Statistical mechanics of strange metals and black holes

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Talk online: qpt.physics.harvard.edu/talks

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Black hole thermodynamics

The Einstein action for gravity in 3+1 dimensions is

$$I_E = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4\right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_E)\,,$$

where $\kappa^2 = 8\pi G_N$ is the gravitational constant, $\mathcal{R}_4$ is the Ricci scalar. The Schwarzschild solution of the saddle-point equations is

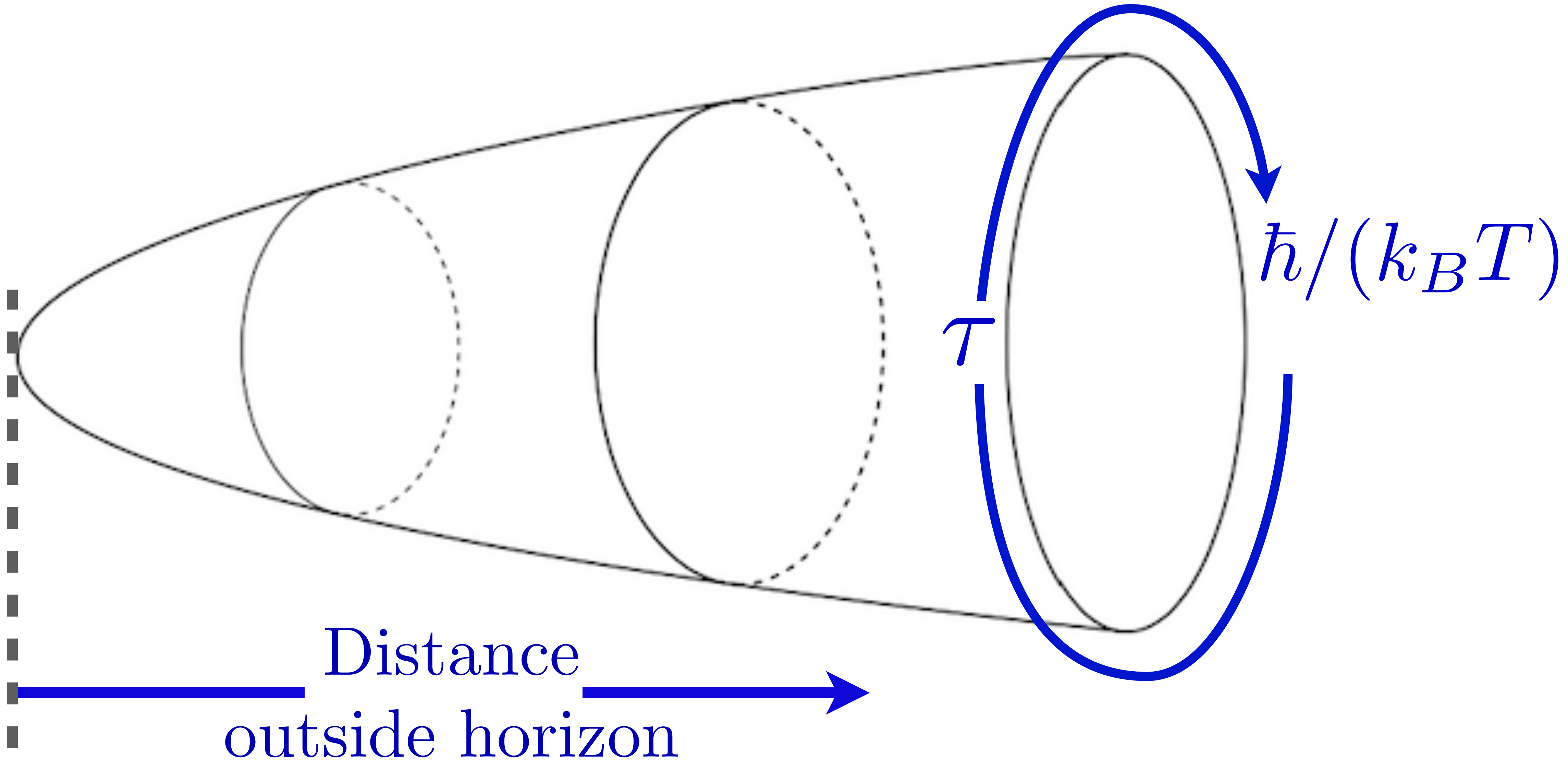
$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_2^2 + \frac{dr^2}{V(r)}$$

where $d\Omega_2^2$ is the metric of the 2-sphere, and

$$V(r) = 1 - \frac{m}{r}.$$

The gravitational mass of the black hole is $M = 2G_N m$. The black hole horizon is at $r = r_0$ where $V(r_0) = 0$; so

$$r_0 = m$$



The $T > 0$ quantum partition function is obtained in a spacetime which is periodic as a function of τ with period $\hbar/(k_B T)$. We have to ensure that there is no singularity at the horizon r_0 where $V(r_0) = 0$. Let us change radial co-ordinates to y, where $r = r_0 + y^2$. Then for small y

$$ds^2 = \frac{4}{V'(r_0)} \left[\frac{(V'(r_0))^2}{4} y^2 d\tau^2 + dy^2 \right] + r_0^2 d\Omega_2^2 = \frac{4}{V'(r_0)} \left[y^2 d\theta^2 + dy^2 \right] + r_0^2 d\Omega_2^2$$

The expression in the square brackets is the metric of the flat plane in polar co-ordinates, with radial co-ordinate y and angular co-ordinate $\theta = V'(r_0)\tau/2$. Smoothness requires periodicity in θ with period 2π, and so

$$4\pi T = V'(r_0) = \frac{1}{m}\,.$$

The free energy $\beta F = I_E$, where $\beta = 1/T$. So the entropy is

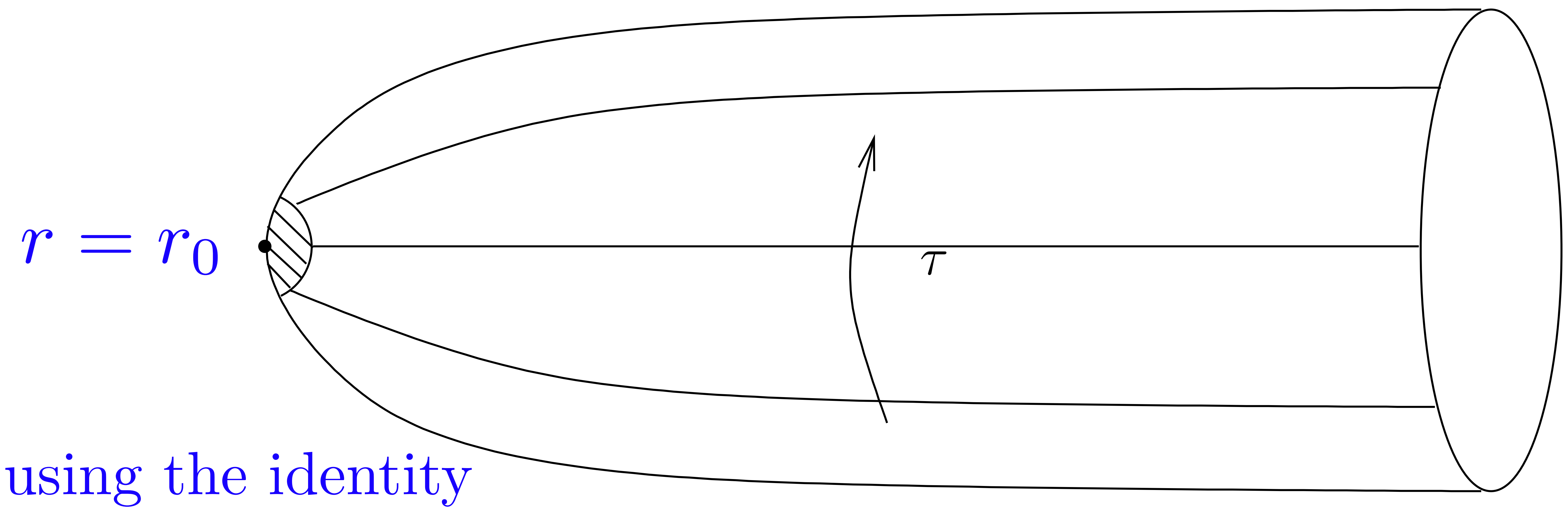
$$S = -\frac{\partial F}{\partial T} = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I_E$$

However, the metric is τ-independent, and the only explicit dependence of the action is via $I_E = \beta H$. Such an action implies $S = 0$.

The entire contribution to the entropy comes from the vicinity of the co-ordinate singularity at $r = r_0$. We evaluate the action is the small region around this point

$$I_{\rm grav} = I_E + I_{GH} \quad , \quad I_{GH} = \int_{\partial} d^3x \sqrt{g_b} \left[-\frac{1}{\kappa^2} \mathcal{K}_3 \right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_{\rm grav}) \,,$$

where $\mathcal{K}_3$ is the extrinsic scalar curvature of the 3-dimensional boundary of spacetime. I_{GH} is the Gibbons-Hawking boundary term, deduced by the requirement that the Euler-Lagrange equations of $I_{\rm grav}$ co-incide with the Einstein equations, with no additional boundary terms. The entire contribution to the entropy will come from I_{GH}.



