\, dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$



The control (i.e. the drift) has to be stronger where the domain is more curved.

3. Applications of blow-up sol.

Pb1. Let $f \in L^\infty$. If $q > 1$ the Dirichlet problem

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Classical example when $q = 2$:

$$\begin{cases} -\Delta u + |\nabla u|^2 = f(x) \\ u_{|_{\partial\Omega}} = 0 \end{cases} \qquad v = e^{-u} - 1 \qquad \begin{cases} -\Delta v = -f(x)(v+1) \\ v_{|_{\partial\Omega}} = 0 \end{cases}$$

If $f \leq -\lambda_1$ (first eigenvalue), then there is no solution.

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If $f \leq -\lambda_1$ (first eigenvalue), then there is no solution.

But:

- There is a solution if $\|f\|_\infty$ is sufficiently small
- If $\lambda > 0$, there always exists a solution of

$$\begin{cases} \lambda u - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0\,. \end{cases}$$

[Kazdan-Kramer, Boccardo-Murat-Puel, Ferone-Murat, Dall'Aglio-Giachetti-Puel, Sirakov...]

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$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

- What happens to the solutions of (4) when $\lambda \to 0$?

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- What happens to the solutions of (4) when $\lambda \to 0$?

Pb2. Similar question holds for the asymptotic behaviour as $t \to +\infty$ of

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), & u(0) = u_0 \quad \text{in } \Omega\,. \end{cases} \tag{5}$$

- If the stationary problem has no solution, what happens as $t \to +\infty$?

Pb1. For $\lambda > 0$, consider the solution u_λ of

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega \\ u_\lambda = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}$$

• What happens to the solutions of (4) when $\lambda \to 0$?

Pb2. Similar question holds for the asymptotic behaviour as $t \to +\infty$ of

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), & u(0) = u_0 \quad \text{in } \Omega\,. \end{cases} \tag{5}$$

• If the stationary problem has no solution, what happens as $t \to +\infty$?

Recall: the stochastic interpretation of (4), (5) suggests

$$\lim_{\lambda \to 0} \lambda\, u_\lambda = \lim_{t \to \infty} \frac{u(t)}{t}$$

and that this limit be a constant.
[Bensoussan-Frehse, Arisawa-PL Lions, Alvarez-Bardi, Barles-Souganidis...]

Recall that if X_t is a process satisfying the SDE

$$dX_t = a(X_t) + \sqrt{2} dB_t \,, \qquad X_0 = x \in \Omega \,,$$

the sol. u_λ of (4) can be represented as

$$u_\lambda(x) = \inf_{\mathcal{A}} E_x \left\{ \int_0^{\tau_x} \left[f(X_t) + \gamma_q |a(X_t)|^{\frac{q}{q-1}} \right] \mathrm{e}^{-\lambda t} \, dt \right\}$$

where E_x is the expectation conditioned to $X_0 = x$, τ_x is the exit time from Ω.

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Indeed, if f is very negative inside, the control will try to push the process in the interior to realize the minimum: this can lead to the state constraint problem and the so-called ergodic behaviour:

$$\lim_{\lambda\to 0} \lambda \int_0^\infty f(X_t) e^{-\lambda t} dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X_t) dt \quad \forall x \in \Omega$$

So the explanations come from the interpretation in terms of stochastic control problems. Indeed we will see:

- The existence of a stationary solution depends on the ergodic constant for state constraint problems
- The behaviour of u_λ and of $u(t)$ is described by boundary blow-up solutions
- The profile of boundary blow-up solutions determines the rate of convergence for the large time behaviour (new rates induced by boundary conditions)

NB: This is the same for any Dirichlet type condition $u = g_{|_{\partial\Omega\times(0,T)}}$ with g continuous.

First remark: existence of stationary solutions depends on the ergodic constant for state constraint pb.

Recall: there exists a unique constant c_0 such that the problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} v(x) = +\infty \,, \end{cases}$$

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Crucial point: There exists a solution of the Dirichlet problem

$$\begin{cases} -\Delta \hat{u} + |\nabla \hat{u}|^q = f(x) & \text{in } \Omega \\ \hat{u} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

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• The reason is simple: $c_0 > 0$ gives the existence of subsolutions to (6).
(recall PL Lions [Arma '80]: if $\exists$ a subsolution $\Rightarrow$ $\exists$ a solution)

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• The reason is simple: $c_0 > 0$ gives the existence of subsolutions to (6). (recall PL Lions [Arma '80]: if $\exists$ a subsolution $\Rightarrow$ $\exists$ a solution)

• When $q = 2$ we have $c_0 = \lambda_1(-\Delta + f)$ and this is consistent with [Kazdan-Kramer] (eigenvalues are particular cases of ergodic constants !)

So, the behaviour of u_λ sol. of

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega \\ u_\lambda = 0 \end{cases}$$

depends on the value of ergodic constant c_0.

