Equations de Navier-Stokes avec des Conditions aux Limites Non Homogènes. Des Solutions Fortes aux Solutions très Faibles en passant par les Sobolev Fractionnaires

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1. Les équations de Stokes

$$(S)\ \begin{cases} -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{et} \quad \nabla \cdot \boldsymbol{u} = h & \text{dans } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{sur } \Gamma. \end{cases}$$

2. Les équations d'Oseen

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• Solutions généralisées de (NS)

Si $h = 0$, on sait depuis Leray (34) que si

$$\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega) \quad \text{et} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma), \quad \text{avec} \quad p \geq 2$$

et

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma = 0, \qquad \forall i = 0, ..., I, \tag{1}$$

où Γ_i sont les composantes connexes de la frontière Γ, $i = 0, \ldots, I$, alors il existe une solution

$$(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

satisfaisant (NS).

Serre (85) a montré l'existence de solution faible

$$(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega) \quad \text{pour tout} \quad \frac{3}{2} < p < 2$$

avec les mêmes hypothèses sur h et $\boldsymbol{g}$.

Plus récemment, Kim (09) a étendu le résultat d'existence de Serre au cas $\frac{3}{2} \leq p < 2$, quand Γ est connexe ($I = 0$) et si h et g sont suffisamment petits dans une norme appropriée.

• Solutions très faibles de (NS)

L'existence de solutions très faibles $(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$, pour

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad h = 0 \quad \text{et} \quad \boldsymbol{g} \in \mathbf{L}^2(\Gamma)$$

arbitrairement grands et sans la condition (1) de nullité des flux, a d'abord été établie par Marusic-Paloka (00) avec Ω simplement connexe et de classe $\mathcal{C}^{1,1}$.
Mais la preuve de ce résultat devient correcte seulement si l'on suppose que la condition (1) a lieu ou si plus généralement, la condition

$$\sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq \delta. \tag{2}$$

est satisfaite (car ici $h = 0$).
Le même résultat a été prouvé par Kim (09) pour des forces extérieures $\boldsymbol{f} \in [\mathbf{W}_0^{1,3/2}(\Omega) \cap W^{2,3}(\Omega)]'$ arbitrairement grandes et pour $h \in [W^{1,3/2}(\Omega)]'$ et $\boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$ suffisamment petits, avec Γ supposée connexe ($I = 0$).
Observons pour ce dernier que les espaces choisis pour h et $\boldsymbol{f}$ ne sont pas corrects.

L'objet de ce travail est de généraliser la théorie des solutions très faibles

$$(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega),$$

avec $1 < p < \infty$, pour les problèmes stationnaires de Stokes, d'Oseen et de Navier-Stokes, avec des conditions aux limites de type Dirichlet non homogènes. Cela passe par une définition rigoureuse des traces des functions de $\mathbf{L}^p(\Omega)$ (voir Amrouche-Girault (94) ou Amrouche- Rodriguez-Bellido (10)).

On s'intéresse également aux questions de régularité et d'unicité des solutions très faibles

Nous considérerons enfin le cas où les données et donc les solutions appartiennent à des espaces de Sobolev fractionnaires.

We introduce the spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla\cdot\boldsymbol{\varphi} = 0\}, \quad \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})^3;\ \nabla\cdot\boldsymbol{\varphi} = 0\},$$

and for any $1 < r,\, p < \infty$,

$$\begin{aligned} \mathbf{L}^p_\sigma(\Omega) &= \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla\cdot\boldsymbol{v} = 0\}, \\ \mathbf{X}_{r,p}(\Omega) &= \{\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_0(\Omega);\ \nabla\cdot\boldsymbol{\varphi} \in W^{1,p}_0(\Omega)\} \end{aligned}$$

and we set $\mathbf{X}_{p,p}(\Omega) = \mathbf{X}_p(\Omega)$.

lemma 1

i) The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{L}^p_\sigma(\Omega)$.

ii) The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathbf{X}_{r,p}(\Omega)$ and for all $q \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{X}_{r',p'}(\Omega)$, we have

$$\langle \nabla q,\ \boldsymbol{\varphi}\rangle_{[\mathbf{X}_{r',p'}(\Omega)]'\times\mathbf{X}_{r',p'}(\Omega)} = -\langle q,\ \nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W^{1,p'}_0(\Omega)}. \quad (3)$$

It is then easy to prove the following characterization:

$$(\mathbf{X}_{r',p'}(\Omega))' = \Big\{ \boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1;\ \mathbb{F}_0 \in \mathbb{L}^r(\Omega),\ f_1 \in W^{-1,p}(\Omega),\\ \text{with} \quad \mathbb{F}_0 = (f_{ij})_{1\le i,j\le 3}\Big\}. \tag{4}$$

As a consequence of Lemma 1 ii) and the Sobolev embeddings, we have the embeddings

$$\mathbf{W}^{-1,r}(\Omega) \hookrightarrow (\mathbf{X}_{r',p'}(\Omega))' \hookrightarrow \mathbf{W}^{-2,p}(\Omega), \tag{5}$$

where the second embedding holds if $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$.
Giving a meaning to the trace of a very weak solution of a Stokes problem is not trivial. Remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ (\nabla \cdot \boldsymbol{\psi})|_\Gamma = 0\}$$

that can also be described (see Amrouche-Girault (94)) as:

$$\mathbf{Y}_{p'}(\Omega) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,p'}(\Omega);\ \boldsymbol{\psi}|_\Gamma = \mathbf{0},\ \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\Big|_\Gamma = 0\}. \tag{6}$$

Note also that if $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$, then div $\boldsymbol{\psi} \in W_0^{1,p'}(\Omega)$ and the range space of the normal derivative $\gamma_1 : \mathbf{Y}_{p'}(\Omega) \to \mathbf{W}^{1/p,p'}(\Gamma)$ is

$$\mathbf{Z}_{p'}(\Gamma) = \{\boldsymbol{z} \in \mathbf{W}^{1/p,p'}(\Gamma);\ \boldsymbol{z} \cdot \boldsymbol{n} = 0\}.$$

Secondly, we shall use the spaces:

$$\begin{aligned} \mathbf{T}_{p,r}(\Omega) &= \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \Delta \boldsymbol{v} \in (\mathbf{X}_{r',p'}(\Omega))'\}, \\ \mathbf{T}_{p,r,\sigma}(\Omega) &= \{\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}, \end{aligned}$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\mathbf{T}_{p,r}(\Omega)} = \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\mathbf{X}_{r',p'}(\Omega)]'}.$$

When $p = r$, these spaces are denoted as $\mathbf{T}_p(\Omega)$ and $\mathbf{T}_{p,\sigma}(\Omega)$, respectively.