We evaluate I_{GH} by using the identity

$$\int_\partial d^3x\sqrt{g_b}\,\mathcal{K}_3 = \frac{\partial}{\partial n}\int_\partial d^3x\sqrt{g_b}$$

where n is the Gaussian normal co-ordinate of the boundary. Evaluating at $y = \epsilon$, we have

$$\int_\partial d^3x\sqrt{g_b} = 2\pi\epsilon\mathcal{A}$$

where $\mathcal{A} = 4\pi r_0^2$ is the area of the horizon. Combining everything, we have the famous result of Hawking

$$S = \frac{2\pi\mathcal{A}}{\kappa^2} = \frac{\mathcal{A}}{4G_N}.$$

Charged black holes

We consider a charged black hole in Einstein-Maxwell theory of g and a U(1) gauge flux $F = dA$

$$I_{EM} = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4 + \frac{1}{4g_F^2}F^2\right] \quad , \quad \mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}g\mathcal{D}A\,\exp(-I_{EM} - I_{GH})\,.$$

The saddle-point equations now yield a solution as before with

$$V(r) = 1 + \frac{\Theta^2}{r^2} - \frac{m}{r} \quad ; \quad A_\tau = i\mu\left(1 - \frac{r_0}{r}\right) \quad ; \quad \Theta = \frac{\kappa r_0}{\sqrt{2}g_F}\mu \quad ; \quad \mathcal{Q} = \frac{4\pi\mu r_0}{g_F^2} \quad ; \quad S = \frac{2\pi\mathcal{A}}{\kappa^2}$$

where $\mathcal{Q}$ is the total charge, the chemical potential is μ, and as before the horizon is where $V(r_0) = 0$, the temperature $T = V'(r_0)/(4\pi)$, and $\mathcal{A} = 4\pi r_0^2$.

This defines a two parameter family of charged black hole solutions of I_{EM} determined by T and $\mathcal{Q}$.

Charged black holes

Now we take the limit $T \to 0$ at fixed $\mathcal{Q}$. Then we find the remarkable feature that the horizon radius remains finite

$$R_h \equiv r_0(T \to 0, \mathcal{Q}) = \frac{\mathcal{Q}\kappa g_F}{4\pi}$$

In this limit, entropy becomes

$$S(T \to 0, \mathcal{Q}) = \frac{4\pi R_h^2}{G_N} + \gamma T \quad , \quad \gamma \equiv \frac{4\pi^2 R_h^3}{G_N}$$

For the near-horizon metric, it is useful to introduce the co-ordinate ζ

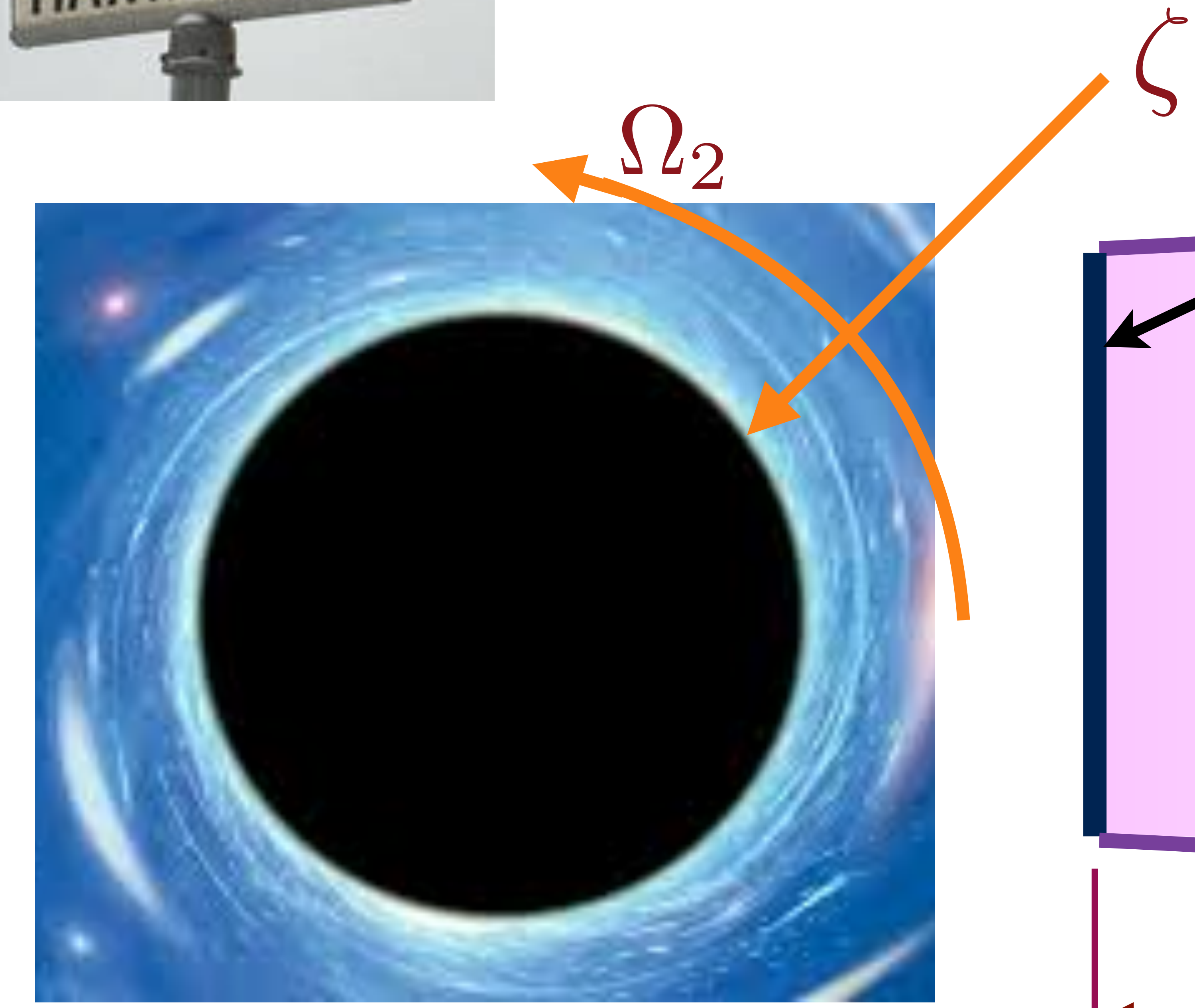
$$r = R_h + \frac{R_h^2}{\zeta}$$

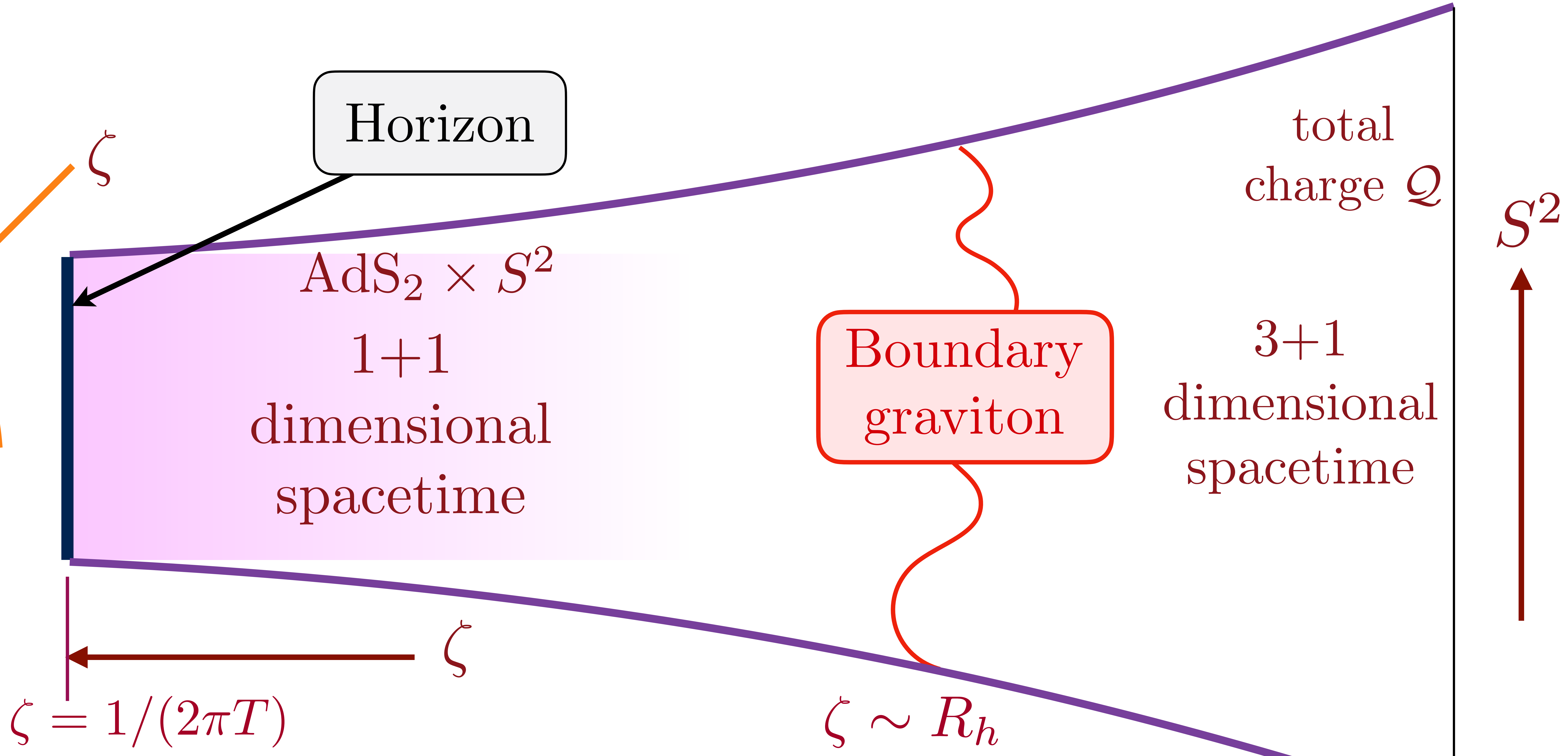
so that the horizon at $T = 0$ is at $\zeta = \infty$. Then in the near-horizon regime $R_h \ll \zeta < \infty$ the $T = 0$ metric is

$$ds^2 = R_h^2 \frac{d\tau^2 + d\zeta^2}{\zeta^2} + R_h^2 d\Omega_2^2$$

This spacetime is $\text{AdS}_2 \times S^2$.