Easy case: $c_0 > 0 \iff$ there exists a solution of the limit problem

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Interesting case: $c_0 \le 0$. Then we have

$$\begin{cases} u_\lambda \to -\infty & \text{(complete blow-up)} \\ \lambda\, u_\lambda \to c_0 \end{cases}$$

(and rescaling u_λ we obtain the sol. v of the ergodic problem)

Theorem

Let $1 < q \leq 2$, *and* $f \in L^\infty$. *Let* u_λ *be sol. of (4).*

(i) *If* $c_0 > 0$ *(hence there exists a sol.* $\hat{u}$ *with* $\lambda = 0$*), then* $u_\lambda \to \hat{u}$ *uniformly as* $\lambda \to 0$.

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(ii) *If* $c_0 \le 0$, *then*

$$\begin{cases} u_\lambda \to -\infty & \text{for every } x \in \Omega, \\ \lambda\, u_\lambda(x) \to c_0 & \text{for every } x \in \Omega, \end{cases}$$

and

$$v_\lambda := u_\lambda + \|u_\lambda\|_\infty \to v_0 \quad \text{locally uniformly,}$$

where c_0 *is the unique constant such that*

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} v(x) = +\infty\,, \end{cases}$$

admits a solution, and v_0 *is the unique solution such that* $\min\limits_{\Omega} v_0(x) = 0$.

Proof of this result relies on two fundamental points:

- Interior gradient estimates (Bernstein's technique):
  $|\nabla u_\lambda|$ is (locally) uniformly bounded.
  $\rightarrow$ hence $\lambda\, u_\lambda$ must converge to a constant
- Uniqueness of the ergodic constant c_0 and of v_0 (strong max. principle) imply convergence for the whole sequence.

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NB: One can prove that $u_\lambda(x) - u_\lambda(x_0) \to v - v(x_0)$ for any point x_0.

The fact that the same convergence holds for $u_\lambda + \|u_\lambda\|_\infty$ gives an idea of the barrier effect:
$u_\lambda + \|u_\lambda\|_\infty$ is locally uniformly bounded, i.e. maximum points of $|u_\lambda|$ remain sufficiently far from boundary ("the blow-up comes from the interior")

Large time behaviour

[Work in progress with G. Barles and T. Tabet Tchamba]

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), & u(0) = u_0\,. \end{cases}$$

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$$\lim_{t\to\infty} \frac{u(t)}{t} = c_0$$

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Rmk: the typical result (e.g. periodic case) is that

$u(t) - c_0 t$ is bounded $\longrightarrow$ rate of convergence: $|\frac{u(t)}{t} - c_0| = O(\frac{1}{t})$

$u(t) - c_0 t \to v$ where v is a sol. of the ergodic problem

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[Barles-Souganidis, Namah-Roquejoffre,..]

- This is also true when $q > 2$ (Tabet Tchamba).

In the range $1 < q \leq 2$ we have a different behaviour.

- $u - c_0 t$ is bounded when $c_0 < 0$ and $\frac{3}{2} < q \leq 2$
- if $c_0 < 0$ and $1 < q \leq \frac{3}{2}$, or if $c_0 = 0$, then

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The reason for this new situation is that

- the blow-up rate is influenced by the profile of blow-up solutions
- when $c_0 = 0$ the blow-up rate becomes slower in any case

Blow-up rate

We prove indeed the following rate of convergence:

- Case $c_0 < 0$:

$$u - c_0 t = O(1) \qquad \text{when } \tfrac{3}{2} < q \leq 2.$$
$$u - c_0 t = O(\log t) \qquad \text{when } q = \tfrac{3}{2}$$
$$u - c_0 t = O(t^{\frac{3-2q}{2-q}}) \qquad \text{when } 1 < q < \tfrac{3}{2}$$

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- Case $c_0 = 0$:

$$u - c_0 t = O(\log t) \qquad \text{when } q = 2$$
$$u - c_0 t = O(t^{2-q}) \qquad \text{when } 1 < q < 2$$

NB: The following bounds are locally uniformly.

Of course this gives the rate of convergence of $\frac{u(t)}{t}$:

- Case $c_0 < 0$

$$\begin{cases} \frac{u(t)}{t} - c_0 = O(\frac{1}{t}) & if \frac{3}{2} < q \leq 2 \\ \frac{u(t)}{t} - c_0 = O\left(\frac{\log t}{t}\right) & if\, q = \frac{3}{2} \\ \frac{u(t)}{t} - c_0 = O\left(\frac{1}{t^{\frac{q-1}{2-q}}}\right) & if\, 1 < q < \frac{3}{2} \end{cases}$$

- Case $c_0 = 0$

$$\frac{u(t)}{t} - c_0 = O\left(\frac{1}{t^{q-1}}\right) \qquad \text{for any } 1 < q \leq 2$$

Idea of the proof: compare $u - c_0 t$ with a suitable translation of a blow-up sol. v of the ergodic problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} v(x) = +\infty \,, \end{cases}$$

Indeed, set $\tilde{u} = u - c_0 t$, it solves the equation

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} + |\nabla \tilde{u}|^q = f - c_0 \\ \tilde{u}_{|_{\partial\Omega\times(0,T)}} = -c_0\, t \end{cases}$$

and one expects $u - c_0 t \simeq v(x) + \ldots.$ (error terms).

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and one expects $u - c_0 t \simeq v(x) + ...$ (error terms).

