Well Posedness and Derivation of Multi-Fluid Models

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Outline

A model with an algebraic closure (common pressure)

$$
\begin{aligned}
\alpha_+ + \alpha_- &= 1,\\
\partial_t(\alpha^+\rho^+) + \operatorname{div}(\alpha^+\rho^+u^+) &= 0,\\
\partial_t(\alpha^-\rho^-) + \operatorname{div}(\alpha^-\rho^-u^-) &= 0,\\
\partial_t(\alpha^+\rho^+u^+) + \operatorname{div}(\alpha^+\rho^+u^+\otimes u^+) + \alpha^+\nabla P &= \mathbf{0},\\
\partial_t(\alpha^-\rho^-u^-) + \operatorname{div}(\alpha^-\rho^-u^-\otimes u^-) + \alpha^-\nabla P &= \mathbf{0},\\
P = P_-(\rho_-) = P_+(\rho_+),
\end{aligned}
$$

with

$$0 \le \alpha_\pm \le 1.$$

See for instance M. Ishii (1975), D.A. Drew and S.L. Passman (1998).

The model with algebraic closure

Non-conservative, non-hyperbolic system if $0 \leq |u^+ - u^-| < c_m$ with

$$c_m^2 = c_-^2 c_+^2 ((\alpha^+ \rho^+)^{1/3} + (\alpha^- \rho^-)^{1/3})^3 / (\alpha^+ \rho^- c_-^2 + \alpha^- \rho^+ c_+^2).$$

In general, c_m is large compared to u^+ and u^- and thefore flow belongs to non-hyperbolic region.

See: H.B. STEWART, B. WENDROFF, *J. Comp. Physics*, 363–409, (1984) (Appendix I).

The model with algebraic closure with extra terms

$$
\begin{aligned}
\alpha_+ + \alpha_- &= 1,\\
\partial_t(\alpha^+\rho^+) + \mathrm{div}\,(\alpha^+\rho^+u^+) &= 0,\\
\partial_t(\alpha^-\rho^-) + \mathrm{div}\,(\alpha^-\rho^-u^-) &= 0,\\
\partial_t(\alpha^+\rho^+u^+) + \mathrm{div}\,(\alpha^+\rho^+u^+\otimes u^+) + \alpha^+\nabla P + \pi\nabla\alpha^+ &= \mathbf{0},\\
\partial_t(\alpha^-\rho^-u^-) + \mathrm{div}\,(\alpha^-\rho^-u^-\otimes u^-)) + \alpha^-\nabla P + \pi\nabla\alpha^- &= \mathbf{0},\\
P = P_-(\rho_-) = P_+(\rho_+),
\end{aligned}
$$

with

$$0 \le \alpha_\pm \le 1.$$

The model with algebraic closure

In litterature, use Bestion term

$$\pi = \delta \frac{\alpha^+\alpha^-\rho^+\rho^-}{\alpha^+\rho^- + \alpha^-\rho^+}(u^+ - u^-)^2.$$

with $\delta > 1$ to get hyperbolicity for small relative velocity.

– See paper by M. NDJINGA, A. KUMBARO, F. DE VUYST,
P. LAURENT-GENGOUX, ISMF (2005) for geometric discussions: number of intersecting points of parabola and hyperbola in quarter plane.
Extension of H.B. STEWART, B. WENDROFF's approach.

– See for direct study: analytical calculations.
D.B., B. DESJARDINS, J.–M. GHIDAGLIA, E. GRENIER. Low Mach Number Limit and Bi-Fluid Systems. In preparation (2011).

A low mach number model

$$\begin{aligned}
\alpha^- + \alpha^+ &= 1,\\
\partial_t(\alpha^+) + \operatorname{div}(\alpha^+ u^+) &= 0,\\
\partial_t(\alpha^-) + \operatorname{div}(\alpha^- u^-) &= 0,\\
\rho^+(\partial_t(\alpha^+ u^+) + \operatorname{div}(\alpha^+ u^+ \otimes u^+)) + \alpha^+ \nabla P + \pi \nabla \alpha^+ &= \mathbf{0},\\
\rho^-(\partial_t(\alpha^- u^-) + \operatorname{div}(\alpha^- u^- \otimes u^-)) + \alpha^- \nabla P + \pi \nabla \alpha^- &= \mathbf{0},
\end{aligned}$$

with ρ^- and ρ^+ constants and P the Lagrangian multiplier associated to the constraint $\alpha^+ + \alpha^- = 1$.

A model with an algebraic closure

Hyperbolic with Bestion closure namely:

$$\pi = \delta \frac{\alpha^+\alpha^-\rho^+\rho^-}{\alpha^+\rho^- + \alpha^-\rho^+}(u^+ - u^-)^2$$

with $\delta > 1$.

Rq: We will see a model which shares the same form:
The two-layers shallow-water system between rigid lids: See slide 14.
In this model, $\pi = 0$ and a term $cst\nabla\alpha^+$ appears in the $+$ momentum component.

A model with an algebraic closure

- Local well-posedness on an associated low mach number limit system, see [1]
- Global weak solutions if degenerate viscosities and capillarity terms, see [2]
- Invariant regions, see [2]
- Global weak solutions in one space dimension if degenerate viscosities, see [3]

[1] D. B., M. RENARDY. Well-Posedness of Two-Layer Shallow-Water Flow Between Two Horizontal Rigid Plates. *Nonlinearity* , 24, 1081–1088, (2011).

[2] D. B., B. DESJARDINS, J.–M. GHIDAGLIA, E. GRENIER. On Global Weak Solutions to a Generic Two-Fluid Model. *Arch. Rational Mech. Anal.* Volume 196, Number 2, 599-629, (2009).

[3] D.B., X. HUANG, J. LI. A Global Weak Solution to a One-Dimensional Non-Conservative Viscous Compressible Two-Phase System. Submitted (2010).

A model with a PDE closure (equation on fraction)

$$\begin{aligned}
\alpha_+ + \alpha_- &= 1, \\
\partial_t \alpha^+ + u_{\text{int}} \cdot \nabla \alpha^+ &= \frac{1}{\lambda_P}(P^+ - P^-), \\
\partial_t(\alpha^+ \rho^+) + \text{ div }(\alpha^+ \rho^+ u^+) &= 0, \\
\partial_t(\alpha^- \rho^-) + \text{ div }(\alpha^- \rho^- u^-) &= 0, \\
\partial_t(\alpha^+ \rho^+ u^+) + \text{ div }(\alpha^+ \rho^+ u^+ \otimes u^+) + \alpha^+ \nabla P^+ + P_{\text{int}} \nabla \alpha^+ &= \frac{1}{\lambda_u}(u^+ - u^-), \\
\partial_t(\alpha^- \rho^- u^-) + \text{ div }(\alpha^- \rho^- u^- \otimes u^-) + \alpha^- \nabla P^- + P_{\text{int}} \nabla \alpha^- &= \frac{1}{\lambda_u}(u^- - u^+),
\end{aligned}$$

with u_{int} and P_{int} respectively interface velocity and interface pressure explicitely given in terms of the unknowns.

A model with a PDE closure (equation on fraction)

If $\lambda_u \to 0$, One-velocity field. See works by F. DIAS, D. DUTYKH and J.–M. GHIDAGLIA (2010) on a two-fluid model for violent aerated flows.