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

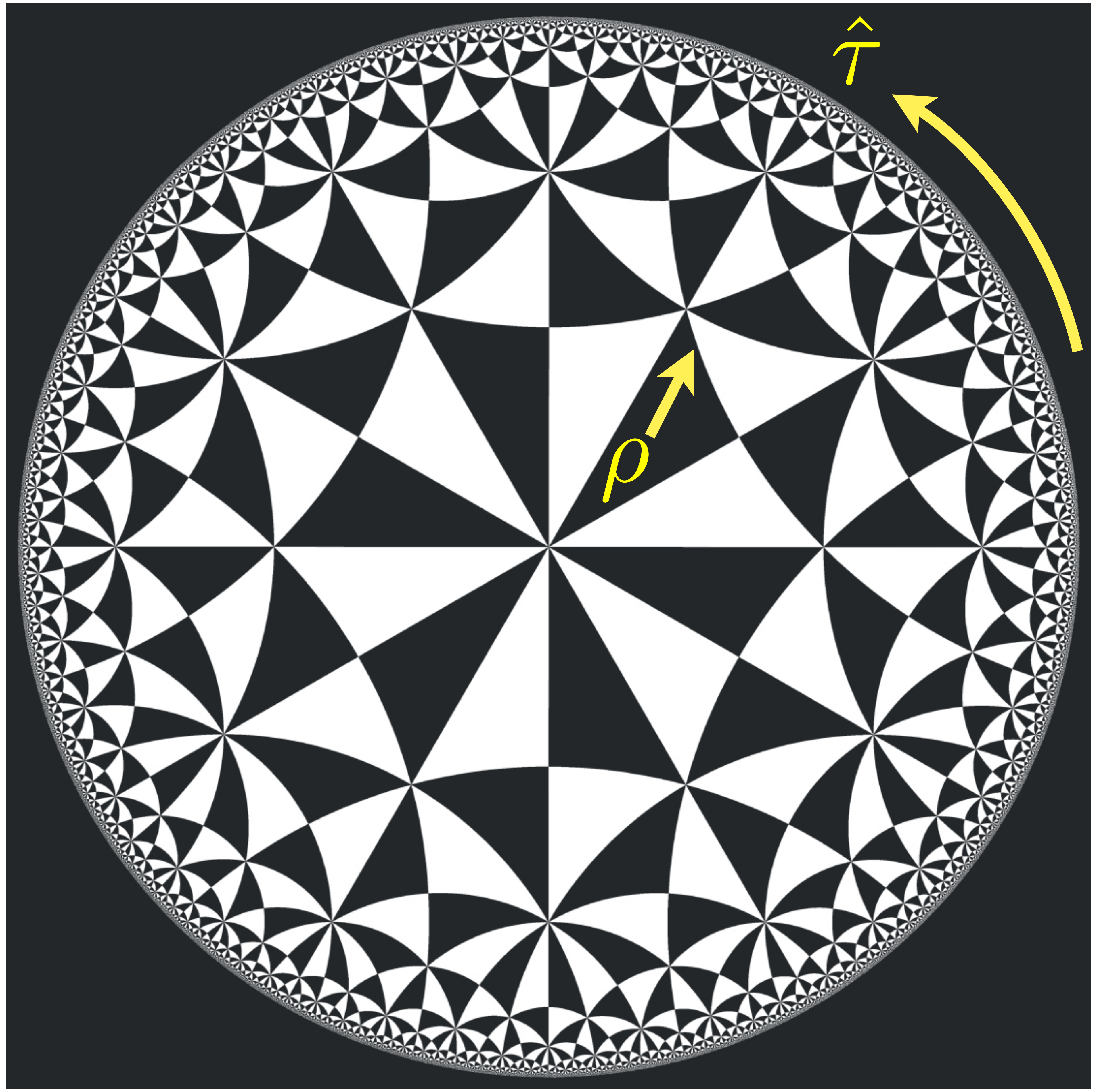
The AdS_2 metric

$$ds^2 = \frac{d\tau^2 + d\zeta^2}{\zeta^2}$$

is invariant under isometries which are SL(2,R) transformations. Verify that the co-ordinate change

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d}, \quad ad - bc = 1,$$

with a,b,c,d real, leaves the AdS_2 metric invariant.



The co-ordinate transformation

$$\zeta = \frac{1}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})} \quad , \quad \tau = \frac{\sinh(2\pi T\rho)\sin(2\pi T\hat{\tau})}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})}$$

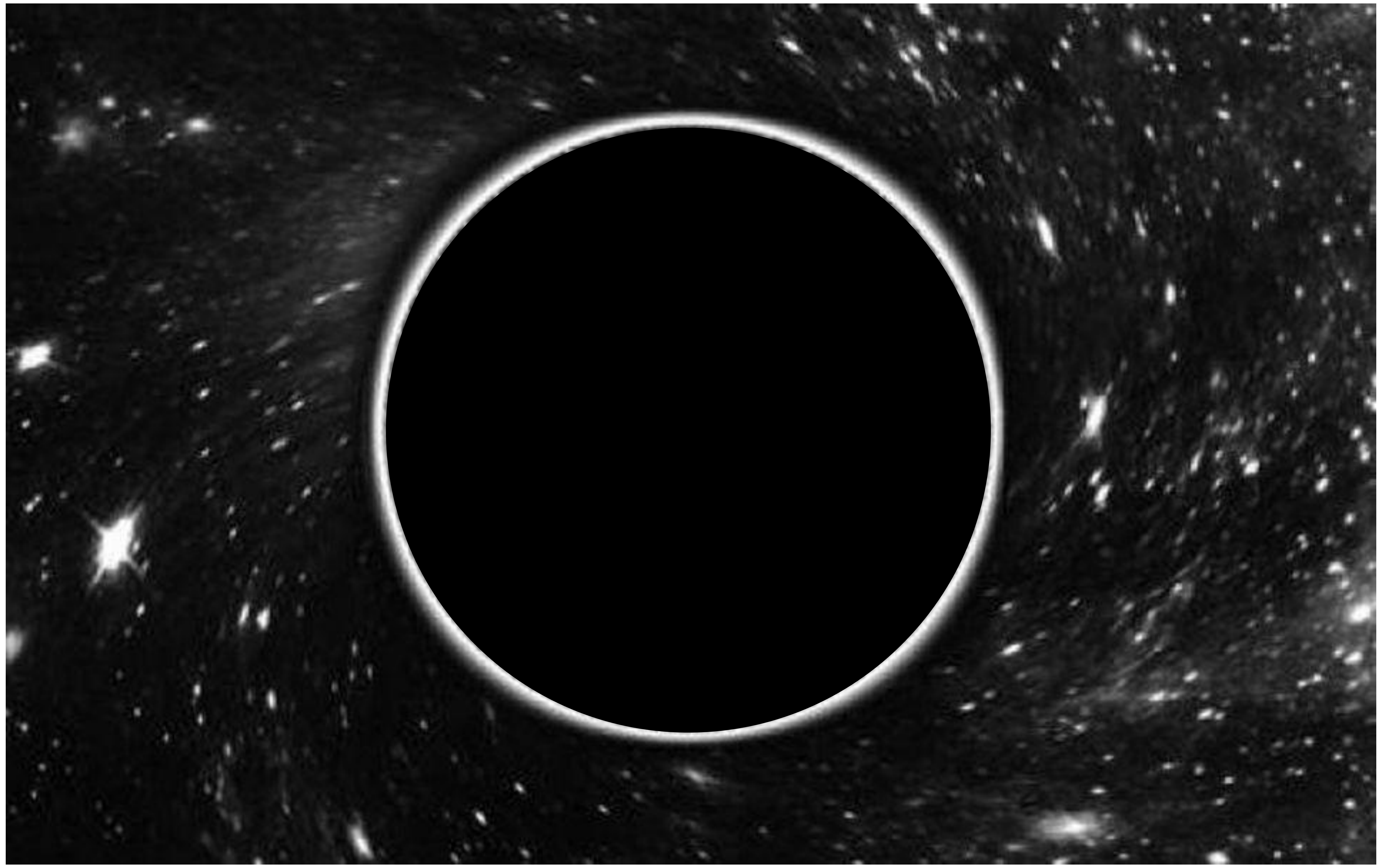
maps the metric to

$$ds^2 = 4\pi^2 T^2 \left[d\rho^2 + \sinh^2(2\pi T\rho) d\hat{\tau}^2\right]$$

Quantum gravity and holography

Quantum Black holes

- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area. $S = A k_B c^3/(4G\hbar)$.



