Finite time blow up in the Nordheim equation for bosons

M. Escobedo

Universidad del País Vasco

& BCAM, Bilbao.

Joint work with J. J. L. Velázquez.

Nordheim Equation for bosons

Homogeneous isotropic gas: the density function of particles of energy ϵ at time t, is independent of the space variable x: $f \equiv f(t, \epsilon)$.

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t, \epsilon_1) &= \int_{D(\epsilon_1)} w\left(\epsilon_1, \epsilon_3, \epsilon_4\right) q(f) d\epsilon_3 d\epsilon_4 \\
q\left(f\right) &= f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \\
f_i &= f(\epsilon_i), \;\; i = 1, 2, 3, 4, \\
\epsilon_2 &= \epsilon_3 + \epsilon_4 - \epsilon_1 \\
D\left(\epsilon_1\right) &= \{(\epsilon_3, \epsilon_4) : \epsilon_3 > 0, \epsilon_4 > 0, \epsilon_3 + \epsilon_4 \geq \epsilon_1 > 0\} \\
w\left(\epsilon_1, \epsilon_3, \epsilon_4\right) &= \frac{\min\left(\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right)}{\sqrt{\epsilon_1}}.
\end{aligned}
$$

L. W. Nordheim, 1928.

Why?

The question is: do the solutions of that equation blow up in finite time or not?

Mathematical problem: blow up for a Boltzmann type equation.

Motivated by the Bose Einstein condensation.

Related with formation of a Dirac measure at zero energy.

Gas of Bosons at Equilibrium I

- Density of non relativistic bosons in non interacting isotropic gas at equilibrium:

$$F(p) = \frac{1}{e^{\beta(|p|^2-\mu)}-1}, \quad \beta > 0, \ \mu \leq 0.$$

- $\beta = 1/T$, where T is the temperature of the gas, p momentum of the particles.
- If we define: $M(F) = \int_{\mathbb{R}^3} F(p)dp$, $E(F) = \int_{\mathbb{R}^3} |p|^2 F(p)dp$, then:

$$M(F) \leq C_0\, E(F)^{\frac{3}{5}}, \qquad C_0 = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}} \left(\frac{4\pi}{3}\right)^{\frac{3}{5}}.$$

- If $M \leq C_0 E^{\frac{3}{5}}$, there is F such that $M(F) = M$ and $E(F) = E$.

Remark: $\quad M \leq C_0\, E^{\frac{3}{5}} \Longleftrightarrow T \geq T_{cr}$

where $T_{cr} = T_{cr}(M)$ is a critical temperature.

Equilibrium II

Another family of equilibria (super critical):

$$F(p) = \frac{1}{e^{\beta |p|^2} - 1} + \alpha\, \delta(p), \;\; \alpha > 0$$

are such that: $M(F) > C_0\, E(F)^{\frac{3}{5}} \quad (\Longleftrightarrow T < T_{cr}.)$

The presence of a Dirac measure is the precise formulation (in this setting) of the presence of a B-E condensate (S. N. Bose 1924, A. Einstein 1925):

Below a critical temperature a macroscopical fraction of particles appears at the minimum energy level of the system.

All the particles in the condensate are described by the same wave function.

Predicted in '24-'25 and first observed by E. Cornell, C. Wieman & al. in 1995.

Non equilibrium.

For a spatially homogeneous, isotropic, weakly interacting gas:

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t,\epsilon_1) &= Q(f)(t,\epsilon_1) \equiv \int_{D(\epsilon_1)} w\left(\epsilon_1,\epsilon_3,\epsilon_4\right) q(f) d\epsilon_3 d\epsilon_4 \\
q\left(f\right) &= f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \\
\epsilon &= |p|^2, \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1 \\
D\left(\epsilon_1\right) &= \left\{(\epsilon_3,\epsilon_4) : \epsilon_3 > 0, \epsilon_4 > 0, \epsilon_3 + \epsilon_4 \geq \epsilon_1 > 0\right\} \\
w\left(\epsilon_1,\epsilon_3,\epsilon_4\right) &= \frac{\min\left(\sqrt{\epsilon_1},\sqrt{\epsilon_2},\sqrt{\epsilon_3},\sqrt{\epsilon_4}\right)}{\sqrt{\epsilon_1}}
\end{aligned}
$$

L. W. Nordheim, Proc. R. Soc. Lond. A, 1928.

For all the equilibria F we have: $Q(F) = 0$.