NB: The bound from above is trivial: $u - c_0 t \leq v + \|u_0\|_\infty$.

The problem is the bound from below since $u \to -\infty$.

Idea: construct a subsolution using the graph $v(x)$ as a propagating front.

Example of our construction: if Ω is star-shaped we take

$$\tilde{v}(x,t) = r(t)^{\frac{2-q}{q-1}} v(r(t)x)$$

$$\text{with } r(t) < 1, \;\; r(t) \uparrow 1 \text{ as } t \to \infty$$

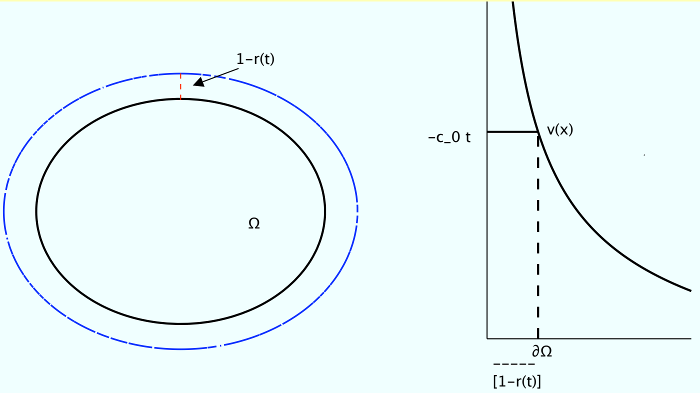
This corresponds to a translation of the profile:

(i) $\tilde{v}$ is defined on $\frac{\Omega}{r(t)} \supset \Omega$

(ii) The graph of v moves with velocity $1 - r(t)$

(iii) The velocity $r(t)$ is chosen in a way that $\tilde{v}$ is comparable to $u - c_0 t$ on the boundary: $r(t)^{\frac{2-q}{q-1}} v(r(t)x) \simeq -c_0 t$ on $\partial\Omega \times (0,t)$
(recall $c_0 \leq 0$)

This will fix the velocity $r(t)$..... $1 - r(t) \simeq \frac{1}{t^{\frac{1}{\alpha}}}$)


1−r(t)
Ω
−c_0 t
v(x)
∂Ω

[1−r(t)]

Computing the equation for $\tilde{v}(x,t) = r(t)^{\frac{2-q}{q-1}} v(r(t)x)$ we find

$$\tilde{v}_t - \Delta \tilde{v} + |\nabla \tilde{v}|^q = r(t)^{\frac{q}{q-1}}(f - c_0) + r'(t)....$$

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$$\tilde{v}_t - \Delta \tilde{v} + |\nabla \tilde{v}|^q \simeq f - c_0 - (1 - r(t))(f - c_0)$$

Since $\tilde{u} = u - c_0 t$ satisfies

$$\tilde{u}_t - \Delta \tilde{u} + |\nabla \tilde{u}|^q = f - c_0$$

the correction term is $H(t) \simeq \|f - c_0\| \int_0^t (1 - r(s))ds$:

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Optimality of the bound: if $f - c_0 < -\delta < 0$, then also

$$\tilde v - \delta\, H(t) \quad \text{is a supersolution} \Rightarrow u - c_0 t \leq \tilde v - \delta\, H(t)$$

which gives us the optimality of the convergence rate.

The velocity $1 - r(t)$ is fixed by the boundary rate:

$$\text{on } \partial\Omega \times (0, T), \qquad \tilde{v}(x,t) \simeq u - c_0 t = -c_0 t$$

Recall at the boundary $\tilde{v} \simeq d(x)^{-\alpha} \simeq (1 - r(t))^{-\alpha}$. This gives

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$$u - c_0 t \gtrsim \tilde{v}(x,t) - H(t)\,, \quad H \simeq \int_0^t (1 - r(s))ds$$

we get

$$u - c_0 t \gtrsim v(r(t)x) - O(t^{1-\frac{1}{\alpha}})$$

Rmk: $u - c_0 t$ is locally uniformly bounded only if $1 - \frac{1}{\alpha} < 0$ i.e. $\alpha < 1$
(this means $q > \frac{3}{2}$ and corresponds to the case $v \in L^1$:
$H(t)$ is the integral of $1 - r(t)\ =\ $ area below the graph of v).

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Things to be done, work in progress...

- Inhomogeneous diffusions

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\,\sigma(X_t)dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

with associated HJB equation

$$-\mathrm{tr}\left(A(x)D^2u\right) + \lambda u + |\nabla u|^q = f(x)$$

where $A(x) = \sigma(x)\sigma^T(x)$.

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where $A(x) = \sigma(x)\sigma^T(x)$.
If $A(x)$ elliptic and smooth, one can use the same approach replacing the distance function $d(x)$ with the solution of the first order equation

$$\begin{cases} A(x)\nabla\rho\nabla\rho = \gamma\,|\nabla\rho|^q & \text{in } \Omega \\ \rho > 0\,, \\ \rho = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- general diffusions, possibly non smooth and/or possibly degenerate ?
- singular domains (link with Wiener criteria for the Brownian motion)?