See for instance: R. ABGRALL, C. BERTHON, F. COQUEL, S. DELLACHERIE, D.A. DREW and S.L. PASSMAN, Th. GALLOUËT, M. ISHII, Ph. LE FLOCH, R. SAUREL and others for modeling and numerics.

A model with a PDE closure (equation on fraction)

- Viscous multi-fluid model as limit of viscous mono-fluid model: (One-velocity field), see [4].

[4] D.B., X. HUANG. A Multi-Fluid Compressible System as the Limit of Weak-Solutions of the Isentropic Compressible Navier-Stokes Equations. To appear *Arch. Rational Mech. Anal.* (2011).

Local well posedness

LOCAL WELL POSEDNESS WITH NO-IRROTATIONALITY CONDITION

Collaboration with M. RENARDY: Paper [1]

Model and Theorem

The model (SW) in $\Omega = T^2$ or R^2

$$\begin{aligned} h_t + \operatorname{div}(h\mathbf{v}_1) &= 0, \\ -h_t + \operatorname{div}((1-h)\mathbf{v}_2) &= 0, \\ (\mathbf{v}_1)_t + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 + \frac{\rho - 1}{\rho}\nabla h + \frac{1}{\rho}\nabla p &= \mathbf{0}, \\ (\mathbf{v}_2)_t + (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 + \nabla p &= \mathbf{0}. \end{aligned}$$

Remark. Indices 1 and 2 refer to the bottom and top layer respectively. Density of bottom layer $\rho = \rho_1/\rho_2 > 1$, the top one equals 1. The depth of the bottom layer is $h_1 = h$ and top $h_2 = 1 - h$. Gravity g is taken equal to 1.

Theorem. Let $\rho > 1$ and $s > 2$. Assuming that $(h_0, \mathbf{v}_1^0, \mathbf{v}_2^0) \in (H^s)^5$ with $0 < h_0 < 1$ are such that

$$|\mathbf{v}_1^0 - \mathbf{v}_2^0|^2 < (\rho - 1)(h_0 + \rho(1 - h_0))/\rho. \tag{1}$$

is satisfied and, moreover, $\operatorname{div}(h_0\mathbf{v}_1^0 + (1-h_0)\mathbf{v}_2^0) = 0$. Then, there exists $T_{\max} > 0$, and a unique maximal solution $(h, \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}([0, T_{\max}); (H^s)^5)$ (and a corresponding pressure p) to the system (SW), which satisfies the initial condition $(h, \mathbf{v}_1, \mathbf{v}_2)|_{t=0} = (h_0, \mathbf{v}_1^0, \mathbf{v}_2^0)$.

Framework and idea

Non-irrotational case: First result to the authors's knowledge.

Main result: Local well-posedness under optimal restrictions on the data by rewriting the system in an appropriate form which fits into the abstract theory of T.J.R. Hughes, T. Kato and J.E. Marsden related to second order quasi-linear hyperbolic systems.

Idea: Isolate the "essential" part, using the total derivative $\partial_t + \mathbf{V} \cdot \nabla$ operator with $\mathbf{V}$ the weighted average velocity $\mathbf{V} = (\rho(1-h)\mathbf{v}_1 + h\mathbf{v}_2)/(\rho(1-h)+h)$.

Some Mathematical comments

Assumption equivalent to the one obtained by P. GUYENNE, D. LANNES, J.–C. SAUT [GLS2010] in the one-dimensional case (see $(24)_3$) and better than the one obtained in the irrotational case (see $(44)_3$). With our notation, Condition $(44)_3$ in [GLS2010] reads

$$\|\mathbf{v}_1^0 - \mathbf{v}_2^0\|_\infty^2 < (\rho - 1)(1 + \rho - (\rho - 1)\|2h_0 - 1\|_\infty)/2\rho.$$

We note we obtain if we replace the L^∞ norms by point values.

Methods in GLS2010:

In one-dimension, explicit relation between $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_2 = -h\mathbf{v}_1/(1-h)$. In the bi-fluid framework, no gravity inside, see B.L. KEYFITZ's works (reduction indicated due to C.M. DAFERMOS) related to singular shocks, Riemann problems and loss of hyperbolicity.

In irrotational-two dimensional case, non-local relation between $\mathbf{v}_1 = \nabla\Phi_1$ and $\mathbf{v}_2 = \nabla\Phi_2$ through

$$\mathrm{div}(h\nabla\Phi_1) = -\mathrm{div}((1-h)\nabla\Phi_2).$$

The interesting difficulty being to define an appropriate symmetrizer.

Some physical comments

Physical point of view: Condition arises from the competition between the Kelvin-Helmholtz instability and the stabilizing effect of gravity.

Same condition obtained in the study of long wave linear stability of density stratified two layer flow with a constant velocity in each layer (take the limit $k \to 0$ with surface tension coefficient $\gamma = 0$ and $g = 1$ in (3.6) of Funada-Joseph):

$$|\mathbf{v}_1 - \mathbf{v}_2|^2 \leq \left[\frac{\tanh(kh_1)}{\rho_1} + \frac{\tanh(kh_2)}{\rho_2}\right]\frac{1}{k}[(\rho_1 - \rho_2)g + \gamma k^2]$$

See also the recent fundamental mathematical paper D. LANNES in the nonlinear framework. Note that papers T. FUNADA – D.D. JOSEPH and D. LANNES concern potential flows.

Applications to bifluid systems and simulations.

Remark: Same kind of result in 3 dimension for $s > 5/2$ with application for the two-fluid models of a suspension page 903 (with no viscosity $\mu = 0$) in R. CAFLISH, G. PAPANICOLAOU (*SIAM J. Appl. Math* (1983)).

Remark: If no gravity and nothing more, well posedness only for analytical data (See E. GRENIER, *Comm. Partial Diff. Eqs* (1996)).

Remark: Important to deal with non-irrotational data in bifluid framework. For instance Bestion closure in the momentum equations.

$$P_{\text{int}} \nabla \alpha_i = \delta \frac{\alpha_1 \alpha_2 \rho_1 \rho_1}{\alpha_2 \rho_1 + \alpha_1 \rho_2} (u_1 - u_2)^2 \nabla \alpha_i$$

with $\delta \geq 1$.

Remark: Important from a numerical point of view: Iterative scheme!