(In the Boltzmann equation for classical particles: $q\left(f\right) = f_3 f_4 - f\, f_2$)

Condensation and evolution

Since a BE condensation is observed in the experiments... one expects a Dirac mass to appear in finite time in the density function of the particles that solves the Nordheim equation.

In E. Levich & al. 1977: first attempt to describe delta formation in Nordheim's equation.

Nordheim evolution equation makes sense for radial distributions of the form $f(t) + n(t)\delta$ (D. V. Semikoz & al. 1995), and even for general radial measures (X. Lu '05, see below).

First relation between B-E condensation and blow up for the Nordheim equation: B. S. Svistunov in J. Moscow Phys. Soc. 1991.

No (regular) equilibrium $\to$ blow up in finite time ?? (H. Brezis et al. Adv. Diff. Eq. 1996 for a non linear heat equation).

Existence of Mild Solutions

Theorem. Suppose that $f_0 \in L^{\infty}\left(\mathbb{R}^+; (1+\epsilon)^{\gamma}\right)$ with $\gamma > 3$. There exists $T > 0$, depending only on $\|f_0(\cdot)\|_{L^{\infty}\left(\mathbb{R}^+;(1+\epsilon)^{\gamma}\right)}$, and there exists a unique mild solution, $f \in L^{\infty}_{loc}\left([0,T); L^{\infty}\left(\mathbb{R}^+; (1+\epsilon)^{\gamma}\right)\right)$ in the sense of the previous definition.
The solution f satisfies (mass & energy conservation):

$$\int_0^{\infty} f_0(\epsilon)\, \epsilon^r d\epsilon = \int_0^{\infty} f(t,\epsilon)\, \epsilon^r d\epsilon\,, \quad t \in (0,T)\ , \quad r = \frac{1}{2}, \frac{3}{2}.$$

The function f is in $W^{1,\infty}\left((0,T); L^{\infty}\left(\mathbb{R}^+\right)\right)$ and it satisfies the equation for $a.e.\ \epsilon \in \mathbb{R}^+$ and for any $t \in (0, T_{\max})$. Moreover, f can be extended as a mild solution to a maximal time interval $(0, T_{\max})$ with $T_{\max} \leq \infty$. If $T_{\max} < \infty$ we have:

$$\lim_{t \to T_{\max}^-} \|f(t,\cdot)\|_{L^{\infty}(\mathbb{R}^+)} = \infty.$$

Blow up Theorem

Theorem. Suppose that $f_0 \in L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)$ with $\gamma > 3$. Define:

$$M = 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon}\,d\epsilon, \quad E = 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon^3}\,d\epsilon$$

Let $f \in L^\infty_{loc}\left([0,T_{\max});L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)\right)$ be the mild solution of the radial Nordheim equation with initial data f_0, where $T_{\max}$ is the maximal existence time. Suppose that:

$$M > C_0\, E^{\frac{3}{5}}.$$

Then:

$$T_{\max} < \infty,$$

and

$$\lim_{t \to T_{\max}^-} ||f(t)||_{L^\infty} = +\infty.$$

Previous references:

1.- E. Levich & V. Yakhot Study a more general Boltzmann type equation:

$$\frac{\partial f}{\partial t}(t, \epsilon) = Q(f)(t, \epsilon) + \widetilde{Q}(f)(t, \epsilon)$$

$\widetilde{Q}$ describes collisions of bosons with a heat bath of fermions.

Collisions with the heat bath form a peak for f at small values of ε.

1.-A (Phys. Rev. B.'77): $Q(f) + \widetilde{Q}(f)$ drive f close to an equilibrium of $Q(f)$. Then $\widetilde{Q}(f)$ is dominant $\longrightarrow$ self similar Delta formation in infinite time.