We also introduce the space

$$\mathbf{H}_{p,r}(\mathrm{div};\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \nabla\cdot\boldsymbol{v} \in L^r(\Omega)\},$$

which is endowed with the graph norm. The following lemma will help us to prove a trace result:

lemma 2

i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r}(\Omega)$ and in $\mathbf{T}_{p,r}(\Omega) \cap \mathbf{H}_{p,r}(\mathrm{div};\Omega)$ respectively.

ii) The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\mathbf{T}_{p,r,\sigma}(\Omega)$.

The following lemma proves that the tangential trace of functions $\boldsymbol{v}$ of $\mathbf{T}_{p,r,\sigma}(\Omega)$ belongs to the dual space of $\mathbf{Z}_{p'}(\Gamma)$, which is

$$(\mathbf{Z}_{p'}(\Gamma))' = \{\boldsymbol{\mu} \in \mathbf{W}^{-1/p,p}(\Gamma);\ \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\}.$$

Besides, we recall that we can decompose $\boldsymbol{v}$ into its tangential, $\boldsymbol{v}_\tau$, and normal parts, that is: $\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}$.

lemma 3 (tangential traces)

Let Ω be a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. Let $1 < p < \infty$ and $r > 1$ be such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. The mapping $\gamma_\tau : \boldsymbol{v} \mapsto \boldsymbol{v}_\tau|_\Gamma$ on the space $\mathcal{D}(\overline{\Omega})^3$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\mathbf{T}_{p,r}(\Omega)$ into $\mathbf{W}^{-1/p,p}(\Gamma)$, and the following Green formula holds

$$\begin{aligned} \langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)} &= \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} \, d\boldsymbol{x} - \\ &- \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}, \end{aligned} \tag{7}$$

for any $\boldsymbol{v} \in \mathbf{T}_{p,r}(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{Y}_{p'}(\Omega)$.

We can also prove that

$$\boldsymbol{\mathcal{D}}(\overline{\Omega}) \quad \text{is dense in} \quad \mathbf{H}_{p,r}(\mathrm{div};\Omega)$$

and the mapping

$$\gamma_{\boldsymbol{n}} : \boldsymbol{v} \mapsto \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma}$$

is continuous from

$$\mathbf{H}_{p,r}(\mathrm{div};\Omega) \quad \text{into} \quad W^{-1/p,p}(\Gamma)$$

and we have the Green formula:

$$\text{for any} \quad \boldsymbol{v} \in \mathbf{H}_{p,r}(\mathrm{div};\Omega) \quad \text{and} \quad \varphi \in W^{1,p'}(\Omega),$$

we have

$$\int_{\Omega} \boldsymbol{v}\cdot\nabla\varphi \, d\boldsymbol{x} + \int_{\Omega} \varphi \,\mathrm{div}\, \boldsymbol{v} \, d\boldsymbol{x} = \langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi \,\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} \, .$$

3. Very weak solutions and regularity for Stokes problem

We focus on the study of the stationary Stokes problem (S) with the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \qquad (8)$$

Basic results on weak and strong solutions of problem (S) may be summarized in the following theorem (see Cattabriga (61) and Amrouche-Girault (94)).

Theorem 4 (Generalized solutions for Stokes system)

i) For every $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition (8), the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and $q \in L^p(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$, depending only on p and Ω, such that:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} &\leq C\,(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \\ &+ \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}). \end{aligned} \tag{9}$$

Theorem 5 (Strong solutions for Stokes system)

ii) Moreover, if

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma),$$

then

$$\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega), \quad q \in W^{1,p}(\Omega)$$

satisfy an analogous estimate to (9) with the corresponding norms.

We wonder about minimal necessary assumptions on $\boldsymbol{f}$, h and $\boldsymbol{g}$, in order that a very weak solution, that is,

$$(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$$

exists.

We are interested here in the case of singular data satisfying the following assumptions:

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \; h \in L^r(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma) \tag{10}$$

with

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3} \quad \text{and} \quad r \leq p.$$

Observe that the space $(\mathbf{X}_{r',p'}(\Omega))'$ is an intermediate space between $\mathbf{W}^{-1,r}(\Omega)$ and $\mathbf{W}^{-2,p}(\Omega)$.

Definition (Very weak solution for the Stokes problem)

A pair

$$(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$$

is a **very weak solution** of (S) if the following equalities hold: For any $\boldsymbol{\varphi} \in \mathbf{Y}_{p'}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot \Delta\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}\rangle_\Gamma,$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla\pi\, d\boldsymbol{x} = -\int_\Omega h\,\pi\, d\boldsymbol{x} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma, \qquad (11)$$

with

$$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\mathbf{X}_{r',p'}(\Omega)]' \times \mathbf{X}_{r',p'}(\Omega)}$$

and

$$\langle \cdot, \cdot \rangle_\Gamma = \langle \cdot, \cdot \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1/p,p'}(\Gamma)}.$$

Note that

$$W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$$

and

$$\mathbf{Y}_{p'}(\Omega) \hookrightarrow \mathbf{X}_{r',p'}(\Omega)$$

if

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3},$$

which means that all the brackets and integrals have a sense. We can then prove that, if $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy (10), then $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is a very weak solution of (S) if and only if $(\boldsymbol{u}, q)$ satisfies the system (S) in the sense of distributions.