Algebraic computations

We take the divergence of the last two equations and obtain

$$\begin{aligned}
(\frac{\partial}{\partial t} + (\mathbf{v}_1 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_1 + \frac{\rho - 1}{\rho}\Delta h + \frac{1}{\rho}\Delta p &= -\,\mathrm{tr}\,((\nabla \mathbf{v}_1)^2), \\
(\frac{\partial}{\partial t} + (\mathbf{v}_2 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_2 + \Delta p &= -\,\mathrm{tr}\,((\nabla \mathbf{v}_2)^2).
\end{aligned}$$

We can eliminate p and combine these two equations in the form

$$\begin{aligned}
\rho(\frac{\partial}{\partial t} + (\mathbf{v}_1 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_1 \quad - \quad & (\frac{\partial}{\partial t} + (\mathbf{v}_2 \cdot \nabla))\,\mathrm{div}\,\mathbf{v}_2 + (\rho - 1)\Delta h \\
= \quad & -\rho\,\mathrm{tr}\,((\nabla \mathbf{v}_1)^2) + \,\mathrm{tr}\,((\nabla \mathbf{v}_2)^2).
\end{aligned}$$

We introduce the following weighted average of the velocity ("Favre velocity"):

$$V = \frac{\rho(1 - h)\mathbf{v}_1 + h\mathbf{v}_2}{\rho(1 - h) + h}.$$

Algebraic computations

With this, we can write in the form

$$
\begin{aligned}
& (\frac{\partial}{\partial t} + (V \cdot \nabla))\, \mathrm{div}\,(\rho \mathbf{v}_1 - \mathbf{v}_2) + (\rho - 1)\Delta h \\
& + \frac{\rho}{h + \rho(1-h)} (\mathbf{v}_1 - \mathbf{v}_2) \cdot (h \nabla\, \mathrm{div}\, \mathbf{v}_1 + (1-h)\nabla\, \mathrm{div}\, \mathbf{v}_2) \\
= \quad & -\rho\, \mathrm{tr}\,((\nabla \mathbf{v}_1)^2) + \, \mathrm{tr}\,((\nabla \mathbf{v}_2)^2).
\end{aligned}
$$

Combining the first two equations, we find

$$
\mathrm{div}\,(h\mathbf{v}_1 + (1-h)\mathbf{v}_2) = 0.
$$

Using this, we find

$$
\begin{aligned}
& (\frac{\partial}{\partial t} + (V \cdot \nabla))\, \mathrm{div}\,(\rho \mathbf{v}_1 - \mathbf{v}_2) + (\rho - 1)\Delta h \\
& - \frac{\rho}{h + \rho(1-h)} ((\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla)^2 h = \mathbf{f}_1(\mathbf{v}_1, \mathbf{v}_2, h, \nabla \mathbf{v}_1, \nabla \mathbf{v}_2, \nabla h),
\end{aligned}
$$

where $\mathbf{f}_1$ depends only on the arguments indicated.

Algebraic computations

Next, we multiply the first equation of System (”mass equation”) by $\rho(1-h)/(h+\rho(1-h))$, the second equation by $h/(h+\rho(1-h))$ and subtract. The result is

$$h_t + (\boldsymbol{V}\cdot\nabla)h + \frac{(1-h)h}{h+\rho(1-h)}\,\mathrm{div}\,(\rho\mathbf{v}_1 - \mathbf{v}_2) = 0.$$

We can now combine to find

$$\Big(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla)\Big)^2 h = \frac{(1-h)h}{h+\rho(1-h)}\Big((\rho-1)\Delta h - \frac{\rho}{h+\rho(1-h)}((\mathbf{v}_1-\mathbf{v}_2)\cdot\nabla)^2 h\Big) + \mathbf{f}_2(\mathbf{v}_1,\mathbf{v}_2,h,\nabla\mathbf{v}_1,\nabla\mathbf{v}_2,\nabla h,h_t).$$

For given $\mathbf{v}_1$ and $\mathbf{v}_2$, this is a second order hyperbolic equation for h as long as

$$|\mathbf{v}_1-\mathbf{v}_2|^2 < (\rho-1)(h+\rho(1-h))/\rho.$$

Iterative scheme

Use an abstract result established by T.J.R. HUGHES, T. KATO and J.E. MARSDEN. We begin with a quote of the abstract theorem, see pages 275–276. This result concerns evolution problems of the form

$$\dot{u} = A(t,u)u + f(t,u),$$

where u takes values in a Banach space, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup, and f is a "perturbation" term. We say that $A \in G(X, M, \omega)$ if

$$\|e^{At}\|_{L(X)} \leq Me^{\omega t}.$$

The construction of the solution is by an iteration of the form

$$\dot{u}^{n+1} = A(t,u^n)u^{n+1} + f(t,u^n),$$

with fixed initial condition $u^n(0) = u_0$.

Hughes-Kato-Marsden Theorem

Theorem. Let $Y \subset Z \subset Z' \subset X$ be four real Banach spaces, all of them reflexive and separable, with continuous and dense inclusions. We assume that
1) Z' is an interpolation space between Y and X (i.e. linear operators which are bounded on both Y and X are also bounded on Z').
Let $N(X)$ be the set of all norms on X equivalent to the given one. On $N(X)$ we introduce a distance function

$$d(\|\cdot\|_\alpha, \|\cdot\|_\beta) := \ln \max\{\sup_{z\neq 0} \|z\|_\alpha / \|z\|_\beta, \sup_{z\neq 0} \|z\|_\beta / \|z\|_\alpha\}.$$

Let W be an open set in Y. We assume that there is a real number β and positive numbers $\lambda_N,\ \mu_N, \ldots$ such that the following hold for all $t, t' \in [0,T]$ and $w, w' \in W$.
2) $N(t,w) \in N(X)$, and

$$\begin{aligned} d(N(t,w), \|\cdot\|_X) &\leq \lambda_N, \\ d(N(t',w'), N(t,w)) &\leq \mu_N[|t'-t| + \|w'-w\|_Z]. \end{aligned}$$

Hughes-Kato-Marsden Theorem

3) There is an isomorphism $S(t,w) \in B(Y,X)$, with

$$\|S(t,w)\|_{Y,X} \leq \lambda_S, \ \|S(t,w)^{-1}\|_{X,Y} \leq \lambda_S',$$
$$\|S(t',w') - S(t,w)\|_{Y,X} \leq \mu_S[|t'-t| + \|w'-w\|_Z].$$

4) $A(t,w) \in G(X_{N(t,w)}, 1, \beta)$.
5) $S(t,w)A(t,w)S(t,w)^{-1} = A(t,w) + B(t,w)$, where $B(t,w)$ is a bounded operator in X and $\|B(t,w)\|_X \leq \lambda_B$.
6) $A(t,w) \in B(Y,Z)$ with

$$\|A(t,w)\|_{Y,Z} \leq \lambda_A, \ \|A(t,w') - A(t,w)\|_{Y,Z'} \leq \mu_A \|w'-w\|_{Z'}.$$

Moreover, the mapping $t \to A(t,w) \in B(Y,X)$ is continuous in norm.
7) $f(t,w) \in Y$, with

$$\|f(t,w)\|_Y \leq \lambda_f, \ \|f(t,w') - f(t,w)\|_{Z'} \leq \mu_f \|w'-w\|_{Z'},$$

and the mapping $t \to f(t,w) \in X$ is continuous.

Hughes-Kato-Marsden Theorem

If all of the above assumptions are satisfied, and $u_0 \in W \subset Y$, then there is a $T' \in (0, T]$ such that System (I) has a unique solution u on $[0, T']$ with $u \in C([0, T']; W) \cap C^1([0, T']; X)$. Here T' may depend on all the constants involved in the assumptions and on the distance between u_0 and the boundary of W. The mapping $u_0 \to u(t)$ is Lipschitz continuous in the Z'-norm, uniformly for $t \in [0, T']$. The solution is obtained by the iteration.