1.-B (J. Low Temp.'77): When $f >> 1$: neglect $\widetilde{Q}$ (quadratic) in front of Q (cubic). Consider a simplified version of the Nordheim equation. Explicitely solvable. $\longrightarrow$ self similar Delta formation in finite time: $\sqrt{\varepsilon} f(t, \varepsilon) \rightharpoonup C\delta(\varepsilon)$, as $t \to t_0$.

2.- D. V. Semikoz & I. I. Tkachev (PRL '95) On the ground of numerics and previous work by B. S. Svistunov (J. Moscow Phys. Soc. '91) , propose that the Nordheim equation has solutions of the self similar form:

$$f(t,\epsilon) = C(t_c - t)^{-\alpha}\phi\left(\frac{\epsilon}{(t_c - t)^{\beta}}\right)$$

for some constants $t_c > 0$, C, α, β and ϕ a bounded integrable function satisfying:

$$-\frac{\alpha}{\beta}\phi + x\,\phi' = \int\int_{D(x)} q(\phi)w(x, x_2, x_3, x_4)\,dx_3 dx_4$$

3.- Similar in R. Lacaze, P. Lallemand, Y.Pomeau & S. Rica (Physica D '01).

• Several mathematical results on the existence of solutions by X. Lu ('02, '05), A. Nouri '07, L. Arkeryd & al. '12. A general presentation by H. Spohn '10.

Proof of the blow up result

Two different parts:

• In the first we prove a local criteria for blow up.

• In the second we prove that every "super critical" solution satisfies the local criterium at some finite time.

The first part uses mainly a monotonicity argument and some measure theory to describe the "local" properties of the solutions.

The second part uses more functional analysis arguments. In particular the entropy and dissipation of entropy.

The local criterium for blow up

Theorem. Let $M > 0,\ E > 0,\ \nu > 0,\ \gamma > 3.$
There exist $\rho = \rho\left(M, E, \nu\right) > 0,\ K^{*} = K^{*}\left(M, E, \nu\right) > 0$ and $\theta_{*} > 0$,
(θ_{*} independent on $M,\ E,\ \nu$), such that for any $f_0 \in L^{\infty}\left(\mathbb{R}^{+}; \left(1+\epsilon\right)^{\gamma}\right)$ with $M(f_0) = M$, $E(f_0) = E$ satisfying:

$$(i) \qquad \int_0^R f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon \geq \nu R^{\frac{3}{2}} \ \text{ for } 0 < R \leq \rho,$$

$$(ii) \qquad \int_0^{\rho} f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon \geq K^{*}\rho^{\theta_{*}},$$

the unique mild solution $f \in L^{\infty}_{loc}\left(\left[0, T_{\max}\right); L^{\infty}\left(\mathbb{R}^{+}; \left(1+\epsilon\right)^{\gamma}\right)\right)$ with initial data f_0 and maximal existence time $T_{\max}$ satisfies:

$$T_{\max} < +\infty, \ \text{ and } \ \lim \sup_{t \to T_{\max}^{-}} \left\| f\left(\cdot, t\right)\right\|_{L^{\infty}(\mathbb{R}^{+})} = \infty.$$

Proof of the local criterium

The proof has several steps:

① Monotonicity of the cubic part of the collision integral. For any $f \in L^1(\mathbb{R}^+)$:

$$\cdot \int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_3 d\epsilon_4 \, w(\epsilon_1, \epsilon_3, \epsilon_4) \;\; q_3(f)(\epsilon_1) \sqrt{\epsilon_1} \, \varphi(\epsilon_1) =$$

$$= \int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 \, f_1 \, f_2 \, f_3 \mathcal{G}_\varphi(\epsilon_1 \, \epsilon_2 \, \epsilon_3),$$

where: $q_3(f) = f_3 f_4 (f + f_2) - f \, f_2 (f_3 + f_4)$

$$\cdot \;\; \varphi \;\; \text{convex} \Longrightarrow \mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) \geq 0; \quad \varphi \;\; \text{concave} \Longrightarrow \mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) \leq 0$$