T.B. BAKKER / DR. J.P. VAN DER SCHAAR

J. D. Bekenstein, PRD **7**, 2333 (1973)
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Remarkable features:

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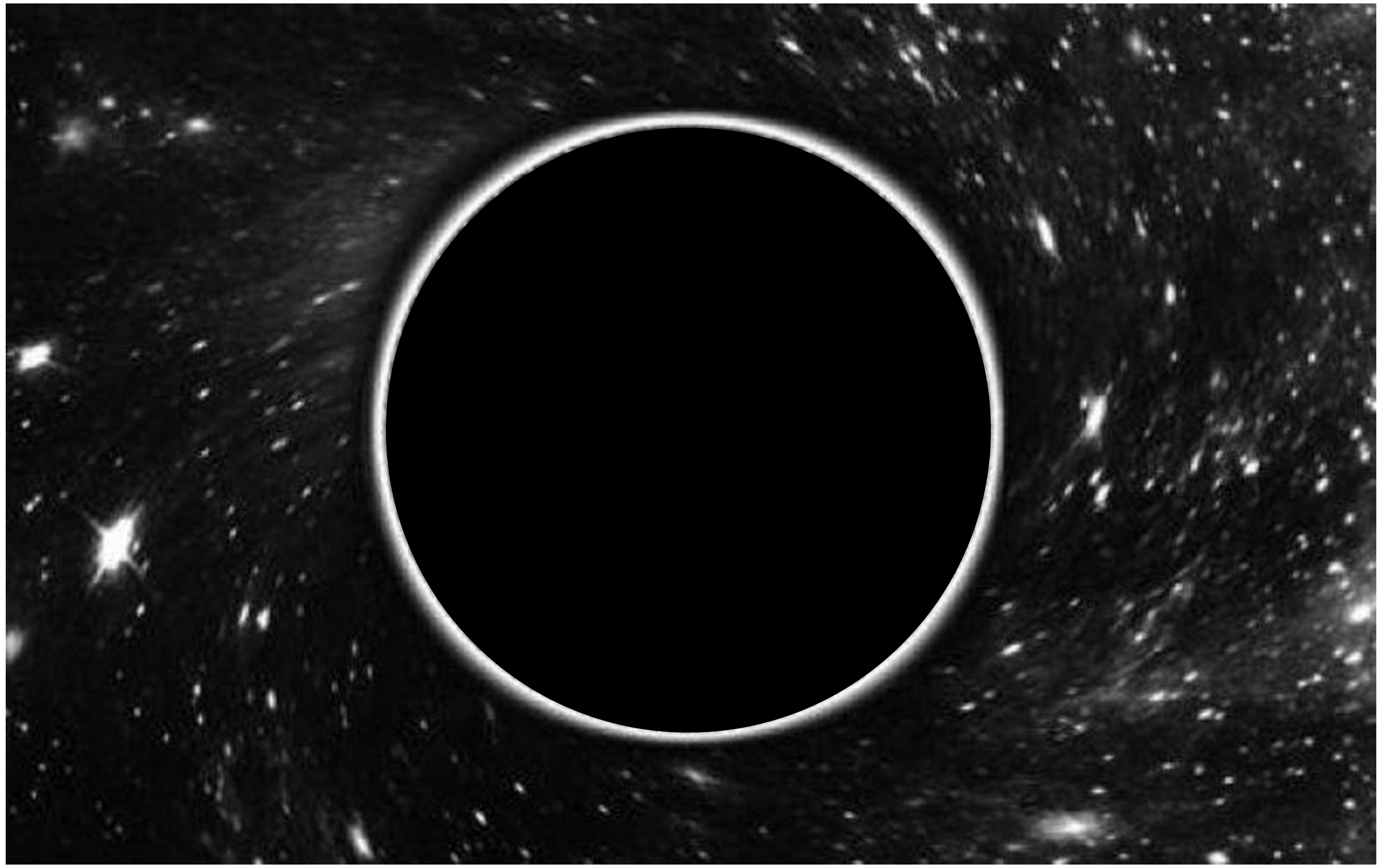

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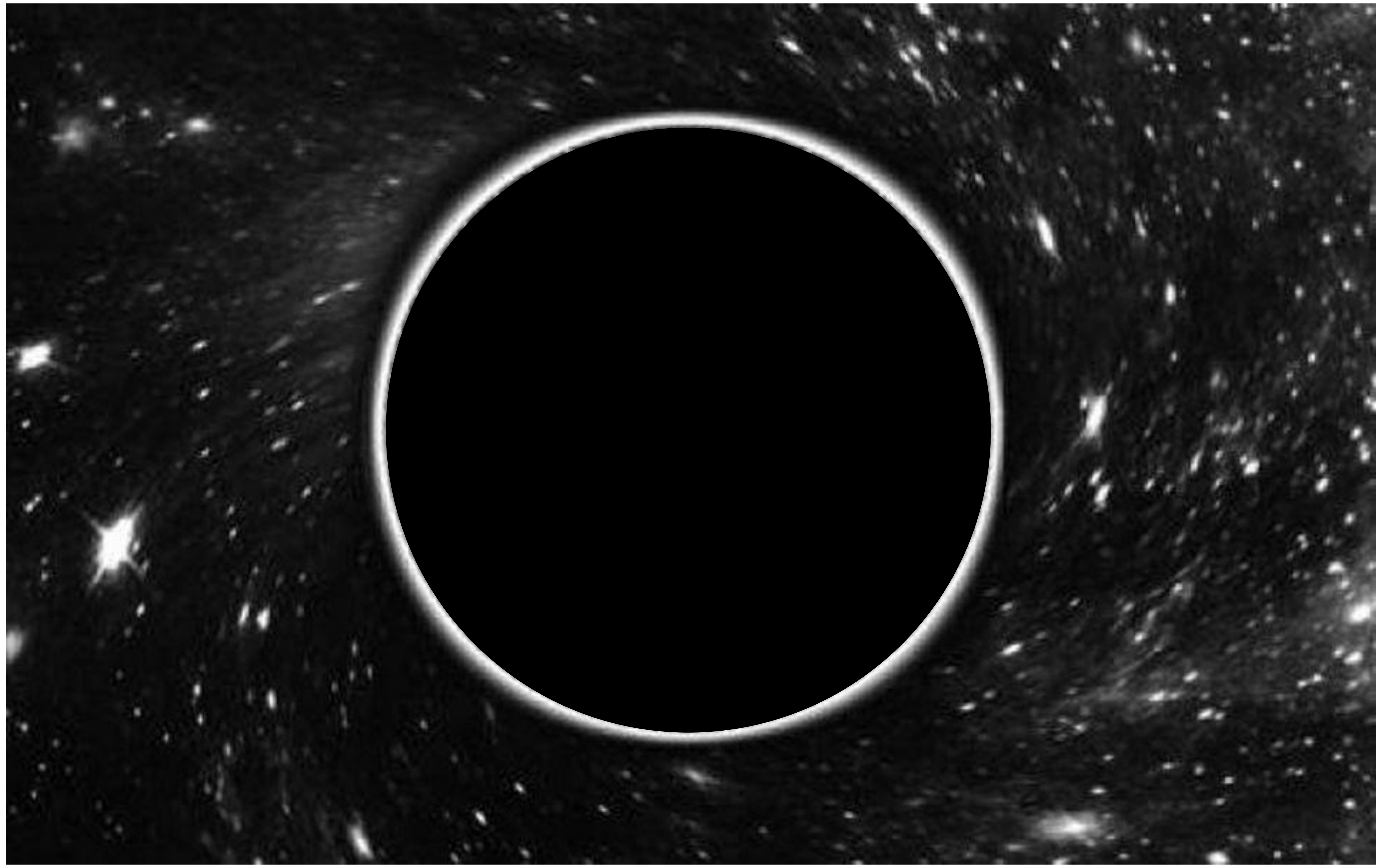

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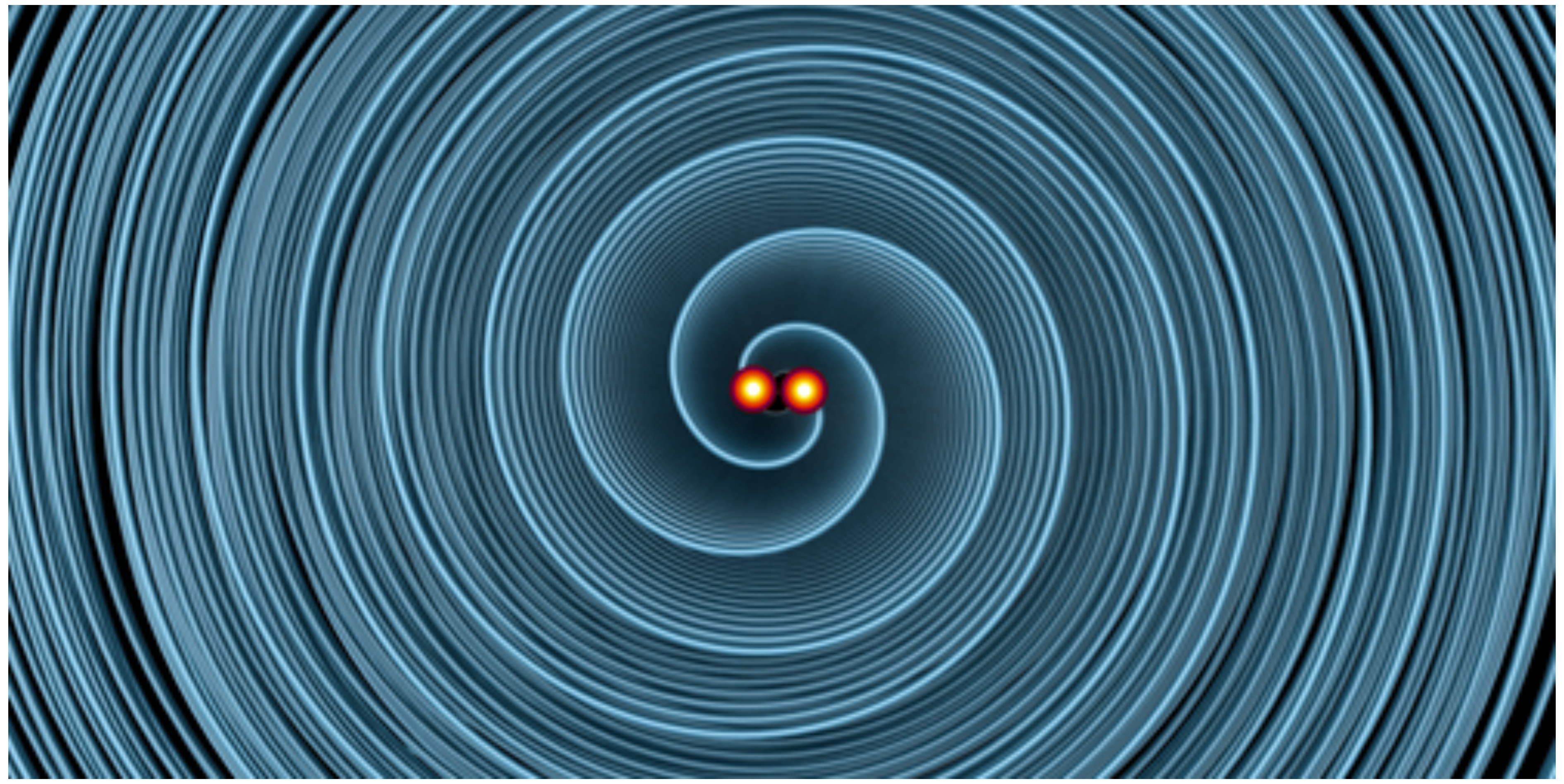
Black Holes Obey Information-Emission Limits