Application

To apply the result, we shall view $\mathbf{w}_i$ as determined by ω_i, ϕ_i, and $\mathbf{v}_i$ given by

$$\mathbf{v}_i = \mathbf{w}_i + \mathbf{q}_i + \nabla\phi_i.$$

Thus, $\mathbf{v}_i \in (H^s)^2$ is determined by $\omega_i \in H^{s-1}$, $\mathbf{q}_i \in \mathbb{R}^2$, $h \in H^s$, and $h_t \in H^{s-1}$. We set

$$u = (h, g, \omega_1, \omega_2, \mathbf{q}_1, \mathbf{q}_2),$$

where g represents h_t.
The spaces are given as

$$\begin{aligned} Y &= H^s \times H^{s-1} \times (H_0^{s-1})^2 \times \mathbb{R}^4, \\ Z &= Z' = H^{s-1} \times H^{s-2} \times (H_0^{s-2})^2 \times \mathbb{R}^4. \\ X &= H^1 \times L^2 \times (L_0^2)^2 \times \mathbb{R}^4. \end{aligned}$$

Here the subscript 0 denotes functions of zero average. We define W to be a sufficiently small neighborhood of the initial data in Y; in particular W must be small enough so that h and $1-h$ have strict lower bounds and (1) (with $\mathbf{v}_i$ given by through Helmotz decomposition) holds uniformly on W.

Application

We can set

$$S = ((-\Delta + 1)^{(s-1)/2})^4 \times (Id)^4.$$

Consider

$$u = (h, g, \omega_1, \omega_2, \mathbf{q}_1, \mathbf{q}_2), \qquad \tilde{u} = (\tilde{h}, \tilde{g}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2).$$

We define

$$A(\tilde{u})u = \begin{pmatrix} g, \\ -2(\tilde{V}\cdot\nabla)g - (\tilde{V}\cdot\nabla)^2 h + \dfrac{(1-\tilde{h})\tilde{h}}{\tilde{h}+\rho(1-\tilde{h})}\Big((\rho-1)\Delta h \\ -\dfrac{\rho}{\tilde{h}+\rho(1-\tilde{h})}((\tilde{\mathbf{v}}_1-\tilde{\mathbf{v}}_2)\cdot\nabla)^2 h\Big), \\ -(\tilde{\mathbf{v}}_1\cdot\nabla)\omega_1 - \omega_1 \operatorname{div}\tilde{\mathbf{v}}_1, \\ -(\tilde{\mathbf{v}}_2\cdot\nabla)\omega_2 - \omega_2 \operatorname{div}\tilde{\mathbf{v}}_2, \\ \mathbf{0}, \\ \mathbf{0} \end{pmatrix}$$

where $\tilde{\phi}_i$, $\tilde{\mathbf{v}}_i$ and $\tilde{V}$ are given in terms of $\tilde{u}$ through the relations (mass equation, algebraic relation...).

Application

Moreover, we define

$$
\begin{aligned}
(N(\tilde{u})u)^2 &= \|h\|^2 + \left\|\left(\frac{(1-\tilde{h})\tilde{h}}{\tilde{h}+\rho(1-\tilde{h})}\right)^{1/2}(\rho-1)^{1/2}\nabla h\right\|^2 \\
&- \left\|\frac{(\rho(1-\tilde{h})\tilde{h})^{1/2}}{\tilde{h}+\rho(1-\tilde{h})}((\tilde{\mathbf{v}}_1-\tilde{\mathbf{v}}_2)\cdot\nabla)h\right\|^2 + \|g+(\tilde{V}\cdot\nabla)h\|^2 \\
&+ \|\omega_1\|^2 + \|\omega_2\|^2 + |\mathbf{q}_1|^2 + |\mathbf{q}_2|^2.
\end{aligned}
$$

The verification of the assumptions is quite routine using the definition of W, $N(w)$, S and $A(w)$. Assumption 4 follows from the Lumer-Phillips theorem (dissipativity of $A(w)$ and surjectivity of $A(w)-\lambda_0\mathrm{Id}$ for some $\lambda_0>0$ with the appropriate constants and norm), and the remaining assumptions can be verified using the fact that H^{s-1} is a Banach algebra, as well as a multiplier in lower order Sobolev spaces. The reader is referred to paper by HUGHES-KATO-MARSDEN for an application to nonlinear elasto-dynamics of the Theorem for which a similar system is involved.

Remarks

In D.B., B. DESJARDINS, J.–M. GHIDAGLIA, E. GRENIER. Low Mach Number Limit and Bi-Fluid systems. In preparation (2011).

1) Appropriate unknowns imply adequate variables to study low Mach number limits. System closed to the non-isentropic system.

2) With bestion term, the incompressible bi-fluid system gives

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla)\right)^2 \alpha_+ = \\
&\quad \frac{(1-\alpha_+)\alpha_+}{\alpha_+ + \rho(1-\alpha_+)}\Big(\frac{\delta\rho}{\alpha_+ + (1-\alpha_+)\rho}|u_+ - u_-|^2 \Delta\alpha_+ \\
&\qquad\qquad - \frac{\rho}{\alpha_+ + \rho(1-\alpha_+)}((u_+ - u_-)\cdot\nabla)^2\alpha_+\Big) \\
&\quad + \mathbf{f}_2(u_+, u_-, \alpha_+, \nabla u_+, \nabla u_-, \nabla\alpha_+, (\alpha_+)_t)
\end{aligned}
$$

with $\mathbf{f}_2(u_+, u_+, \alpha_+, \nabla u_+, \nabla u_-, \nabla\alpha_+, (\alpha_+)_t) = 0$

Effect of viscosity and surface tension

GLOBAL WEAK SOLUTIONS AND INVARIANT REGIONS

Collaboration with B. DESJARDINS, J.M. GHIDAGLIA, E. GRENIER: paper [2]

Collaboration with X. HUANG, J. LI: paper [3]

The model

Introducing viscosity and capillarity effects on bifluid system and write:

$$
\begin{aligned}
&\alpha^+ + \alpha^- = 1\,,\\
&\partial_t(\alpha^\pm \rho^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm u^\pm) = 0\,,\\
&\partial_t(\alpha^\pm \rho^\pm u^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm u^\pm \otimes u^\pm)\\
&\qquad + \alpha^\pm \nabla p = \mathrm{div}(\alpha^\pm \tau^\pm) + \sigma^\pm \alpha^\pm \rho^\pm \nabla\Delta\left(\alpha^\pm \rho^\pm\right)\,.
\end{aligned}
$$

with

$$\tau^\pm = 2\mu^\pm D(u^\pm) + \lambda^\pm \,\mathrm{div}\, u^\pm \mathrm{Id}$$

$$p = p_\pm(\rho^\pm) = A^\pm(\rho^\pm)^{\gamma^\pm} \quad \text{where } \gamma^\pm \text{ are given constants greater than } 1$$

Assume

$$\mu_\pm(\rho^\pm) = \mu^\pm \rho^\pm, \qquad \lambda_\pm(\rho^\pm) = 0.$$

Multi-dimensional case

With surface tension term: Paper [2].