(Proved independently by X. Lu)

We deduce that, if $\boxed{g = 4\pi\sqrt{2\epsilon}\, f}$ and f solves the Nordheim equation, then:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^+} g(\epsilon_1)\varphi(\varepsilon_1)d\epsilon_1 &= \int_{(\mathbb{R}_+)^3} \frac{g_1g_2g_3}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)d\epsilon_1 d\epsilon_2 d\epsilon_3 \\ &+ \frac{1}{2}\int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 w(\epsilon_1,\epsilon_2,\epsilon_3)\frac{g_1g_2}{\sqrt{\epsilon_2}}\left(\varphi_3+\varphi_4-\varphi_1-\varphi_2\right)\end{aligned}$$

We seem to be in good shape but:

$\mathcal{G}_\varphi\left(\epsilon_1,\epsilon_2,\epsilon_3\right)$ vanishes along the diagonal $\left\{(\epsilon_1,\epsilon_2,\epsilon_3)\in\left(\mathbb{R}^+\right)^3 : \epsilon_1=\epsilon_2=\epsilon_3\right\}$.

Very vague intuition:

· If g is "far from a Dirac measure", then $(\epsilon_1,\epsilon_2,\epsilon_3)$ is far from the diagonal and $\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)$ does not vanish.

· The quadratic term prevents g to get close to a Dirac measure.

② Technical Lemma.

Suppose that $b > 1$ and define, for all $k = 1, 2, \ldots$:

$$\mathcal{I}_k\left(b\right) = b^{-k}\left(b^{-1}, 1\right], \quad \mathcal{I}_k^{(E)} = \mathcal{I}_{k-1}\left(b\right) \cup \mathcal{I}_k\left(b\right) \cup \mathcal{I}_{k+1}\left(b\right)$$

$$\mathcal{P}_b = \left\{ A \subset [0,1] : A = \bigcup_j \mathcal{I}_{k_j}\left(b\right) \text{ for some set of indexes } \{k_j\} \subset \{1, 2, ...\} \right\}.$$

$$\text{If} \quad A = \cup_{j=1}^{\infty} \mathcal{I}_{k_j}\left(b\right) \quad \text{define:} \quad A^{(E)} = \bigcup_{j=1}^{\infty} \mathcal{I}_{k_j}^{(E)}\left(b\right).$$

Given $0 < \delta < \frac{2}{3}$, define $\eta = \min\left\{\left(\frac{1}{3} - \frac{\delta}{2}\right), \frac{\delta}{6}\right\}$.

For any function $g \in L^{\infty}(0,1)$ at least one of the following properties is satisfied:

(i) There exists an interval $\mathcal{I}_k\left(b\right)$ such that: $\displaystyle\int_{\mathcal{I}_k^{(E)}(b)} g d\epsilon \geq (1-\delta) \int_0^1 g d\epsilon$

(ii) There exists two sets $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{P}_b$ such that $\mathcal{U}_2 \cap \mathcal{U}_1^{(E)} = \varnothing$ and:

$$\min \left\{ \int_{\mathcal{U}_1} g d\epsilon, \int_{\mathcal{U}_2} g d\epsilon \right\} \geq \eta \int_0^1 g d\epsilon.$$

In that case the set $\mathcal{U}_1$ can be written in the form: $\mathcal{U}_1 = \bigcup_{j=1}^{L} \mathcal{I}_{k_j}\left(b\right),$ for some set of integers $\{k_j\} \subset \{1, 2, 3, ...\}$ and some finite L, and we also have:

$$\mathcal{I}_{k_m}\left(b\right) \cap \left(\cup_{j=1}^{m-1} \mathcal{I}_{k_j}^{(E)}\left(b\right) \right) = \varnothing \quad , \quad m = 2, 3, ...L$$

$$\sum_{j=1}^{L} \left(\int_{\mathcal{I}_{k_j}(b)} g d\epsilon \right)^2 \leq \left(\int_{\mathcal{I}_{k_1}(b)} g d\epsilon \right)^2 + \sum_{j=2}^{L} \int_{\mathcal{I}_{k_1}(b)} g d\epsilon \int_{\mathcal{I}_{k_j}(b)} g d\epsilon,$$

$$\int_{\mathcal{I}_{k_1}(b)} g d\epsilon < (1-\delta) \int_0^1 g d\epsilon.$$

Last part of the proof

Using the monotonicity property:

$$\int_0^{\rho/2} g_0(\epsilon)d\epsilon \geq m_0 \Longrightarrow \exists T_0 > 0; \int_0^{\rho} g(t,(\epsilon)d\epsilon \geq \frac{m_0}{4} \; \forall t \in [0,T_0].$$