April 22, 2021 • *Physics* 14, s47 –Christopher Crockett

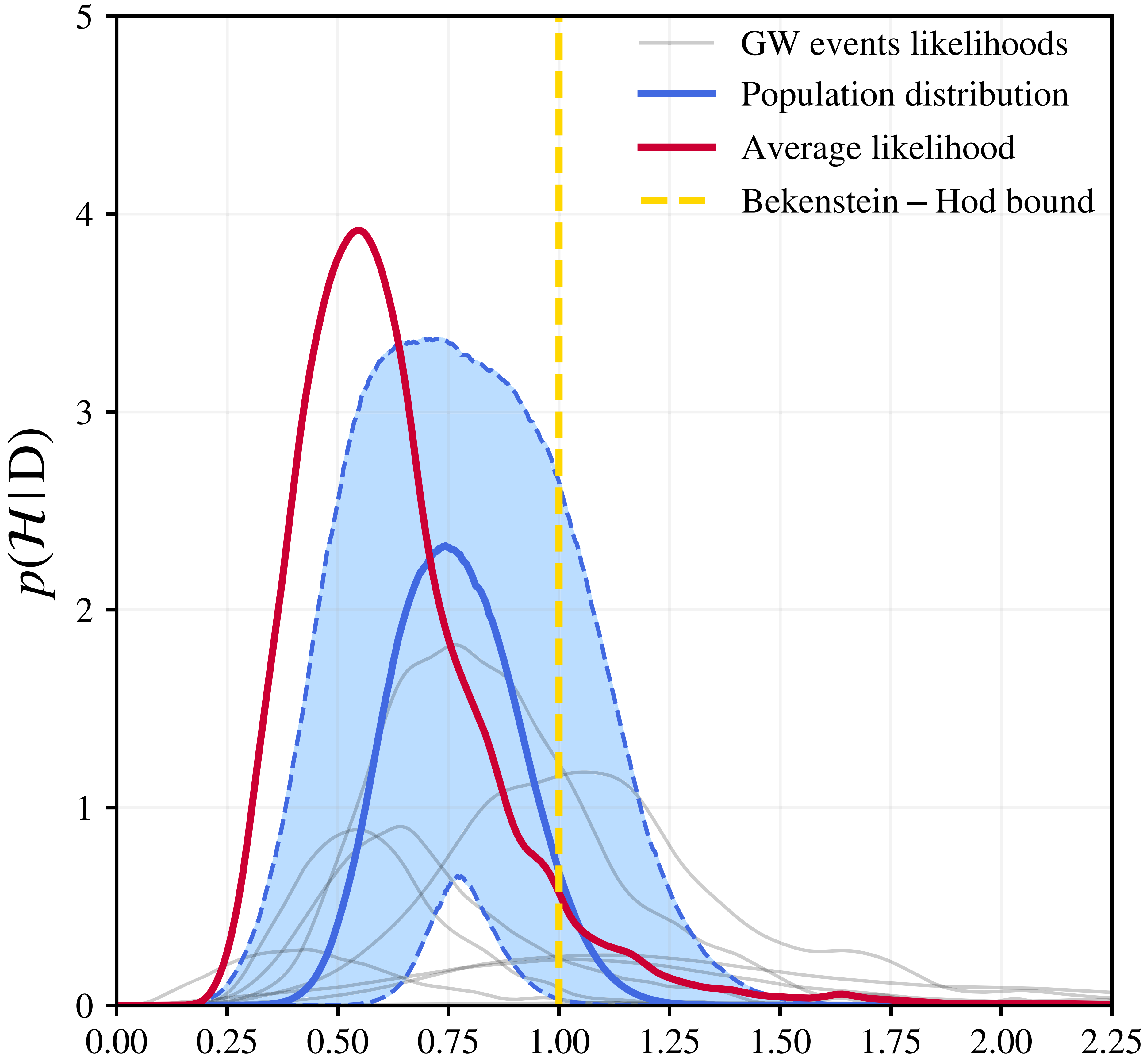
G. Carullo, D. Laghi, J. Veitch, W. Del Pozzo, Phys. Rev. Lett. **126**, 161102 (2021)

An analysis of the gravitational waves emitted from black hole mergers confirms that black holes are the fastest known information dissipaters.



Gravity wave observations of 8 different black holes show a relaxation time

$$\tau \sim \frac{\hbar}{k_B T}$$



$$\mathcal{H} = \frac{1}{\pi}\frac{\hbar/\tau}{k_B T}$$

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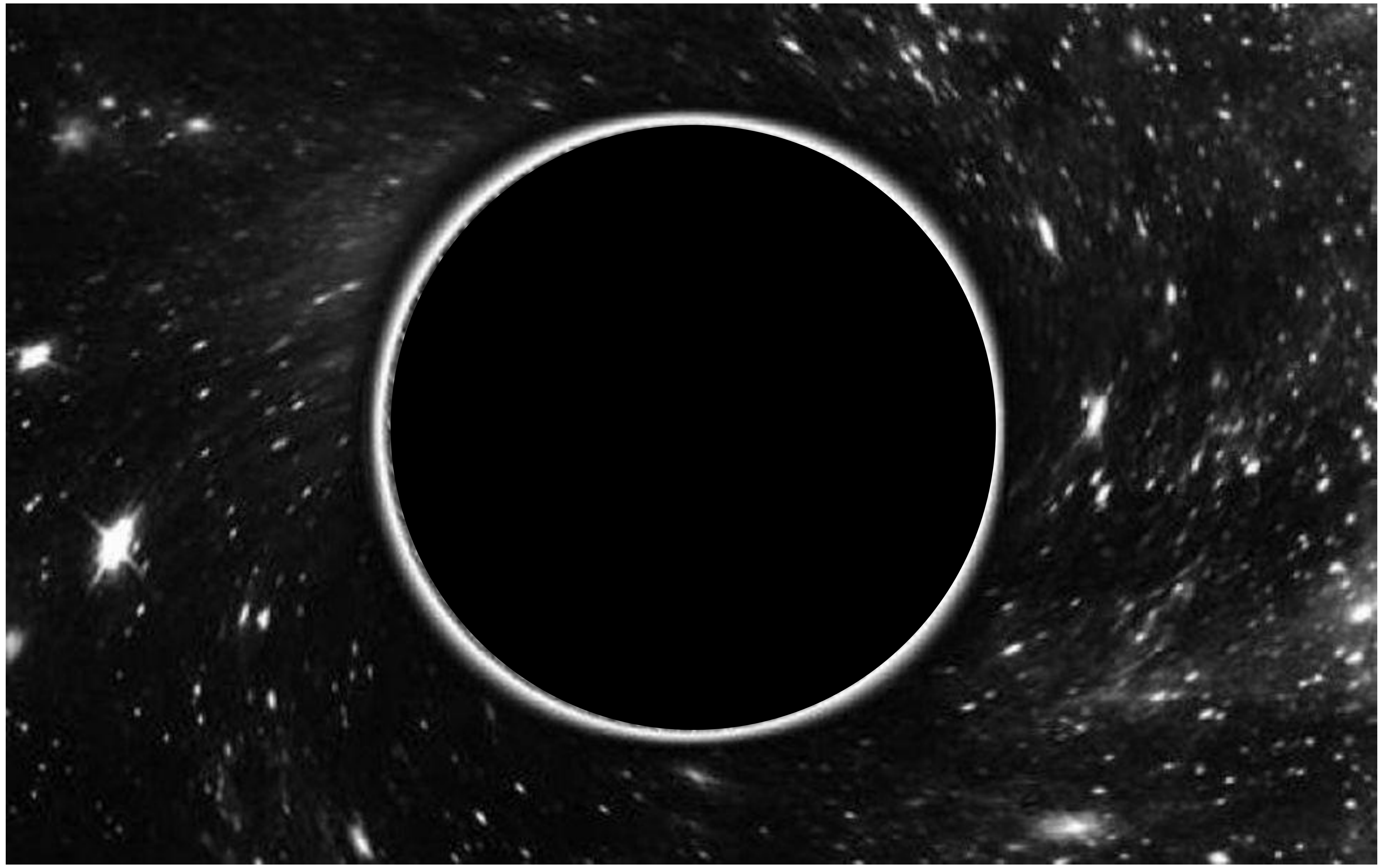