Definition of weak solutions. We shall say that $(\rho^\pm, \alpha^\pm, u^\pm)$ is a weak solution on $(0, T)$ if the following three conditions are fulfilled:

- the following regularity properties hold

$$\alpha^\pm \rho^\pm |u^\pm|^2 \in L^\infty(0, T; L^1(\Omega)),$$

$$\nabla\sqrt{\alpha^\pm \rho^\pm} \in L^\infty(0, T; L^2(\Omega)^3),$$

$$\sqrt{\alpha^\pm \rho^\pm}\nabla u^\pm \in L^2((0, T) \times \Omega)^{3\times 3},$$

$$\sqrt{\sigma^\pm}\nabla(\alpha^\pm \rho^\pm) \in L^\infty(0, T; L^2(\Omega)^3),$$

$$\sqrt{\sigma^\pm}\Delta(\alpha^\pm \rho^\pm) \in L^2(0, T; L^2(\Omega)),$$

  with following time continuity properties

$$\alpha^\pm \rho^\pm \in C([0, T]; H^s(\Omega)), \quad \text{for all} \quad s < 1/2,$$
$$\alpha^\pm \rho^\pm u^\pm \in C([0, T]; H^{-s}(\Omega)^3) \quad \text{for some positive} \quad s,$$

Multi-dimensional case

- the initial conditions holds in $\mathcal{D}'(\Omega)$.
- "Mass" equations hold in $\mathcal{D}'((0,T)\times\Omega)$ and momentum equations multiplied by $\alpha^\pm\rho^\pm$ hold in $\mathcal{D}'((0,T)\times\Omega)^3$: for all $\psi^\pm\in C^\infty([0,T]\times\Omega)^d$ and denoting $R^\pm=\alpha^\pm\rho^\pm$, one has

$$
\begin{aligned}
&\int_\Omega {R^\pm}^2 u^\pm(t,\cdot)\cdot\psi^\pm(t,\cdot) - \int_\Omega R_0^\pm m_0^\pm\cdot\psi(0,\cdot)\\
&= \int_0^t\int_\Omega \Big[D(\psi^\pm) : \Big(R^\pm u^\pm\otimes R^\pm u^\pm - 2\nu^\pm {R^\pm}^2 D(u^\pm)\\
&+\sigma^\pm R^\pm\nabla R^\pm\otimes\nabla R^\pm\Big) + \sigma^\pm|\nabla R^\pm|^2\psi^\pm\cdot\nabla R^\pm - \alpha^\pm R^\pm\psi^\pm\cdot\nabla p\\
&-{R^\pm}^2\,\mathrm{div}\, u^\pm(u^\pm\cdot\psi^\pm) - 2\nu^\pm R^\pm\psi\cdot\big(D(u^\pm)\cdot\nabla R^\pm\big) + {R^\pm}^2 u^\pm\cdot\partial_t\psi^\pm\\
&-\sigma^\pm\,\mathrm{div}\,\psi^\pm\Big(\Delta {R^\pm}^3/3 - R^\pm|\nabla R^\pm|^2\Big)\Big]
\end{aligned}
$$

Multi-dimensional case

Theorem. Assume $1 < \gamma^{\pm} < 6$ and that the initial data $(\alpha^{\pm}, R_0^{\pm}, m_0^{\pm})$ satisfy

$$R_0^{\pm} \geq 0, \alpha_0^{\pm} \in [0,1] \text{ such that } \alpha_0^{+} + \alpha_0^{-} = 1$$

$$\frac{|m_0^{\pm}|^2}{R_0^{\pm}} = 0 \text{ on } \{x \in \Omega : R_0^{\pm}(x) = 0\}.$$

and are taken in such a way that $\int_\Omega \frac{|m_0^{\pm}|^2}{R_0^{\pm}} < +\infty$, that the initial density fraction $R_0^{\pm}$ satisfies

$$R_0^{\pm} \in L^1(\Omega), \qquad \nabla\sqrt{R_0^{\pm}} \in L^2(\Omega)^3.$$

Then, there exists a global in time weak solution.

Multi-dimensional case

Sketch of proof and difficulties:

- Non conservative system and non-hyperbolic associated inviscid system....
- Strongly degenerate system.....
- Rewrite system using the $R^{\pm}$ variables and multiply momentum eqs by $R_{\pm}$.
- Combine energy estimate AND BD entropy estimate with implicit function.
- Difficulty to pass to the limit in the pressure term (product) $\Longrightarrow$ constraints on $\gamma_{\pm}$.

Remark: Up to now, nothing if constant viscosities......
P.–L. LIONS's (E. FEIREISL) framework??

Degenerate viscosities: It may give an extra estimate on the gradient of conserved quantities: Identified by D.B., B. DESJARDINS, C.K. LIN, *Comm. Partial Diff. Eqs* (2001) and generalized D.B., B. DESJARDINS, *CRAS*, section mécanique (2004), *J. Maths Pures Appl* (2007) (relation between viscosities).

Multi-dimensional case: Why a degenerate viscosity may help?

Simple example in one-dimensional case:

$$\partial_t \rho + \partial_x(\rho u) = 0$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \nu \partial_x(\rho \partial_x u) + \partial_x p(\rho) = 0$$

with p an increasing function.
Energy estimate reads:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|u|^2 + \pi(\rho) + \int_\Omega \rho|\partial_x u|^2 = 0$$

with $\pi(\rho)$ the potential associated to thepressure.
Differentiating the mass equation with respect to x, multiplying by ν and adding with the momentum eqs, we find

$$\partial_t(\rho\mathcal{V}) + \partial_x(\rho u \mathcal{V}) + \partial_x p(\rho) = 0$$

with $\mathcal{V} = u + \nu\partial_x \log\rho$. This gives the mathematical BD entropy equality:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\mathcal{V}|^2 + \pi(\rho) + \frac{\nu}{2}\int_\Omega p'(\rho)|\partial_x\sqrt{\rho}|^2 = 0$$

$\Longrightarrow$ information on $\sqrt{\rho}\partial_x \log\rho = 2\partial_x\sqrt{\rho}$ (using Energy and BD entropy).

Multi-dimensional case: Why a degenerate viscosity may help?

In multi-fluid setting, after some calculations taking care of the non-conservative term, it gives an extra information on $\nabla\sqrt{R}^{\pm}$ if initially it is the case.

Mathematical difficulties comes from the degenerate framework:
Multiplication by $R^{\pm}$ the momentum eqs.

Multi-dimensional case: constraint on $\gamma_\pm$?

We write

$$\alpha_n^\pm \nabla p_n = \frac{\alpha_n^\pm \gamma_+ \gamma_- p_n}{(\gamma_- \alpha_n^+ + \gamma_+ \alpha_n^-)\rho_n^- \rho_n^+} (\rho_n^- \nabla R_n^+ + \rho_n^+ \nabla R_n^-).$$

When $1 < \gamma^\pm < 6$, we get $\alpha_n^\pm \nabla p_n \in L_t^2 L_x^r$ with $r > 1$.

Since $R_n^\pm \in L_t^\infty H_x^1 \cap L_t^2 H_x^2$ then $\alpha_n^\pm R_n^\pm \nabla p_n \in L_{t,x}^p$ for some $p > 1$.