If we define:

$$B_\ell = \left\{ t \in [0,T_0] : \int_{[0,R_\ell]} g\left(\epsilon,t\right)d\epsilon \geq \left(R_\ell\right)^{\theta_1} \right\}, \; R_\ell = 2^{-\ell}, \;\; \ell = 0,1,2,...$$

then, for L and $\theta_1 > 0$ such that $2^{-\theta_1 L} \leq m_0/4$: $\boxed{B_L = [0,T_0]}$

We look now at the times where the function g satisfies (i) and those where it satisfies (ii).

$\mathcal{I}_1^{(E)}(b, R)$ in red; $\mathcal{I}_5^{(E)}(b, R)$ in blue

For $\theta_2 > 0$ such that $1 - 2\theta_1 - \theta_2 > 0$, define:

$$b_\ell = 1 + (R_\ell)^{\theta_2}, \ \ell = 0, 1, 2, ..., \text{ and the sets}$$

$$A_{n,\ell} = \left\{ t \in [0, T_0] : \text{ such that } \int_{\mathcal{I}_n^{(E)}(b_\ell, R_\ell)} g(t, \epsilon)\, d\epsilon \geq (R_{\ell+1})^{\theta_1} \right\}, \ n = 1, 2, 3, ...$$

$$\mathcal{A}_\ell = \bigcup_{n=1}^{\left[\frac{\log(2)}{\log(b_\ell)}\right]+1} A_{n,\ell} \quad \text{where: } \mathcal{I}_k^{(E)}(b, R) = \left(\frac{R}{b^{k+2}}, \frac{R}{b^{k-1}} \right].$$

We now use: $B_L = \bigcup_{\ell=L}^{\infty} B_\ell \setminus B_{\ell+1}$ and so:

$$T_0 = |B_L| \leq \sum_{\ell=L}^{\infty} |B_\ell \setminus B_{\ell+1}| \leq \sum_{\ell=L}^{\infty} \left(|(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell| + |\mathcal{A}_\ell| \right).$$

The contradiction comes from the estimates of the right hand side.

In the set $(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell$ the function g does not satisfy property (i), so it must satisfy property (ii).

From the monotonicity property (point ①) we deduce that, if $T_{\max} \geq T_0$, there exists $\theta_0 > 0$ such that, if $0 < \min\{\theta_1, \theta_2\} < \theta_0$, we have:

$$|(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell| \leq K \left(1 + T_{\max}\right) R_\ell^{1-3\theta_1-4\theta_2}, \quad 1 - 3\theta_1 - 4\theta_2 > 0$$

for some $K = K\left(M, \theta_1\right)$ and for any $\ell = 0, 1, 2, \ldots$

On the other hand, there exists $\rho \in (0,1)$ such that, if $\ell > \frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$ then:

$$|\mathcal{A}_\ell| \leq K_2 \left(R_\ell\right)^{1-2\theta_1-\theta_2}.$$

$$\begin{aligned} \text{Then:} \quad T_0 = |B_L| \quad &\leq \quad \sum_{\ell=L}^{\infty} \left(|(B_\ell \setminus B_{\ell+1}) \setminus A_\ell| + |A_\ell|\right) \\ &\leq \quad K_3 \sum_{\ell=L}^{\infty} R_\ell^\beta, \quad (\beta = \min\{1-2\theta_1-\theta_2, 1-3\theta_1-4\theta_2\} > 0) \\ &= \quad \frac{K_3}{1-2^{-\beta}} R_L^\beta. \end{aligned}$$

This gives a contradiction since L may be taken as large as we wish.