Proposition 6 (Very weak solution for the Stokes problem, first version)

Let

$$\boldsymbol{f} \in (\mathbf{X}_{p'}(\Omega))', \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

satisfy the compatibility condition (8). Then, the Stokes problem (S) has exactly one solution $\boldsymbol{u} \in \mathbf{T}_p(\Omega)$ and $q \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$, depending only on p and Ω, such that:

$$\|\boldsymbol{u}\|_{\mathbf{T}_p(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big). \quad (12)$$

***Proof* :** The case $\boldsymbol{f} = \mathbf{0}$ and $h = 0$ is considered in Amrouche-Girault (94). Here, we generalize the result as follows:

Step 1: We suppose $\mathbf{g} \cdot \boldsymbol{n} = 0$ on Γ and $\displaystyle\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = 0$. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that: for any $\boldsymbol{w} \in \mathbf{Y}_{p'}(\Omega)$ and any $\pi \in W^{1,p'}(\Omega)$ it holds

$$\begin{aligned}\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} \quad &+ \quad \nabla\pi)\, d\boldsymbol{x} - \langle q, \nabla \cdot \boldsymbol{w}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \\ &= \quad \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{[\mathbf{X}_{p'}(\Omega)]'\times\mathbf{X}_{p'}(\Omega)} - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\rangle_\Gamma - \int_\Omega h\,\pi\, d\boldsymbol{x}.\end{aligned}$$

We can prove (as in Amrouche-Girault (94)) that for any pair $(\mathbb{F}, \varphi)$ in $\mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$, we have

$$\left| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\mathbf{X}_{p'}(\Omega)]' \times \mathbf{X}_{p'}(\Omega)} - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma - \int_\Omega h\, \pi\, d\boldsymbol{x} \right|$$

$$\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|h\|_{L^p(\Omega)} \right) \left(\|\mathbb{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right),$$

being $(\boldsymbol{w}, \pi) \in \mathbf{Y}_{p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ the unique solution of the Stokes (dual) problem:

$$-\Delta \boldsymbol{w} + \nabla \pi = \mathbb{F} \quad \text{and} \quad \nabla \cdot \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = \mathbf{0} \quad \text{on } \Gamma.$$

Note that for any $k \in \mathbb{R}$,

$$\left| \int_\Omega h\, \pi\, d\boldsymbol{x} \right| = \left| \int_\Omega h\, (\pi + k)\, d\boldsymbol{x} \right| \leq \|h\|_{L^p(\Omega)} \|\pi\|_{L^{p'}(\Omega)/\mathbb{R}}$$

and

$$\|\boldsymbol{w}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\pi\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leq C \left(\|\mathbb{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\varphi\|_{W^{1,p'}(\Omega)} \right).$$

From this bound, we deduce that the mapping

$$(\mathbb{F}, \varphi) \to \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \rangle_\Gamma - \int_\Omega h\, \pi\, d\boldsymbol{x}$$

defines an element of the dual space of $\mathbf{L}^{p'}(\Omega) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$ with norm bounded by $C(\|\boldsymbol{f}\|_{[\mathbf{X}_{p'}(\Omega)]'} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Omega)})$. That means that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of (S) satisfying the estimate (12).

Step 2: Now, we suppose that

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \mathbf{g} \cdot \boldsymbol{n}, 1 \rangle_\Gamma.$$

Define $\boldsymbol{u}_0 = \nabla\theta$ with $\theta \in W^{1,p}(\Omega)$ the solution of the Neumann problem:

$$\Delta\theta = h \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{g} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

By Step 1, there exists a unique $(\boldsymbol{z}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying:

$$-\Delta \boldsymbol{z}+\nabla q = \boldsymbol{f}+\nabla h, \quad \nabla\cdot\boldsymbol{z} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{g}-\boldsymbol{u}_0|_\Gamma \quad \text{on}\,\Gamma,$$

where $\nabla h \in (\mathbf{X}_{p'}(\Omega))'$ and $\boldsymbol{g} - \boldsymbol{u}_0|_\Gamma$ satisfies the hypothesis of Step 1. Thus, the pair of functions $(\boldsymbol{u}, q) = (\boldsymbol{z} + \boldsymbol{u}_0, q)$ is the required solution. ■

The following result is a generalization of Proposition 4.11 in Amrouche-Girault (94), where $\boldsymbol{f} = \mathbf{0}$ and $h = 0$.

Theorem 7 (Very weak solution for the Stokes problem, second version)

Let $\boldsymbol{f}, h, \boldsymbol{g}$ be given satisfying (8) and

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \; h \in L^r(\Omega), \; \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$$

with

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3} \quad \text{and} \quad r \leq p.$$

Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R};$$

Moreover, there exists a constant $C > 0$, only depending on p and Ω, such that:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} + \|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq \quad & C \quad \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \\ + \quad & \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big) \end{aligned} \tag{13}$$