We can pass to the limit using strong convergence of $R_n^\pm$ in $C([0,T], H^s(\Omega))$ with $s < 1/2$ and strong convergence of $\alpha_n^\pm$ in L^p for all $p < +\infty$ recalling $\rho_n^\pm$ depends continuously on $\alpha_n^\pm$ and $R_n^\pm$.

The one-dimensional in space case

One-d case WITHOUT surface tension term: Paper [3]

In the one-dimensional in space case, results may be strongly improved:

- No need of surface tension and no multiplication by $R_\pm$ in momentum eqs.
- Range of coefficients $\gamma^\pm$ improved: $\gamma_\pm > 1$
  (same hypothesis than for mono-fluid system!!).
- Original construction of approximate systems (depend on $\gamma_\pm$).
  To control terms coming from non-conservative pressure terms.
- Possibility to prove vanishing of vacuum states in finite time.
- Local strong regularity, in time and space, of at least one velocity result.

$\Longrightarrow$ Generalization to viscous bifluid-model of results known for mono-fluid model.

See: H. Li, J. Li and Z. Xin, *Comm. Math. Phys.*, (2008) for mono-fluid.

The one-dimensional in space case

Why we study viscous models?
See papers by H. BRUCE STEWART and B.B. WENDROFF:
(JCP 1984, JMAA 1986, for some motivations).

Incompressible bifluid framework:
Note that B.L. KEYFITZ, M. SEVER and F. ZHANG use of regularized problem: Both strictly and weakly overcompressive singular shocks are limits of viscous structures (regularization not only in momentum equations!!).

Compressible bifluid framework:
The possibility to construct weak solutions : First step of similar studies than B.L. KEYFITZ, M. SEVER and F. ZHANG in the compressible setting.
Important remark: Here physical viscosities that means only in momentum eqs.

In the mono-fluid setting, see for instance recent works by:
– C.Q. CHEN, M. PEREPELITSA with constant viscosity
– F. HUANG, R. PAN, T. WANG, Y. WANG, Z. ZHAI with degenerate viscosity.

The one-dimensional in space case

Using an original approximate system, we get

Theorem 1: Let $\gamma_\pm > 1$ and adequate initial data (energy spaces). Then for any $T > 0$, there exists a global weak solution $(\alpha_\pm, \rho_\pm, u_\pm)$ to the two-phase system in a usual sense.

Theorem 2: Assume that $\gamma_\pm > 1$. Let $(\alpha_\pm, \rho_\pm, u_\pm)$ be any global weak solution to the two-phase system. Then, there exist some time $T_0 > 0$ (depending on initial data) and a constant $\underline{\rho}$ so that

$$\inf_{x \in \overline{\Omega}} \rho_\pm(x,t) \geq \underline{\rho} > 0, \quad t \geq T_0.$$

Corollary: Let $(t_0, x_0) \in (T_0, T) \times \Omega$, there exist a neighborhood $\mathcal{N}_{t_0,x_0}$ of (t_0, x_0) such that at least one of the solutions $(\alpha_\pm, \rho_\pm, u_\pm)$ becomes strong in this neighborhood $\mathcal{N}_{t_0,x_0}$.

Invariant regions

In paper [2]:
Theorem. For smooth solutions to inviscid equation, the region $\alpha^- \geq 0$ is invariant under the evolution if and only if the fluid + is compressible.

Use theory of invariant regions of CHUEH, CONWAY, SMOLLER (see for instance J. SMOLLER, Springer-Verlag (1983)).

Gives a clear answer to a well known problem in the two-fluid flow numerical simulations. In many numerical papers, the authors mention that their schemes lead to negative values for α^- and therefore made use of the clipping techniques which consists in modifying (increasing) the variable α^- when it begins to be small. This procedure destroys mass conservation and very often leads to nonphysical results. Our result shows that the observed discrepancy does not follow from the numerical method but is already included in the physical model at the continuous level. Although analytically more simple, Equations of State

$$p = p^-(\rho^-), \qquad \rho^+ = \rho_0^+$$

should be rejected.

Some references on other related models.

– F. Bouchut, Y. Brenier, J. Cortes, J.F. Ripoll, A hierarchy of models for two phase flows, *J. Nonlinear Sci.*, 10, 637–660, (2000).

– P.–L. Lions, N. Masmoudi, On a free boundary barotropic model, *Ann. Inst. H. Poincaré Anal. non linéaire*, 16 (1999), 373–410.

– B. Maury, A. Roudneff-Chupin, F. Santambrogio, A macroscopic Crowd Motion Model of the gradient-flow type, *Math. Models and Methods in Applied Sciences,* Vol. 20, No. 10 (2010) 1787-1821.

– L. Navoret, PhD Thesis, Toulouse (2010).

– L. Desvillettes, Some aspects of the modeling at different scales of multiphase flows. *Computer Methods in Applied Mechanics and Engineering.* Volume 199, Issues 21-22, 1 April 2010, Pages 1265-1267

Weak limit and multi-fluid system justification

YOUNG MEASURE AND WEAK SOLUTIONS

Collaboration with X. HUANG: paper [4]

The model

Let us consider the following barotropic compressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu + \lambda)\nabla(\mathrm{div} u) + \nabla P(\rho) = 0, \end{cases}$$

where ρ, u, P denote the density, velocity and pressure respectively. The pressure law is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \quad \gamma > 1),$$

μ and λ are the shear viscosity and the bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0.$$

Remarks and references

Question: Justification of multi-fluid system from mono-fluid one with low regularity. More precisely is there exist weak sequences corresponding to concentrating density which converge to the strong solution of a viscous multi-fluid system?
No oscillations-concentrations in velocity – concentrations in density.

As mentioned in LIONS's book (Remarks 5.8 and 5.9), weak limits of a sequence of solutions of compressible Navier–Stokes system with highly-oscillating density are not in general solutions of the compressible Navier-Stokes system.

References:

- D. SERRE (*Physica D*, (1991)) focusing on the one-dimensional case and providing a formal calculus for the multi-dimensional problem.
- To capture the effect of oscillations, using the renormalization procedure related to the mass equation, M. HILLAIRET (*J. Math Fluid Mech*, 2007) (following the formal calculus in D. SERRE) introduced Young measure as in the work by R. DI PERNA, A. MAJDA to describe a "homogenized system" satisfied in the limit.

Open and known results

In M. HILLAIRET's paper, we still do not know whether the obtained solution of the multi-fluid system are weak limit to finite-energy weak solutions of compressible Navier-Stokes equations.

Two assumptions, by M. HILLAIRET (2007), have been done to formally deduce the multi-fluid system from the weak limit system:

- If the initial young measures are linear combination of m Dirac masses then it is the case for all time.
- The concentration points remains distincts (a kind of stratification): $\rho_i(t,x) \neq \rho_j(t,x)$ for all $i,j = 1,\dots,m$ with $i \neq j$.

Remark: Existence and uniqueness of local strong solution of the viscous multi-fluid system has been established by M. HILLAIRET far from vacuum.