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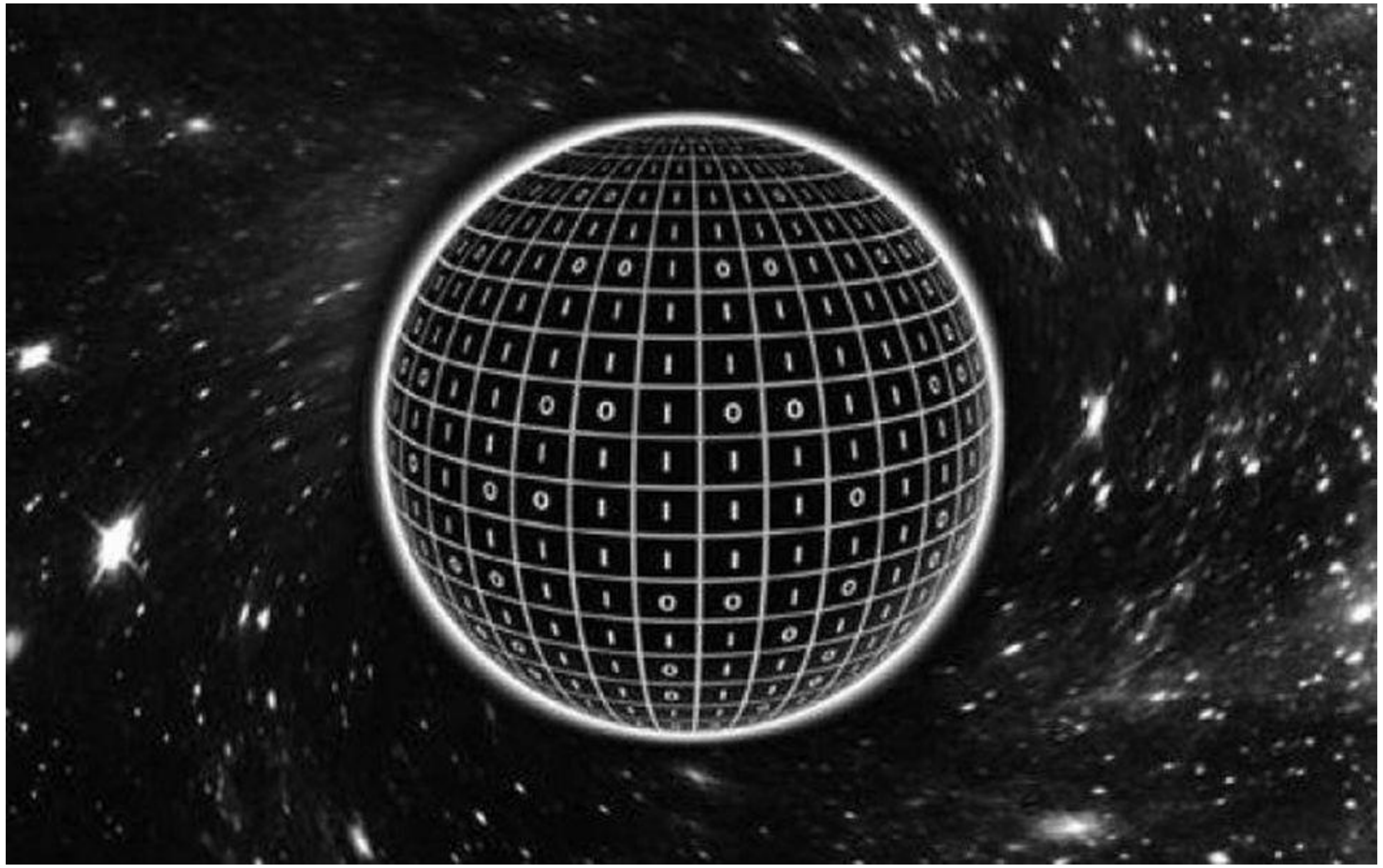


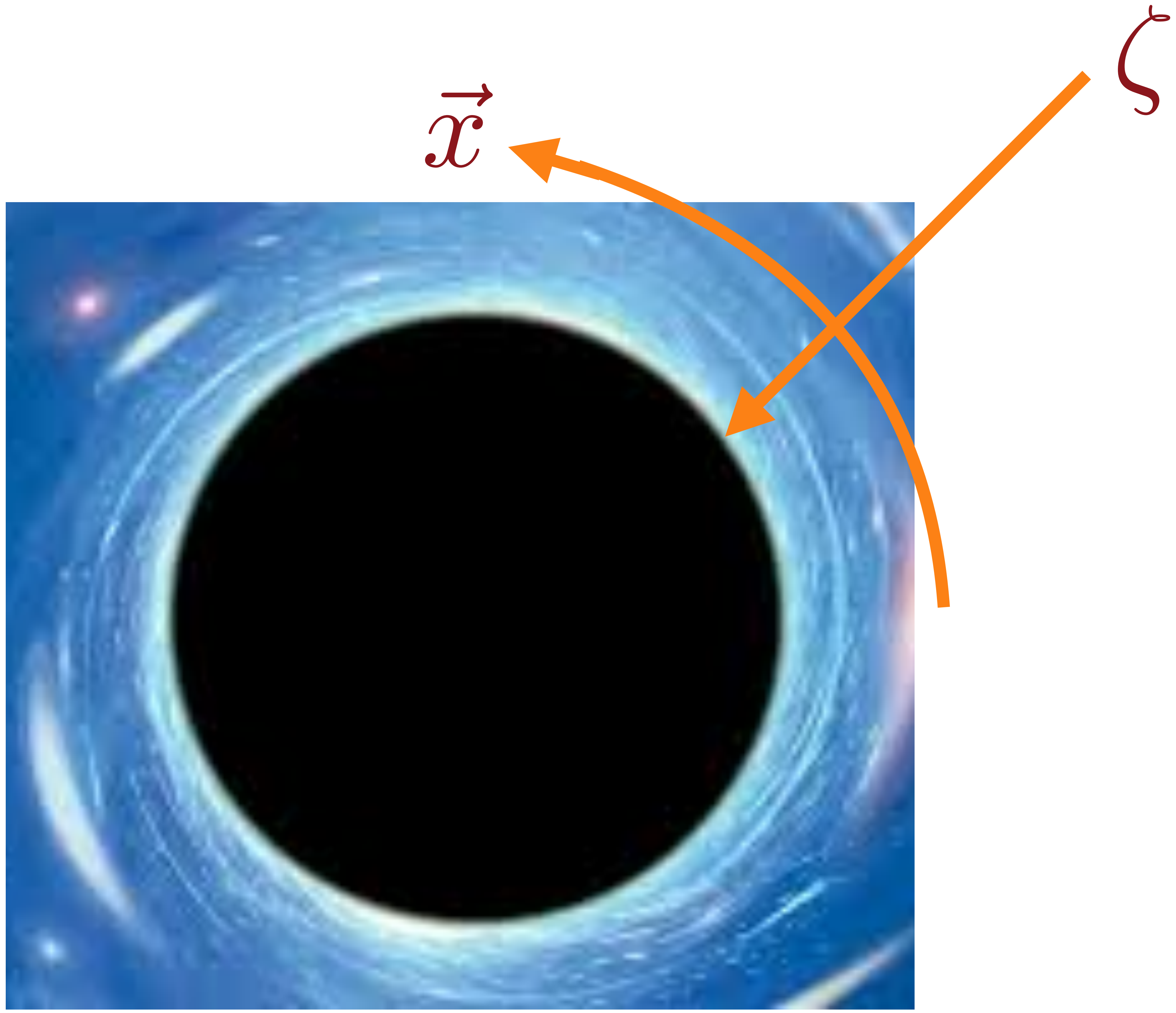
T.B. BAKKER / DR. J.P. VAN DER SCHAAR

A quantum computer simulating a black hole must have:

- Number of qubits proportional to the surface area *i.e.* it is a 'hologram'
- No quasiparticles and Planckian time relaxation to thermal equilibrium.

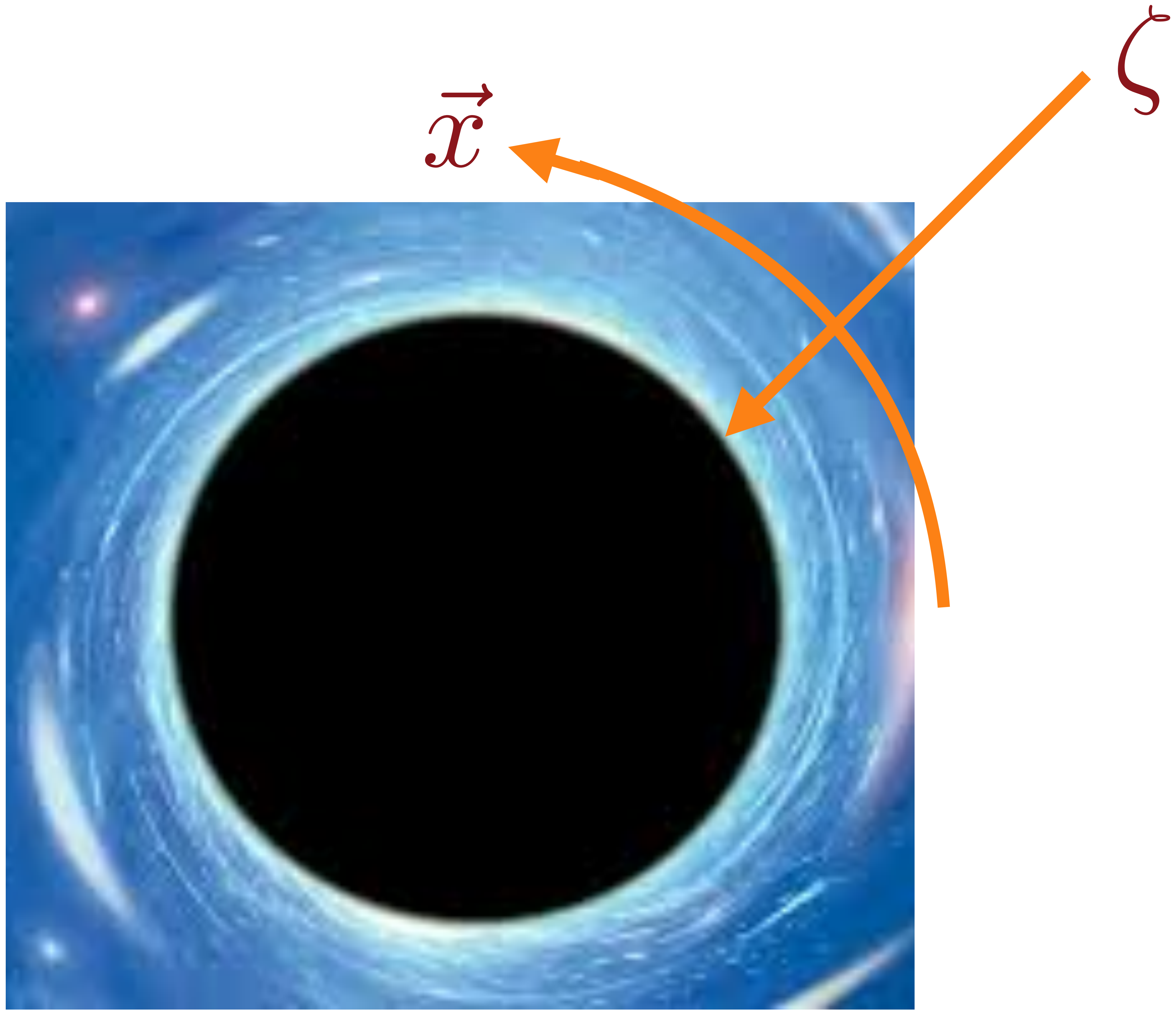
Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge

Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge



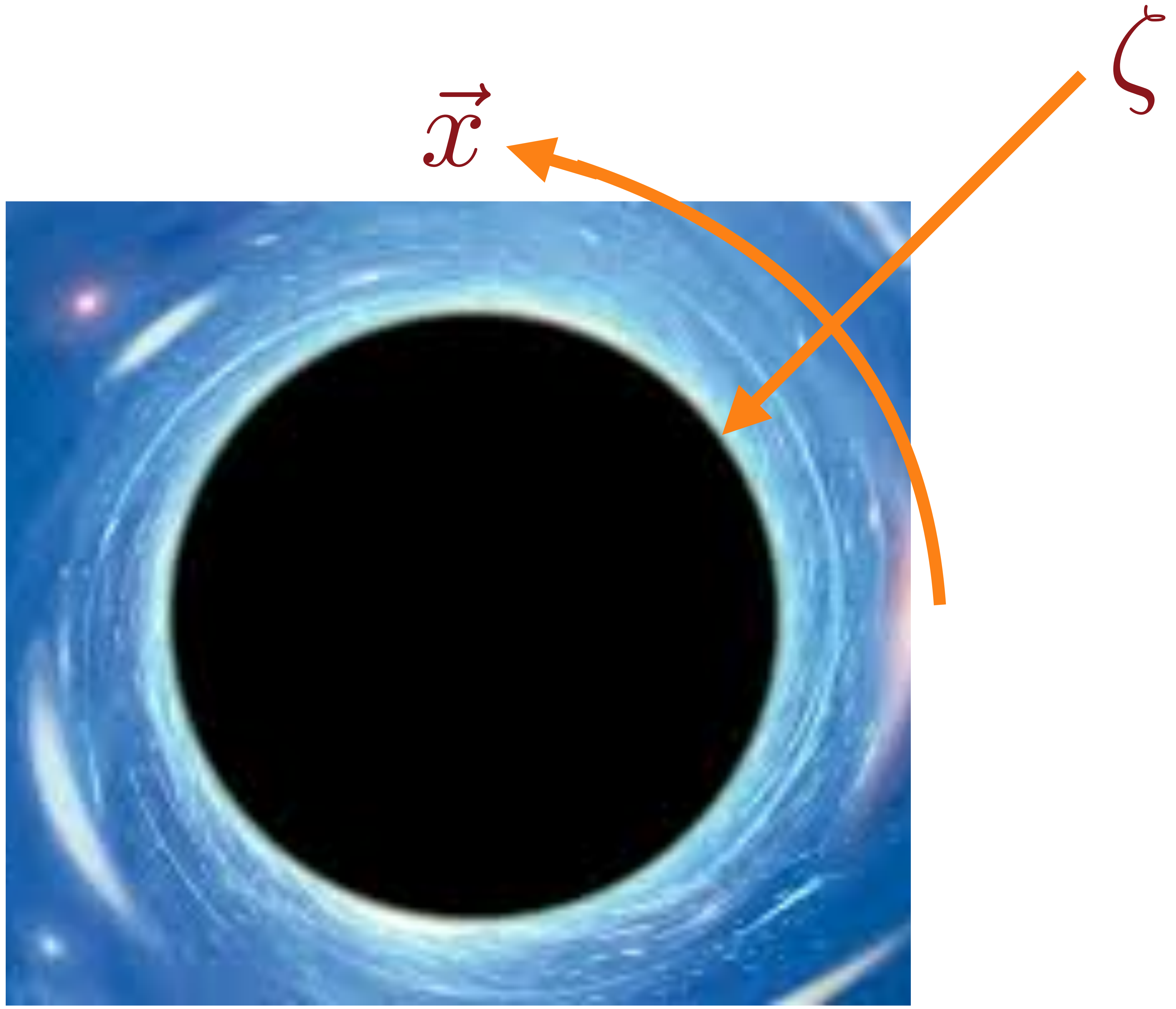
Zooming into the near-horizon region of a charged black hole at low temperature, yields a quantum theory in one space (ζ) and one time dimension

Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge



This should be dual to a quantum computer in 0 space dimensions:

Maxwell's electromagnetism
and Einstein's general relativity
allow black hole solutions with a net charge



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The SYK model!

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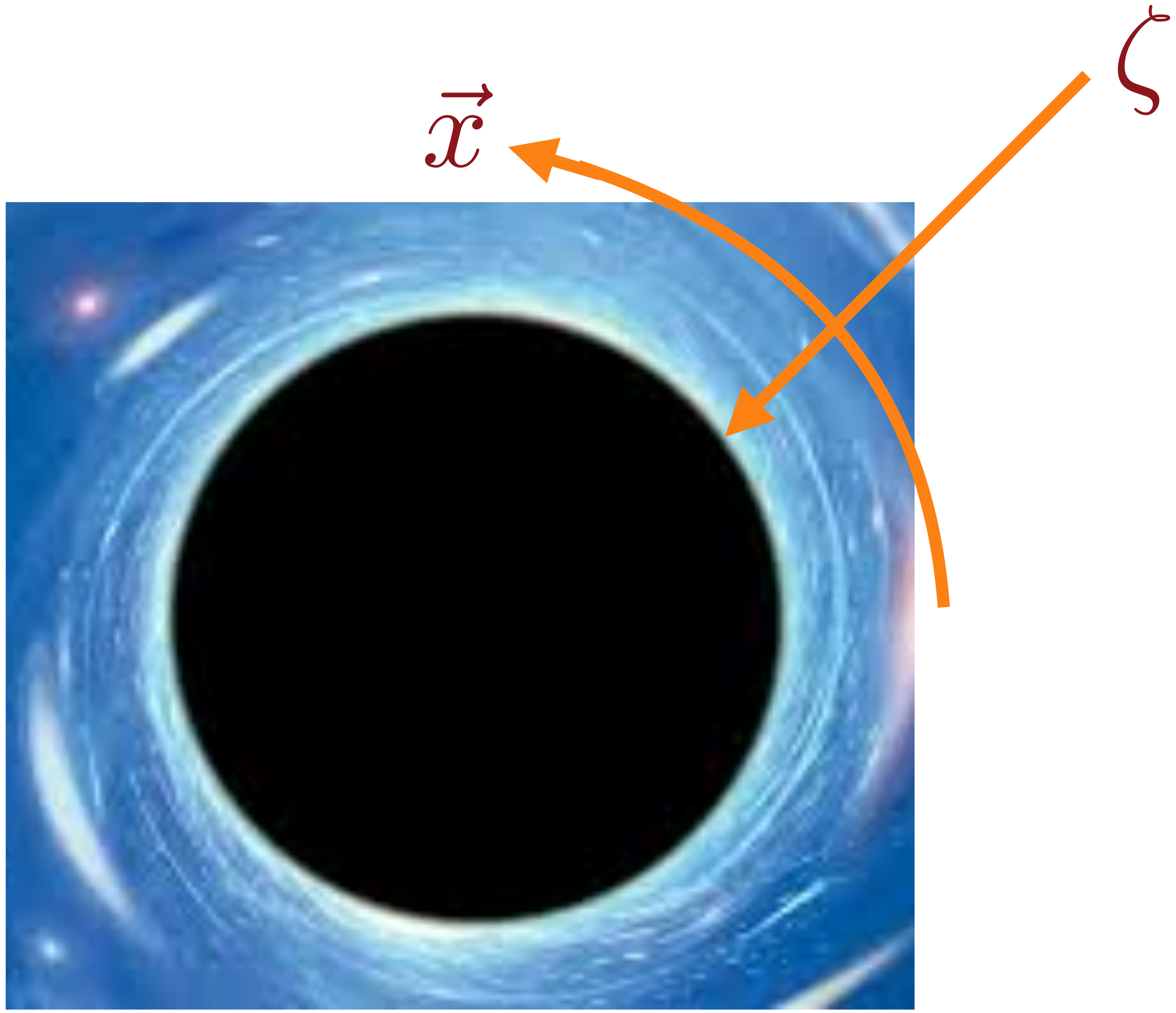


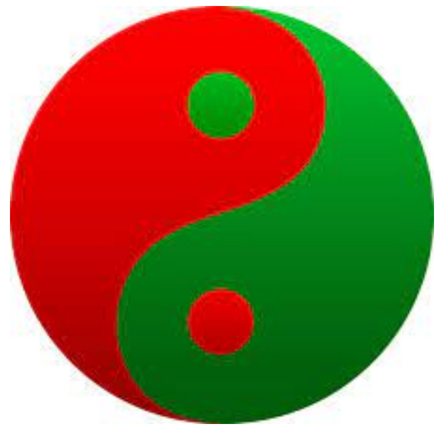
The quantum versions of Maxwell's and Einstein's equations in this two-dimensional spacetime are also the equations describing electron entanglement in the SYK model!