Remark i) Observe that in Galdi-Simader-Sohr (05) (Theorem 3), the domain was of class $\mathcal{C}^{2,1}$ (here it is of class $\mathcal{C}^{1,1}$), and the divergence term was $h \in L^p(\Omega)$ (here of $h \in L^r(\Omega)$). Moreover, our solution is obtained in the space $\mathbf{T}_{p,r}(\Omega)$, which has been clearly characterized, contrary to the space $\widehat{\mathbf{W}}^{1,p}(\Omega)$ appearing in Galdi-Simader-Sohr, which was not characterized, completely abstract and obtained as the closure of $\mathbf{W}^{1,p}(\Omega)$ for the norm

$$\|\boldsymbol{u}\|_{\widehat{\mathbf{W}}^{1,p}(\Omega)} = \|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|A_r^{-1/2}\mathcal{P}_r\Delta\boldsymbol{u}\|_{\mathbf{L}^r(\Omega)},$$

where A_r is the Stokes operator with domain equal to $\mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{L}_\sigma^p(\Omega)$ and $\mathcal{P}_r$ is the Helmholtz projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}_\sigma^r(\Omega)$.
ii) Existence of very weak solution $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ was proved by Kim (09) for

$$\boldsymbol{f} \in [\mathbf{W}_0^{1,p'}(\Omega) \cap W^{2,p'}(\Omega)]', \quad h \in [W^{1,p'}(\Omega)]' \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$$

but the spaces chosen for h and $\boldsymbol{f}$ are not correct either and the equivalence in Theorem 5 of Kim (09) is not valid.

Corollary 8

Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h, $\boldsymbol{g}$ be given satisfying (8) and

$$\mathbb{F}_0 \in \mathbb{L}^r(\Omega), \quad f_1 \in W^{-1,p}(\Omega), \quad h \in L^r(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma).$$

Then the solution $\boldsymbol{u}$ given by Theorem 24 belongs to $\mathbf{W}^{1,r}(\Omega)$. If moreover $f_1 \in L^r(\Omega)$, then q belongs to $L^r(\Omega)$. In both cases, we have analogous estimates to (13).

Remark. It is clear that

$$\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{T}_{p,r}(\Omega) \quad \text{when} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{3},$$

i.e.,

$\mathbf{T}_{p,r}(\Omega)$ is an intermediate space between $\mathbf{W}^{1,r}(\Omega)$ and $\mathbf{L}^p(\Omega)$.

Corollary 9

Let

$$h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

be given, satisfying (8), with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then, there exists at least one solution $\boldsymbol{u} \in \mathbf{T}_{p,r}(\Omega)$ verifying

$$\nabla \cdot \boldsymbol{u} = h \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \textit{on } \Gamma.$$

Moreover, there exists a constant $C = C(\Omega, p, r)$ such that

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C \left(\|h\|_{L^r(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right).$$

The following corollary gives Stokes solutions $(\boldsymbol{u}, q)$ in fractionary Sobolev spaces of type $\mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)$, with $0 < \sigma < 2$.

Corollary 10 (**Solutions in fractionary Sobolev spaces**)

Let s be a real number such that $0 \leq s \leq 1$.
i) Let $\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1$, h and $\boldsymbol{g}$ satisfy the compatibility condition (8) with

$$\mathbb{F}_0 \in \mathbf{W}^{s,r}(\Omega),\ f_1 \in W^{s-1,p}(\Omega),\ \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma),\ h \in W^{s,r}(\Omega),$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$$

satisfying the estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{s,p}(\Omega)} + \|q\|_{W^{s-1,p}(\Omega)/\mathbb{R}} \leq C\,(\|\mathbb{F}_0\|_{\mathbf{W}^{s,r}(\Omega)} + \|f_1\|_{W^{s-1,p}(\Omega)} + \\ + \|h\|_{W^{s,r}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{s-1/p,p}(\Gamma)}).$$

Corollary 10

ii) Assume that

$$\boldsymbol{f} \in \mathbf{W}^{s-1,p}(\Omega), \ \boldsymbol{g} \in \mathbf{W}^{s+1-1/p,p}(\Gamma) \quad \text{and} \quad h \in W^{s,p}(\Omega),$$

fulfill the compatibility condition (8). Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u}, \ q) \in \mathbf{W}^{s+1,p}(\Omega) \times W^{s,p}(\Omega)/\mathbb{R}$$

with

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{s+1,p}(\Omega)} + \|q\|_{W^{s,p}(\Omega)/\mathbb{R}} \quad &\leq \quad C\big(\|\boldsymbol{f}\|_{\mathbf{W}^{s-1,p}(\Omega)} + \|h\|_{W^{s,p}(\Omega)} + \\ &+ \quad \|\boldsymbol{g}\|_{\mathbf{W}^{s+1-1/p,p}(\Gamma)}\big). \end{aligned}$$

The following theorem provides solutions for

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega) \quad \text{and} \quad h \in W^{s-1,p}(\Omega)$$

with $1/p < s < 2$. In particular, if $p = 2$, we obtain solutions in

$$\mathbf{H}^{1/2+\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Omega),$$

for any $0 < \varepsilon \leq 3/2$.

Theorem 11 (Solutions in fractionary Sobolev spaces)

Let s be a real number such that $\frac{1}{p} < s \leq 2$. Let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfy the compatibility condition (8) with

$$\boldsymbol{f} \in \mathbf{W}^{s-2,p}(\Omega), \quad h \in W^{s-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{s-1/p,p}(\Gamma).$$

Then, the Stokes problem (S) has exactly one solution

$$(\boldsymbol{u},\ q) \in \mathbf{W}^{s,p}(\Omega) \times W^{s-1,p}(\Omega)/\mathbb{R}$$

satisfying the corresponding estimate.