Weak-compactness of the effective flux

Important Lemma (due to P.–L. LIONS):

Given $b \in \mathcal{C}^1(\mathbb{R}^+)$ such that $b'(z) = 0$ for z sufficiently large with compact support, let $b((\rho_n)$ converge to $\overline{b}$ in $L^\infty(Q_T)$ endowed with its weak star topology. We have:

$$\lim_{n\to+\infty} \int_0^T \int_\Omega [(p(\rho_n) - (\lambda + 2\mu)\mathrm{div}(u_n))b(\rho_n))\phi(t,x)\, dxdt$$

$$= \int_0^T \int_\Omega [(q - (\lambda + 2\mu)\mathrm{div}(u))\overline{b})\phi(t,x)\, dxdt$$

for all $\phi \in \mathcal{D}(Q_T)$.

The limit system

Assume (ρ_n, u_n) be finite-energy weak solutions to NS eqs and

$$\rho_n \rightharpoonup \rho \quad \text{in } L^\infty(0,T;L^\gamma(\mathbb{T}^3)) \star \text{ weak}, \qquad u_n \rightharpoonup u \quad \text{in } L^2(0,T;H^1(\mathbb{T}^3)),$$
$$\rho \in L^\infty(0,T;L^\gamma(\mathbb{T}^3)), \qquad u \in L^2(0,T;H^1(\mathbb{T}^3)).$$

Then there exists a measurable family of probability measures, we denote $(\nu_{(t,x)})$ such that

1. We have

$$< \nu, \mathrm{Id} >= \rho \text{ and } < \nu, p >= q, \text{ in a sense precised in next slide.}$$

2. For all $b \in C(R^+)$, smooth, with compact support,

$$(< \nu, b >)_t + div(< \nu, b > u) + < \nu, (\mathrm{Id}\, b' - b) > div(u)$$
$$= \frac{< \nu, (\mathrm{Id}\, b' - b) > q - < \nu, (\mathrm{Id}\, b' - b) p >}{\lambda + 2\mu}.$$

3. Finally,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u - (\mu+\lambda)\nabla(\mathrm{div} u) + \nabla q = 0, \end{cases}$$

Definition

Actually, as in E. FEIREISL, we define

$$< \nu, \mathrm{Id} >= \lim_{k\to\infty} < \nu, T_k \circ \mathrm{Id} >, < \nu, p >= \lim_{k\to\infty} < \nu, T_k \circ p >,$$

where $T_k(z) = min\{z, k\}$ is a family of truncation functions. However, if ρ_n is uniformly bounded in both space and time, then 1) in previous slide holds in a classical sense. This will be the case since we will consider weak sequences with uniformly bounded density.

The multi-fluid model studied by M. Hillairet – Strong solution and uniqueness

If Young measures are assumed to be convex combinations of Dirac measures, *i.e*:

$$\nu_{(t,x)} = \sum_{i=1}^{m} \alpha_{i1}(t,x)\delta_{\rho_{i1}(t,x)}, \qquad \forall (t,x) \in (0,T) \times \Omega.$$

Using such hypothesis, the homogeneized compressible Navier-Stokes system reads:

$$\begin{cases} (\alpha_{i1})_t + u_1 \cdot \nabla \alpha_{i1} = f_{\alpha_{i1}}, \qquad i = 1,\ldots,m \\ \alpha_{i1}\left((\rho_{i1})_t + div(\rho_{i1}u_1)\right) = \alpha_{i1} f_{\rho_{i1}}, \qquad i = 1,\ldots,m \\ \partial_t \rho + \mathrm{div}(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \mathrm{div}(\rho u_1 \otimes u_1) + \nabla q = \mu \triangle u_1 + (\mu+\lambda)\nabla(\mathrm{div} u_1), \end{cases}$$

$$f_{\alpha_{i1}} = \frac{\alpha_{i1}(a\rho_{i1}^{\gamma} - q)}{\lambda + 2\mu}, \quad f_{\rho_{i1}} = \frac{\rho_{i1}(q - a\rho_{i1}^{\gamma})}{\lambda + 2\mu}$$

$$0 \le \alpha_{i1}, \qquad \sum_{i=1}^{m} \alpha_{i1} = 1$$

$$\rho = \sum_{i=1}^{m} \alpha_{i1}\rho_{i1}, \qquad q = a\sum_{i=1}^{m} \alpha_{i1}\rho_{i1}^{\gamma},$$

where ρ_{i1}, u_1 denotes the density, velocity respectively and α_{i1} is the coefficients.

The multi-fluid model studied by M. Hillairet – Strong solution and uniqueness

Far from vacuum, kill the red terms:

$$\begin{cases} (\alpha_{i1})_t + u_1 \cdot \nabla \alpha_{i1} = f_{\alpha_{i1}}, & i = 1, \dots, m \\ (\rho_{i1})_t + div(\rho_{i1} u_1) = f_{\rho_{i1}}, & i = 1, \dots, m \\ \partial_t \rho + \mathrm{div}(\rho u_1) = 0, & \\ \partial_t(\rho u_1) + \mathrm{div}(\rho u_1 \otimes u_1) + \nabla q = \mu \triangle u_1 + (\mu + \lambda)\nabla(\mathrm{div} u_1), & \end{cases}$$

$$f_{\alpha_{i1}} = \frac{\alpha_{i1}(a\rho_{i1}^\gamma - q)}{\lambda + 2\mu}, \quad f_{\rho_{i1}} = \frac{\rho_{i1}(q - a\rho_{i1}^\gamma)}{\lambda + 2\mu}$$

$$0 \le \alpha_{i1}, \qquad \sum_{i=1}^m \alpha_{i1} = 1$$

$$\rho = \sum_{i=1}^m \alpha_{i1}\rho_{i1}, \qquad q = a\sum_{i=1}^m \alpha_{i1}\rho_{i1}^\gamma,$$

Sketch of proof

First Result: Solution of the multi-fluid system obtained as weak limit to finite-energy weak solutions of compressible Navier-Stokes equations: Stratification assumption seems to be necessary (to be improved, better defect measures? explicit initial data and use of HOFF-SANTOS's paper (in progress with M. HILLAIRET and X. HUANG)). Model is linked to the BAER-NUNZIATO model.

Weak sequence related to the existence result by B. DESJARDINS. Given initial data

$$\rho_0 \in L^\infty(\mathbb{T}^3), \qquad \rho_0 \geq 0, \qquad u_0 \in H^1(\mathbb{T}^3).$$

There exists $T_0 \in (0,\infty)$ and a weak solution (ρ, u) to the compressible Naiver-Stokes equations with $(\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0)$. For all $0 < T < T_0$,

$$\rho \in L^\infty((0,T)\times\mathbb{T}^3) \cap C([0,T];L^q(\mathbb{T}^3)), \qquad \text{for all q} \in [1,\infty)$$
$$\nabla u \in L^\infty(0,T;(L^2(\mathbb{T}^3))^9).$$
$$\sqrt{\rho}\partial_t u \in (L^2((0,T)\times\mathbb{T}^3))^3, \quad Pu \in L^2(0,T;H^2(\mathbb{T}^3)),$$
$$G = (\lambda+2\mu)\mathrm{div} u - p(\rho) \in L^2((0,T);H^1(\mathbb{T}^3)),$$

where P denotes the projection on the space of divergence-free vector fields.

Remark: The L^∞ bound on ρ_0 is also assumed in D. SERRE (one-dimensional case). This is also required in weak-strong uniqueness by B. DESJARDINS, P. GERMAIN. We will also consider $\rho_0 \geq C > 0$ as in D. SERRE's paper.