Remark: By this local criterium, there exists “sub critical” bounded initial data for which the mild solution blows up in finite time.

Final remark.

The Nordheim equation may also be solved in the space of measures.
We say that f is a weak solution with initial datum f_0 such that

$$g_0(\varepsilon) = 4\pi\sqrt{2\epsilon} f_0\left(\epsilon\right) \in \mathcal{M}_+\left(\mathbb{R}^+; (1+\epsilon)\right),$$

if $g\left(t,\epsilon\right) = 4\pi\sqrt{2\epsilon} f\left(t,\epsilon\right) \in C\left([0,T);\mathcal{M}_+\left(\mathbb{R}^+;(1+\epsilon)\right)\right)$ satisfies:

$$-\int_{\mathbb{R}^+} g_0\left(\epsilon\right)\varphi\left(0,\epsilon\right) d\epsilon = \int_0^T \int_{\mathbb{R}^+} g\partial_t\varphi d\epsilon dt + \int_0^T \int_{\left(\mathbb{R}^+\right)^3} \frac{g_1 g_2 g_3 w}{\sqrt{\epsilon_2\epsilon_3}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 dt +$$

$$+\frac{1}{2}\int_0^T \int_{(\mathbb{R}^+)^3} \frac{g_1 g_2 w}{\sqrt{\epsilon_2}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 dt$$

$\forall\ \varphi \in C_0^2\left([0,T);[0,\infty)\right),\ \ Q_\varphi = \varphi\left(\epsilon_3\right) + \varphi\left(\epsilon_1+\epsilon_2-\epsilon_3\right) - 2\varphi\left(\epsilon_1\right)$

- No problem to give sense as far as the measure g does not charges the origin.
- The first global existence proved by X. Lu in J. Stat. Phys. 2005.

Theorem Let $M > 0,\ E > 0,\ \nu > 0,\ \gamma > 3$. There exist $\rho = \rho\left(M, E, \nu\right) > 0,\ K^* = K^*\left(M, E, \nu\right) > 0,\ T_0 = T_0\left(M, E\right)$ and a numerical constant $\theta_* > 0$ independent on $M,\ E,\ \nu$ such that for any $f_0 \in L^\infty\left(\mathbb{R}^+; (1+\epsilon)^\gamma\right)$ satisfying

$$4\pi\sqrt{2}\int_{\mathbb{R}^+} f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon = M\ ,\quad 4\pi\sqrt{2}\int_{\mathbb{R}^+} f_0\left(\epsilon\right)\sqrt{\epsilon^3}d\epsilon = E$$

$$\int_0^R f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon \geq \nu R^{\frac{3}{2}}\ \text{ for } 0 < R \leq \rho\ ,\quad \int_0^\rho f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon \geq K^*\left(\rho\right)^{\theta_*}$$

there exists a global weak solution f and a positive time $T_* > 0$ such that the following holds:

$$\sup_{0\leq t\leq T_*}\left\|f\left(t,\cdot\right)\right\|_{L^\infty(\mathbb{R}^+)} < \infty,\quad \inf_{T_*<t\leq T_0}\int_{\{0\}}\sqrt{\epsilon}f\left(t,\epsilon\right)d\epsilon > 0.$$

This second result may then be seen as closer to the physical phenomena of condensation in finite time due to the presence of a Dirac measure at the origin.

(In the context of weakly interacting radially symmetric gases, the presence of a condensate corresponds to a Dirac mass at the origin.)

But this is not clear... The underlying question is: how do we precisely define the finite time condensation in mathematical terms?

The BE condensation is a transition between two situations. We have defined precisely the situation before the transition. What is the situation after?