4. Very weak solutions and regularity for the Oseen problem

As for the Navier-Stokes system, we can prove that if

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma),$$

with h and $\boldsymbol{g}$ verifying the compatibility condition

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} \; d\sigma, \tag{14}$$

then the problem (O) has a unique solution

$$(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$$

verifying the following estimate:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big)\ \big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\big)\Big).$$

Theorem 12 (**Strong solutions**)

Consider $p \geq \frac{6}{5}$,

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_s(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma),$$

with

$$s = 3 \quad \text{if} \quad p < 3, \quad s = p \quad \text{if} \quad p > 3 \quad \text{or} \quad s = 3 + \varepsilon \quad \text{if} \quad p = 3,$$

for some arbitrary $\varepsilon > 0$ and satisfying the compatibility condition (14). Then, the unique solution of (O) verifies

$$(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega).$$

Moreover, there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} \; + \; \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \\ + \; \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} \right) \Big).$$

Proof: First, let

$$(\boldsymbol{u}, q) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$$

be the unique solution of Problem (O). For a given

$$\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}(\overline{\Omega}) \quad \text{such that} \quad \nabla \cdot \boldsymbol{v}_\lambda = 0 \quad \text{and} \quad \|\boldsymbol{v}_\lambda - \boldsymbol{v}\|_{\mathbf{L}^s(\Omega)} \leq \lambda,$$

where $\lambda > 0$, let

$$(\boldsymbol{u}_\lambda, q_\lambda) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$$

be the unique solution of the problem (O_λ):

$$-\Delta \boldsymbol{u}_\lambda - \boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda + \nabla q_\lambda = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u}_\lambda = h \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \boldsymbol{g} \quad \text{on } \Gamma$$

(use the Stokes regularity and a bootstrap argument). Secondly, we focus on the obtention of a strong estimate for $(\boldsymbol{u}_\lambda, q_\lambda)$. If $\widetilde{\boldsymbol{v}}$ is the extension by zero of $\boldsymbol{v}$ to $\mathbb{R}^3$ and ρ_ε the classical mollifier, we consider for $\varepsilon > 0$, and $0 < \lambda < \varepsilon/2$:

$$\boldsymbol{v}_\lambda = \boldsymbol{v}_1^\varepsilon + \boldsymbol{v}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{v}_1^\varepsilon = \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}, \quad \boldsymbol{v}_{\lambda,2}^\varepsilon = \boldsymbol{v}_\lambda - \widetilde{\boldsymbol{v}} \star \rho_{\varepsilon/2}. \tag{15}$$

By regularity estimates for the Stokes problem, we have

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \right. \\ \left. + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + \|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}\right). \qquad (16)$$

In order to estimate the term $\|\boldsymbol{v}_\lambda \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)}$, we use (15) and Sobolev embeddings. First:

$$\|\boldsymbol{v}^\varepsilon_{\lambda,2} \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} \leq \|\boldsymbol{v}^\varepsilon_{\lambda,2}\|_{\mathbf{L}^s(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^k(\Omega)} \leq C\,\varepsilon \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)}, \qquad (17)$$

with $\frac{1}{k} = \frac{1}{p} - \frac{1}{s}$.

For the estimate on $\boldsymbol{v}_1^\varepsilon$, we consider two cases: If $p \leq 2$, let $r \in]3, \infty]$ be such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{2}$, and $t \geq 1$ such that $1 + \frac{1}{r} = \frac{1}{3} + \frac{1}{t}$, satisfying:

$$\begin{aligned} \|\boldsymbol{v}_1^\varepsilon \cdot \nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^p(\Omega)} &\leq \|\boldsymbol{v}_1^\varepsilon\|_{\mathbf{L}^r(\Omega)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)} \\ &\leq \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \|\rho_{\varepsilon/2}\|_{L^t(\mathbb{R}^3)} \|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Using the estimate (17), we deduce from (16) that

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} &+ \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \right. \\ &+ \left. \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\big) \big(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\big)\right). \end{aligned}$$

If $p > 2$, using the compact embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$, with $q < p^*$, for any $\varepsilon' > 0$, we known that there exists $C_{\varepsilon'} > 0$ such that

$$\|\nabla \boldsymbol{u}_\lambda\|_{\mathbf{L}^q(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1(\Omega)}.$$

Considering the case $p < 3$ and then the case $p \geq 3$, we can choose the exponent q and fix $\varepsilon > 0$ and $\varepsilon' > 0$ small enough to obtain

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} &+ \|q_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \\ &+ \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)} + C_{\varepsilon'}\|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\|\rho_{\varepsilon/2}\|_{L^t(\Omega)}(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \\ &+ (1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}) \times (\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}))\Big). \end{aligned}$$

Thus, we deduce that there exists a sequence of real numbers k_λ such that

$$(\boldsymbol{u}_\lambda, q_\lambda + k_\lambda) \rightharpoonup (\boldsymbol{u}, q) \quad \text{in} \quad \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$$

with $(\boldsymbol{u}, q)$ is solution of Problem (O) with the corresponding estimate.

Theorem 13 (**Generalized Solutions for the Oseen problem**)

Let $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega)$, $\boldsymbol{v} \in \mathbf{H}_3(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ verify the compatibility condition:

$$\int_\Omega h(\boldsymbol{x})\, d\boldsymbol{x} = \langle \boldsymbol{g}\cdot\boldsymbol{n}, 1\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \qquad (18)$$

Then, the problem (O) has a unique solution

$$(\boldsymbol{u}, q) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}.$$

Moreover, there exists some constant $C > 0$ such that, for

$$\alpha = 1 \quad \text{if} \quad p \geq 2 \quad \text{and} \quad \alpha = 1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \quad if \quad p < 2,$$

we have

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)})^2(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \\ + \alpha\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}).$$

Sketch of the proof: We split it in two cases. If $p \geq 2$, we decompose the solution

$$(\boldsymbol{u}, q) \quad \text{as} \quad (\boldsymbol{z}, \theta) + (\boldsymbol{u}_0, q_0),$$

being

$$(\boldsymbol{u}_0, q_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

satisfying

$$-\Delta \boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_0 = h \quad \text{in} \quad \Omega, \ \ \boldsymbol{u}_0 = \boldsymbol{g} \quad \text{on } \Gamma,$$

and

$$(\boldsymbol{z}, \theta) \in \mathbf{W}^{2,t}(\Omega) \times W^{1,t}(\Omega)$$

satisfying

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \theta = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \quad \text{and} \quad \nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega, \quad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma,$$

where $\frac{1}{t} = \frac{1}{3} + \frac{1}{p}$. The corresponding estimates (see Theorem 12) and the embedding

$$\mathbf{W}^{2,t}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$$

conclude the proof in this case. Secondly, if $p < 2$, we are able to conclude by a duality argument.