Sketch of proof

Prove that weak sequence based on B. DESJARDINS's lemma has extra-regularity. Namely,

$$\mathrm{div} u \in L^1(0,T;L^\infty(\mathbb{T}^3)).$$

B. Desjardins's weak sequence satisfies:

$$\sup_{0<t\leq T} \|\nabla u\|_2 + \int_0^T \int_{\mathbb{T}^3} \rho |\dot{u}|^2 \leq C, \tag{2}$$

with $\dot{u}$ the total time derivative. Write now extra estimates following D. HOFF's Ideas.

First step :

$$\sup_{0<t\leq T} (\|G\|_2 + \|\omega\|_2) \leq C,$$

$$\|\nabla G\|_6 + \|\nabla \omega\|_6 \leq C(\|\rho^{\frac{1}{2}} \dot{u}\|_2 + \|\nabla \dot{u}\|_2),$$

$$G = (2\mu + \lambda) div u - P, \quad \omega = \nabla \times u$$

where G and ω denote the effective viscous flux and vorticity, respectively.

Sketch of proof

Second step : Deduce the following estimate

$$\sup_{0<t\le T}\int_{\mathbb{T}^3}\sigma\rho|\dot u|^2+\int_0^T\int_{\mathbb{T}^3}\sigma|\nabla\dot u|^2\le C$$

where $\sigma=\min(1,t)$.

Use this estimate, the ones in previous slide and the expression of G, to deduce the result that means:

$$\mathrm{div}u\in L^1(0,T;L^\infty(\mathbb{T}^3)).$$

Remark: In Recent HOFF-SANTOS's paper, $\rho_0\in L^\infty$ and $u_0\in H^s$ + smallness assumption and relation betweeg λ and μ is considered. Propagation of singularity result when $s>1/2$: such regularity implies also $\mathrm{div}u\in L^1L^\infty$. If local existence with same kind of estimates thus OK. No direct interpolation easily.

Sketch of proof

Third step : Introduce an adequate defect measure (discussions and properties on such kind of defect measures in M. HILLAIRET's PhD Thesis) to prove that young measures are in fact linear combination of dirac measures.

$$M_\alpha[\Theta, \nu] \text{ the determinant of the } (m+1) \times (m+1) \text{ matrix } \widetilde{M}_\alpha[\Theta, \nu]$$

with elements

$$\left(\widetilde{M}_\alpha[\Theta, \nu]\right)_{i,j} = < \nu, \rho^{(\theta_i+\theta_j)\alpha} >$$

with $\Theta = (\theta_0, \cdots, \theta_m) \in \mathbb{N}^{m+1}$ a weight vector composed with two by two distinct coefficients and α a coefficient.

In the sequel: Choice: $\Theta = (0, \cdots, m)$ and α chosen later on !!

Sketch of proof

Using that $\rho \in L^\infty$, we write, using renormalization procedure:

$$\partial_t(M_\alpha[\Theta,\nu]) + \mathrm{div}(M_\alpha[\Theta,\nu]) + \kappa M_\alpha[\Theta,\nu]\mathrm{div}u + \frac{\mathcal{Q}(\nu)}{\lambda+2\mu} = 0$$

with

$$\kappa = 2\alpha \sum_{i=0}^{m} \theta_i - 1,$$

and

$$\mathcal{Q}(\nu) = \sum_{i,j=0}^{m} (2\alpha\theta_i - 1)\big(\overline{\rho^{(\theta_i+\theta_j)\alpha+\gamma}} - \overline{\rho^{(\theta_i+\theta_j)\alpha}}\,\overline{\rho^\gamma}\big) M_\alpha^{(i,j)}[\Theta,\nu].$$

Note that there exists α , for instance $\alpha = \gamma/(E[m\gamma]+1)$, such that $\mathcal{Q}(\nu) \geq 0$, thus integrating in space, we get

$$\frac{d}{dt}\Big[\int_{\mathbb{T}^3} M_\alpha[\Theta,\nu]\Big] \leq |\kappa||\mathrm{div}u|_\infty \int_{\mathbb{T}^3} M_\alpha[\Theta,\nu].$$

Using that $M_\alpha[\Theta,\nu_0] = 0$ and integrating in time, we get that $M_\alpha[\Theta,\nu] = 0$.

Sketch of proof

This implies that, using characterization given in M. HILLAIRET's PhD thesis

$$\nu = \sum_{i=1}^{m} \alpha_i \delta_{\rho_i}.$$

Assumption: A kind of stratification assumption, namely denoting level sets interval

$$L(f) = [\inf_{z\in Q_T} f(z), \sup_{z\in Q_T} f(z)]$$

we assume

$$\rho_i \in L^\infty, \qquad L(\rho_i) \cap L(\rho_j) = \emptyset \text{ for } i,j = 1,\cdots,m \text{ with } i \neq j$$

then weak limit satisfies a multi-fluid system using expression of ν and then adequate b compactly supported in the limit equation (2).

Note that $\rho \in L^\infty \implies \alpha_i \geq c > 0$ if initially. We then find the multi-fluid system written previously mixing the equation on α_i and $\alpha_i \rho_i$.

Sketch of proof

Fourth step: Use a weak-strong procedure to prove that the strong solution built by M. HILLAIRET corresponds to the weak limit. This use that $\text{div} u \in L^1 L^\infty$ and $\rho \in L^\infty$. If initially the case, we prove that $(\alpha_i, \rho_i, u) = (\alpha_{i1}, \rho_{i1}, u_1)$.

Remark: In fact the strong solution has only to satisfy

$$\alpha_1 \in L^\infty, \rho_1 \in L^\infty, q_1 \in L^\infty, \nabla\alpha_1 \in L^\infty L^3, \nabla\rho_1 \in L^\infty L^3, \nabla u_1 \in L^1 L^\infty, \dot{u}_1 \in L^2 L^3.$$

A kind of generalization of B. DESJARDINS's (1997) and P. GERMAIN's (2009) results.

Sketch of proof

Important remarks:

- Assumptions on ρ_0 same than D. SERRE (1991) in the one-dimensional case.
- Note that Young measures characterization is needed looking at three moments since $(\theta_0, \theta_1) = (0, 1)$ to prove that $\nu_{(t,x)} = \delta_{(\rho(t,x), m(t,x))}$ in vanishing viscosity for compressible Euler flow, see G.Q. CHEN, M. PEREPELITSA, (2009) (Physical viscosity limit of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with finite-energy initial data, 1D case linked to adequate energy estimates and reduction of measure-valued solutions with unbounded support).
- Y. BRENIER, C. DE LELLIS, L. SZÉKELYHIDI JR. (2010): Weak strong uniqueness for measure valued solutions. Argument based on admissible solution of Euler such that $\int_0^T \|D(u)\|_\infty < +\infty$. Linked to the Blow-up criteria for Euler system (see G. PONCE (1985)). For compressible Navier-Stokes equations: Blow-up criteria linked to $\int_0^T \|\mathrm{div} u\|_\infty < +\infty$ or $\int_0^T \|\rho\|_\infty < +\infty$ (see recent papers by X.HUANG, J. LI and Z.P. XIN, B. HASPOT..).