The equations obtained by physicists, describing the weakly interacting isotropic and spatially homogeneous gas of bosons, after are the following:

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,\epsilon) = Q(f)(t,\epsilon) + n_c(t)\mathcal{Q}(n_c, f)(t,\epsilon); \\ n_c'(t) = -n_c(t) \displaystyle\int_0^\infty \mathcal{Q}(n_c, f)(t,\epsilon)\sqrt{\epsilon}d\epsilon \end{cases}$$

where $n_c(t)$ is the density of the condensate and $\mathcal{Q}(n_c, f)$ describes the gas-condensate interaction.
(Khalatnikov ('64), Kirkpatrick&al. ('85), Gardiner&al. ('98), Stoof ('99))

See recently: H. Spohn in Phys.D 2010 for a general presentation, L. Arkeryd & al., Arch. Rat. Mech. 2012 for an existence result on the system).

We easily see that if initially, $n_c(0) = 0$ we may expect to have $n_c(t) = 0$ for an interval of time $t \in [0, T^*)$. A more clear result on finite time condensation would be the existence of (f, n_c) and T_0 such that $n_c(0) = 0$ for $t < T_0$ and $n_c(t) > 0$ for $t > T_0$.
In order to have $n_c(t) > 0$ for some $t > 0$ we need the function $t \to \int_0^\infty \mathcal{Q}(n_c, f)(t,\epsilon)\sqrt{\epsilon}d\epsilon$ to become singular at some finite time.-

Supercritical solutions

① Using the monotonicity property the first condition of the criteria holds:

② Entropy, dissipation of entropy, etc...

$$
\begin{aligned}
S[f] &= \int_{\mathbb{R}^+} \left[(1+f)\log(1+f) - f\log(f)\right]\sqrt{\epsilon}\,d\epsilon \\
D[f] &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (1+f_1)(1+f_2)(1+f_3)(1+f_4)\left[Q_{1,2}-Q_{3,4}\right]\times \\
&\quad \times\left[\log(Q_{1,2})-\log(Q_{3,4})\right]\Phi\, d\epsilon_1 d\epsilon_2 d\epsilon_3 \\
Q_{j,k} &= \frac{f_j}{(1+f_j)}\frac{f_k}{(1+f_k)}; \qquad \Phi = \min\left\{\sqrt{(\epsilon_1)_+}, \sqrt{(\epsilon_2)_+}, \sqrt{(\epsilon_3)_+}, \sqrt{(\epsilon_4)_+}\right\}
\end{aligned}
$$

The equation between T_1 and T_2 gives for some positive constant $C(E,M)$:

$$S\left[f\right]\left(T_2\right) - S\left[f\right]\left(T_1\right) = \int_{T_1}^{T_2} D\left[f\left(\cdot,t\right)\right]dt$$

$$\left|S\left[f\left(t\right)\right]\right| \leq C\left(E,M\right) \quad , \; 0 \leq t < T_{\max}.$$

③ Using the following estimate:

$$D\left[f\right] \geq \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f\left(\epsilon_1\right) f\left(\epsilon_2\right) \Psi\left(\frac{Q\left(\epsilon_3\right)Q\left(\epsilon_4\right)}{Q\left(\epsilon_1\right)Q\left(\epsilon_2\right)} - 1\right)\Phi d\epsilon_1 d\epsilon_2 d\epsilon_3$$

with $\Psi\left(s\right) = s\log\left(1+s\right), \quad Q\left(t,\epsilon\right) = \dfrac{f\left(t,\epsilon\right)}{1+f\left(t,\epsilon\right)}$

We deduce the existence of a sequence $t_n \to +\infty$ such that

$$Q\left(t_n,\epsilon\right) \equiv \frac{f\left(t_n,\epsilon\right)}{1+f\left(t_n,\epsilon\right)} \rightharpoonup e^{-\beta_*(\epsilon+\alpha_*)}.$$

We deduce the existence of $m_* > 0$ and $\rho > 0$ such that, for all $R \in (0,\rho)$ there exists a sequence $t_n \to +\infty$ satisfying:

$$\int_0^R g\left(t_n,\epsilon\right) d\epsilon = 4\pi \int_0^R \sqrt{2\epsilon} f\left(t_n,\epsilon\right) d\epsilon \geq m_*.$$

If R is chosen small enough we may ensure that $m_* > K_* R^{\theta_*}$. And the second condition of the local criterium is fulfilled.