Remark. Estimate (19) can be improved for $p \in [\frac{6}{5}, 6]$, and for any $p > 1$ if $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ as:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} &+ \|q\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \\ &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big)\Big). \end{aligned}$$

Corollary 14 (**Strong solutions for the Oseen problem**)

Consider $1 < p < 6/5$ and

$$\boldsymbol{f} \in \mathbf{L}^p(\Omega), \quad \boldsymbol{v} \in \mathbf{H}_3(\Omega), \quad h \in W^{1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

verifying the compatibility condition (8). Then, the solution given by Theorem 13 satisfies $(\boldsymbol{u}, q) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and the following estimate holds:

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} &+ \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \\ &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\Big(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\Big)\Big). \end{aligned}$$

Using the previous results, we obtain:

Theorem 15 (Very weak solution of Oseen equations)

Let $\quad \boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \quad h \in L^r(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$, be given, satisfying the compatibility condition (8), and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$, with

$s = 3 \quad \text{if } p > 3/2, \; s = p' \text{ if } \; p < 3/2, \text{ or } s = 3 + \varepsilon \quad \text{if } p = 3/2.$

Then, the Oseen problem (O) has a unique solution $(\boldsymbol{u}, q) \in \mathbf{T}_{p,r}(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ verifying the estimates

$$\|\boldsymbol{u}\|_{\mathbf{T}_{p,r}(\Omega)} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right) \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \\ + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big),$$

$$\|q\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)^2 \Big(\|\boldsymbol{f}\|_{[\mathbf{X}_{r',p'}(\Omega)]'} + \|h\|_{L^r(\Omega)} + \\ + \|\boldsymbol{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \Big).$$

Concerning the regularity of solutions for the Oseen equations in fractional Sobolev spaces, we obtain:

Theorem 16 (Regularity for Oseen equations)

Consider $\sigma \in (1/p, 2]$. Let

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \quad h \in W^{\sigma-1,p}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma)$$

be given satisfying the compatibility condition (8), and $\boldsymbol{v} \in \mathbf{H}_s(\Omega)$ with s as in Theorem 15. Then, the Oseen problem (O) has a unique solution

$$(\boldsymbol{u},\ q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega)/\mathbb{R}$$

satisfying

$$\|\boldsymbol{u}\|_{\mathbf{W}^{\sigma,p}(\Omega)} + \|q\|_{W^{\sigma-1,p}(\Omega)/\mathbb{R}} \leq C\big(\|\boldsymbol{f}\|_{\mathbf{W}^{\sigma-2,p}(\Omega)} + \|h\|_{W^{\sigma-1,p}(\Omega)} + \\ + \quad \|\boldsymbol{g}\|_{\mathbf{W}^{\sigma-1/p,p}(\Omega)}\big).$$

5. Very weak solutions and regularity for the Navier-Stokes problem

Now, we present two theorems giving existence of very weak solutions for the Navier-Stokes equations in

$$\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega),$$

first one for the small data case, and second one for arbitrary large $\boldsymbol{f}$ but h and $\boldsymbol{g}$ small enough in a domain possibly multiply-connected.

Theorem 17 (**Very weak solution for Navier-Stokes, small data case**)

Let $\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))'$, $\quad h \in L^{3/2}(\Omega) \quad$ and $\quad \boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$

verify (8). Then,
i) there exists a constant $\alpha_1 > 0$ such that if

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_1,$$

then, there exists a very weak solution, to problem (NS), $(\boldsymbol{u}, q)$ belonging to $\mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$ and verifying the estimates

$$\| \boldsymbol{u} \|_{\mathbf{L}^3} \leq C \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}]'} + \| h \|_{L^{3/2}} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}} \right), \quad (20)$$

$$\begin{aligned} \| q \|_{W^{-1,3}/\mathbb{R}} \quad &\leq \quad C_1 \| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2})]'} + 2(1 + C_2) C \times \quad (21) \\ &\times \quad \left(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}]'} + \| h \|_{L^{3/2}} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}} \right), \end{aligned}$$

where $C > 0$ is the constant given in (20),
$\alpha_1 = \min\left\{(2C)^{-1}, (2C^2)^{-1}\right\}$ and C_1 and C_2 constants of Sobolev embeddings.
ii) Moreover, there exists a constant $\alpha_2 \in \left]0, \alpha_1\right]$ such that if

$$\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \alpha_2,$$

then this solution is unique, up to an additive constant for q.

Proof: We prove existence of a very weak solution by applying Banach's fixed point theorem over the Oseen equations. Indeed, let

$$T : \mathbf{H}_3(\Omega) \to \mathbf{H}_3(\Omega)$$

be the application defined as $\boldsymbol{v} \mapsto T\boldsymbol{v} = \boldsymbol{u}$, where $\boldsymbol{u}$ is the unique solution of (O) provided by Theorem 15. We set

$$\mathbf{B}_r = \{\boldsymbol{v} \in \mathbf{H}_3(\Omega);\ \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)} \leq r\}.$$

We will prove that there exists $\theta \in]0,1[$ such that

$$\|T\boldsymbol{v}_1 - T\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)} \leq \theta \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}. \quad (22)$$

In order to estimate

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)},$$

we observe that for each $i = 1, 2$, $(\boldsymbol{u}_i, q_i)$ is the solution of

$$-\Delta \boldsymbol{u}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{u}_i + \nabla q_i = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_i = h \quad \text{in } \Omega, \quad \boldsymbol{u}_i = \boldsymbol{g} \quad \text{on } \Gamma,$$

with the estimates

$$\begin{aligned} \|\boldsymbol{u}_i\|_{\mathbf{L}^3(\Omega)} &\leq C\left(1 + \|\boldsymbol{v}_i\|_{\mathbf{L}^3(\Omega)}\right) \Big(\| \boldsymbol{f} \|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \\ &+ \| h \|_{L^{3/2}(\Omega)} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/3,3}(\Gamma)} \Big), \end{aligned}$$

being $C > 0$ the constant given in (20). However, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to argue differently. Consider the problem fulfilled by $(\boldsymbol{u}, q) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, q_1 - q_2)$, which is

$$-\Delta \boldsymbol{u} + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{u} + \nabla q = -\boldsymbol{v} \cdot \nabla \boldsymbol{u}_2, \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma,$$

where

$$\boldsymbol{u}_1 = T\boldsymbol{v}_1, \quad \boldsymbol{u}_2 = T\boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2.$$

Using the very weak estimates (20) for the Oseen problem successively for $\boldsymbol{u}$ and for $\boldsymbol{u}_2$, we obtain that

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} &\leq C\left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right) \|(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_2\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} \\ &\leq C^2\beta\,\left(1 + \|\boldsymbol{v}_1\|_{\mathbf{L}^3(\Omega)}\right)\,\left(1 + \|\boldsymbol{v}_2\|_{\mathbf{L}^3(\Omega)}\right)\,\|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)},\end{aligned}$$

where $\beta = \|\, \boldsymbol{f} \,\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|\, h \,\|_{L^{3/2}(\Omega)} + \|\, \boldsymbol{g} \,\|_{\mathbf{W}^{-1/3,3}(\Gamma)}$.
Thus, we obtain estimate (22) if we consider

$$C^2\,\beta\,(1+r)^2\, < 1,$$

and (20)-(21) hold for C_1 the continuity constant of the Sobolev embedding

$$[\mathbf{X}_{3,3/2}(\Omega)]' \hookrightarrow \mathbf{W}^{-2,3}(\Omega)$$

and C_2 the continuity constant of the Sobolev embedding

$$\mathbf{W}_0^{1,3/2}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega).$$

The uniqueness result is a simple consequence of Sobolev embeddings and the Stokes estimates.

Theorem 18 (Very weak solution for Navier-Stokes, arbitrary forces)

Let

$$\boldsymbol{f} \in (\mathbf{X}_{3,3/2}(\Omega))', \quad h \in L^{3/2}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/3,3}(\Gamma)$$

be given, and satisfying the compatibility condition (8). There exists a constant $\delta > 0$ (depending only on Ω) such that the problem (NS) has a very weak solution

$$(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$$

if

$$\|h\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \leq \delta. \tag{23}$$

Sketch of the proof: We decompose (NS) into two problems. One system, denoted (NS_1), for small data:

$$-\Delta \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \nabla q^1_\varepsilon = \boldsymbol{f} - \boldsymbol{f}_\varepsilon, \;\; \nabla \cdot \boldsymbol{v}_\varepsilon = h - h_\varepsilon \;\text{ in } \Omega, \;\; \boldsymbol{v}_\varepsilon = \boldsymbol{g} - \boldsymbol{g}_\varepsilon \text{ on } \Gamma.$$

with $\varepsilon > 0$ and the (NS_2) system:

$$-\Delta \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \boldsymbol{z}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{z}_\varepsilon + \nabla q^2_\varepsilon = \boldsymbol{f}_\varepsilon,$$

$$\nabla \cdot \boldsymbol{z}_\varepsilon = h_\varepsilon \text{ in } \Omega, \quad \boldsymbol{z}_\varepsilon = \boldsymbol{g}_\varepsilon \quad \text{on } \Gamma$$

where

$$\boldsymbol{f}_\varepsilon \in \mathbf{H}^{-1}(\Omega), \quad h_\varepsilon \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{g}_\varepsilon \in \mathbf{H}^{1/2}(\Gamma)$$

satisfy

$$\|\boldsymbol{f} - \boldsymbol{f}_\varepsilon\|_{[\mathbf{X}_{3,3/2}(\Omega)]'} + \|h - h_\varepsilon\|_{L^{3/2}(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_\varepsilon\|_{\mathbf{W}^{-1/3,3}(\Gamma)} \leq \varepsilon$$

and

$$\|h_\varepsilon\|_{L^{3/2}(\Omega)} + \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}| \leq 2\delta,$$

(here, we have used density arguments).

Finally, we use an extension of Hopf's lemma: for any $\alpha > 0$, there exists $\boldsymbol{y}_\varepsilon \in \mathbf{H}^1(\Omega)$, depending on α, such that for $C_1 > 0$ depending only on Ω,

$$\nabla \cdot \boldsymbol{y}_\varepsilon = h_\varepsilon \quad \text{in} \quad \Omega, \quad \boldsymbol{y}_\varepsilon = \boldsymbol{g}_\varepsilon \quad \text{on}\Gamma$$

and for any $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, with $\nabla \cdot \boldsymbol{w} = 0$,

$$\left| \int_\Omega (\boldsymbol{w} \cdot \nabla) \boldsymbol{y}_\varepsilon \cdot \boldsymbol{w} \, d\boldsymbol{x} \right| \leq \left(\alpha + \|h_\varepsilon\|_{L^{3/2}} + C \sum_{i=0}^{i=I} |\langle \boldsymbol{g}_\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}| \right) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)}$$
$$\leq (\alpha + 2C_1\delta) \|\boldsymbol{w}\|^2_{\mathbf{H}^1(\Omega)}. \qquad \blacksquare$$

To finish, we prove some regularity results on very weak solutions for the Navier-Stokes equations by using the regularity results for the Stokes and Oseen problems.

Theorem 19 (Regularity for Navier-Stokes equations)

Let

$$(\boldsymbol{u}, q) \in \mathbf{L}^3(\Omega) \times W^{-1,3}(\Omega)$$

be the solution given by Theorem 18. Then, the following regularity results hold:

i) If

$$\boldsymbol{f} \in (\mathbf{X}_{r',p'}(\Omega))', \quad h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma),$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $\max\{r, 3\} \leq p$, then

$$(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega).$$

ii) Consider $r \geq 3/2$,

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega), \quad h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma).$$

Then

$$(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega).$$

iii) For $1 < r < \infty$), if

$$\boldsymbol{f} \in \mathbf{L}^r(\Omega), \quad h \in W^{1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/r,r}(\Gamma),$$

then

$$(\boldsymbol{u}, q) \in \mathbf{W}^{2,r}(\Omega) \times W^{1,r}(\Omega).$$

iv) Suppose that $3/2 \leq p \leq 3$,

$$\boldsymbol{f} = \nabla \cdot \mathbb{F}_0 + \nabla f_1 \quad \text{for } \mathbb{F}_0 \in \mathbf{W}^{\sigma,r}(\Omega) \quad \text{and} \quad f_1 \in W^{\sigma-1,p}(\Omega),$$

and

$$h \in W^{\sigma,r}(\Omega), \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma),$$

with $\sigma = \frac{3}{p} - 1$, $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $r \leq p$. Then

$$(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega).$$

v) Let σ be such that $1/p < \sigma \leq 1$ and $\sigma \geq 3/p - 1$. Suppose that

$$\boldsymbol{f} \in \mathbf{W}^{\sigma-2,p}(\Omega), \quad h \in W^{\sigma-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{\sigma-1/p,p}(\Gamma).$$

Then

$$(\boldsymbol{u}, q) \in \mathbf{W}^{\sigma,p}(\Omega) \times W^{\sigma-1,p}(\Omega).$$

Remark.

i) Point i) shows in particular that for any $p \geq 3$, if

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma),$$

with

$$h = 0, \int_{\Gamma_i} \boldsymbol{g}\cdot\boldsymbol{n} = 0 \quad \text{for any } i = 1, \ldots, I \quad \text{and} \quad \frac{3p}{3+p} \leq r \leq p,$$

then Problem (NS) has a solution

$$(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega).$$

Serre (85) proves that for any $3/2 < r < 2$ (and then for $r > 3/2$), if

$$\boldsymbol{f} \in \mathbf{W}^{-1,r}(\Omega), \quad \boldsymbol{g} \in \mathbf{W}^{1-1/r,r}(\Gamma), \quad h = 0 \quad \text{and} \quad \int_{\Gamma_i} \boldsymbol{g}\cdot\boldsymbol{n} = 0$$

for any $i = 0, \ldots, I$, then (NS) has a solution

$$(\boldsymbol{u}, q) \in \mathbf{W}^{1,r}(\Omega) \times L^r(\Omega).$$

- Point ii) of Theorem 19 proves that this result holds if $r = 3/2$ without assuming h or the flux $\boldsymbol{g}$ through Γ_i to be equal to 0. Actually, it suffices to assume the smallness condition (23)

ii) From relation (8), condition (23) is automatically fulfilled when the norm $\|h\|_{L^{3/2}(\Omega)}$ is small enough and $I = 0$, that means that the boundary Γ is connected, which is the case considered by Kim (09).

iii) Marusic-Paloka (00) proves Theorem 19 with

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega) \subset (\mathbf{X}_{3,3/2}(\Omega))', \quad h = 0, \quad \boldsymbol{g} \in \mathbf{L}^2(\Gamma) \subset \mathbf{W}^{-1/3,3}(\Gamma)$$

with $\|\boldsymbol{g}\|_{\mathbf{L}^2(\Gamma)}$ small, in a domain Ω simply-connected. In fact, the solution $\boldsymbol{u} \in \mathbf{L}^3(\Omega)$ obtained by Marusic-Paloka (00) is more regular and belongs to $\mathbf{H}^{1/2}(\Omega)$ by point iv) with $p = 2$.

iv) Galdi, Simader and Sohr (05) prove Theorem 18 and Theorem 19 point i) with

$$\boldsymbol{f} = \operatorname{div} \mathbb{F}_0, \quad \mathbb{F}_0 \in \mathbb{L}^r(\Omega), \quad h \in L^p(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{-1/p,p}(\Gamma)$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$ and $\max\{2r, 3\} \leq p$, in a domain Ω of class $\mathcal{C}^{2,1}$, assuming that $\boldsymbol{f}$, h and $\boldsymbol{g}$ are small enough in their respective norms. The smallness condition on $\boldsymbol{f}$ is in fact unnecessary.

For Further Reading

- C. Amrouche and V. Girault, *D*ecomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czechoslovak Mathematical Journal, 44, 119 (1994), pp. 109–140.
- C. Amrouche and M. A. Rodríguez-Bellido, *S*tationary Stokes, Oseen and Navier-Stokes equations with singular data, CRAS, 348, (2010).
- G. P. Galdi, *An Introduction to the Matematical Theory of the Navier-Stokes Equations, Vol 2: Nonlinear Steady Problems.* Springer Tracts in Natural Philosophy, vol. 39. Springer, New York (1994).
- G. P. Galdi, C. G. Simader and H. Sohr, *A* class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$, Math. Ann. , 331 (2005), pp. 41–74.
- H. Kim, *E*xistence and regularity of very weak solutions of the stationary Navier-Stokes equations, Arch. Rational Mech. Anal., **193** (2009), 117-152.
- J. Leray, *E*tude de divesres équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. 12 (1933), pp. 1–82.
- E. Marusič-Paloka, *S*olvability of the Navier-Stokes system with L^2 boundary data, Appl. Math. Optim. 41 (2000), pp. 365-375.
- D. Serre, *E*quations de Navier-Stokes stationnaires avec données peu régulières, Ann. Sc. Norm. Sup. Pisa 10-4, (1983), pp. 543-559.