The Sachdev-Ye-Kitaev (SYK) model

The SYK model has a scale-invariant
entanglement structure:
i.e. electrons are entangled
at all distances

In one set of variables, it models the ***strange***
electrical properties of a material called YBCO

Sachdev, Ye (1993)



In a *dual* set of variables it describes
charged ***black holes***

Sachdev (2010), Kitaev (2015), Maldacena Stanford (2015)

Thermodynamics of the SYK model and charged black holes

SYK model

$$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T)$$

Charged black holes

$$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2}\frac{k_B T}{\hbar}\right)$$

A_0 is the area of the horizon at $T = 0$.

S.S. 2010, 2015

SYK model

$$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T)$$

$$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}$$

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SYK model	Charged black holes
$$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T)$$	$$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2}\frac{k_B T}{\hbar}\right)$$ A_0 is the area of the horizon at $T = 0$.
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$$\lim_{T\to 0}\frac{1}{Nk_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$$	$$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$$

S.S. 2010, 2015

SYK model	Charged black holes
$$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2} \ln \left(\frac{N^{1/3} U}{k_B T} \right) + \ldots$$	$$\frac{S(T)}{k_B} = \frac{1}{\hbar G} \left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2} \frac{k_B T}{\hbar} \right) - \frac{3}{2} \ln \left(\frac{c^2}{A_0^{1/6} (\hbar G)^{1/3} (k_B T/\hbar)} \right) + \ldots$$
$$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}$$	$$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}$$
$$\lim_{T \to 0} \frac{1}{N k_B} \frac{\partial S}{\partial Q} = 2\pi \mathcal{E}$$	$$\lim_{T \to 0} \frac{1}{k_B} \frac{\partial S}{\partial Q} = 2\pi \mathcal{E}$$

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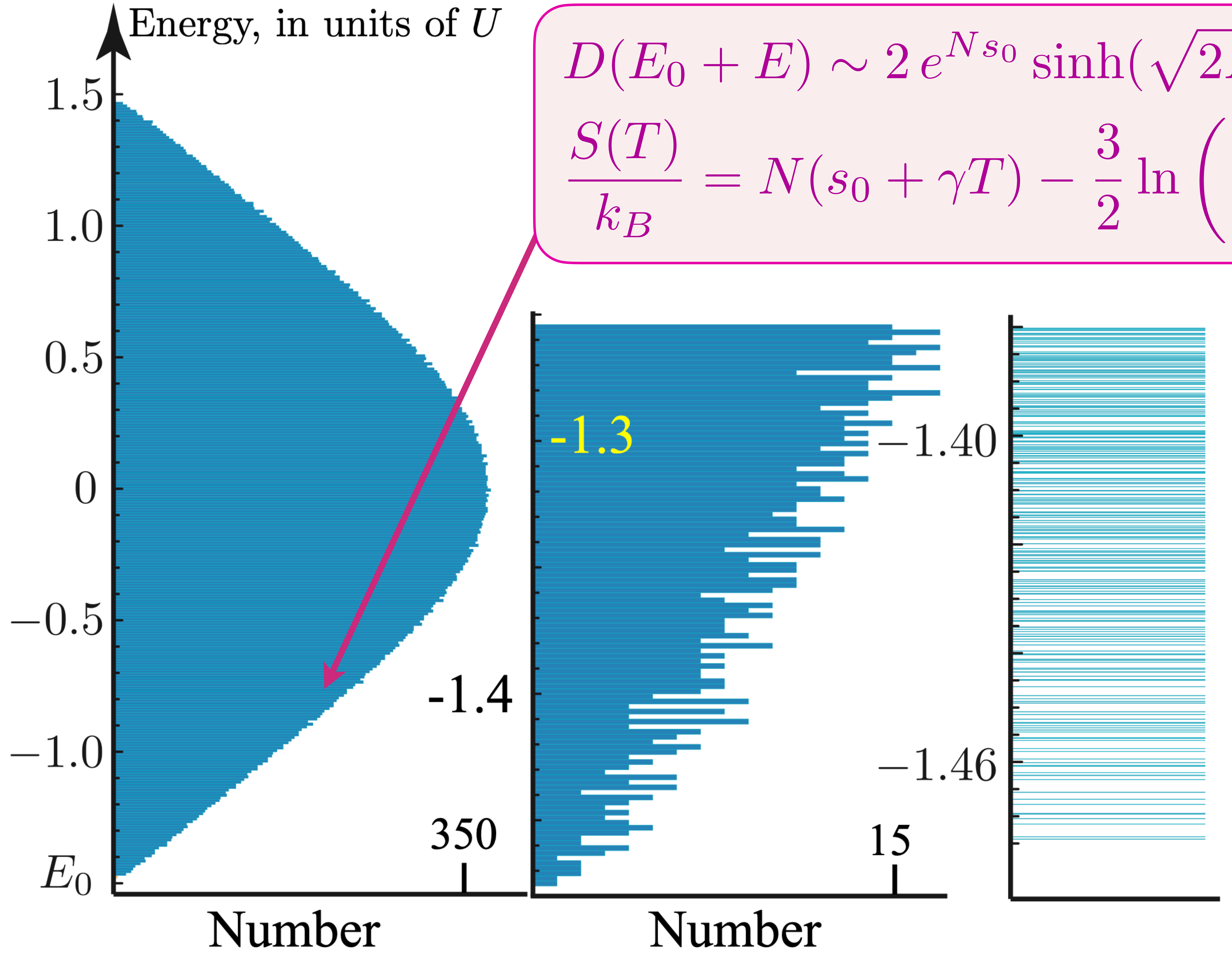
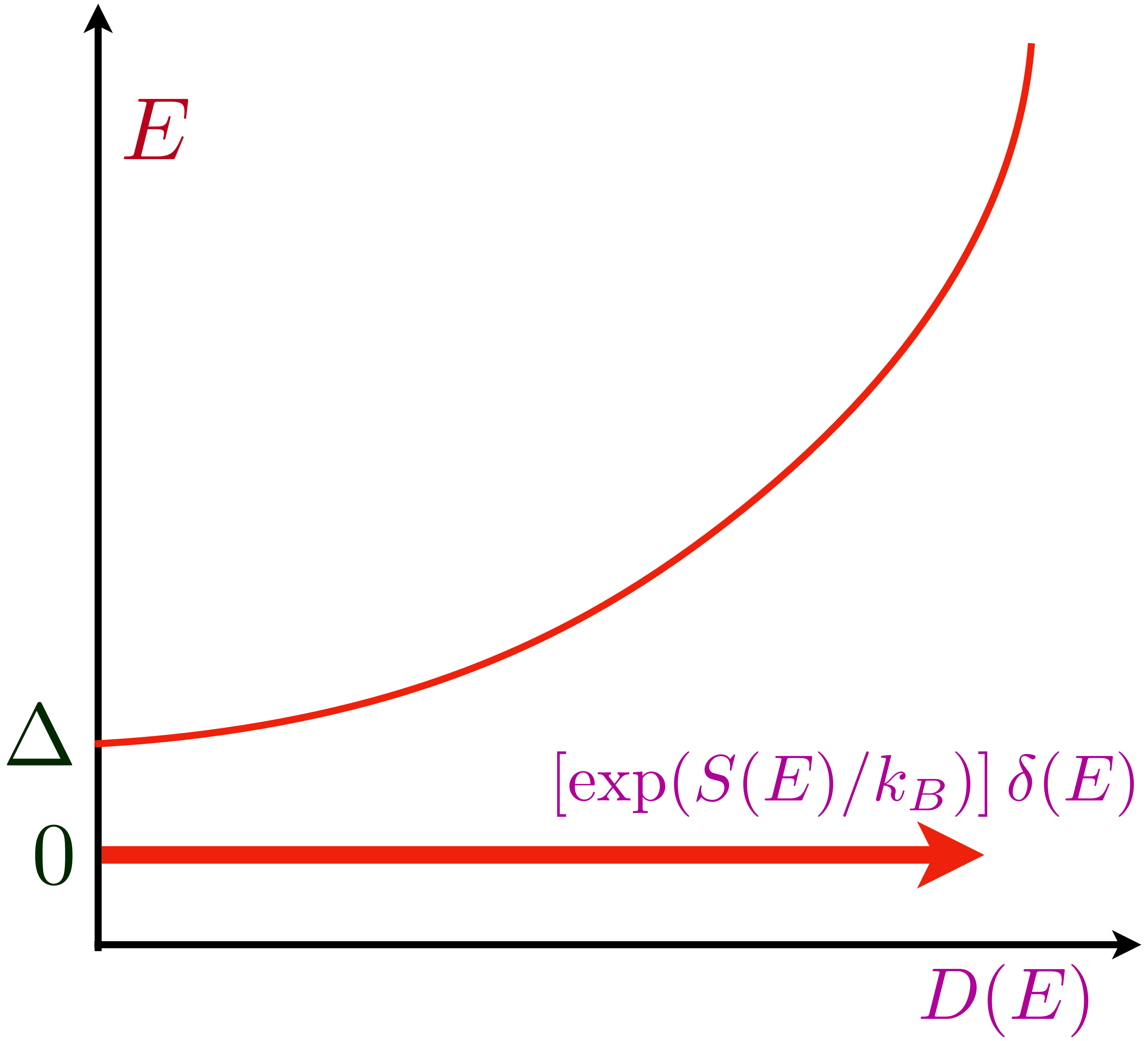
$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{k_B T}\right) + \ldots$	$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi}A_0^{3/2}c^2}{2}\frac{k_B T}{\hbar}\right) - \frac{3}{2}\ln\left(\frac{c^2}{A_0^{1/6}(\hbar G)^{1/3}(k_B T/\hbar)}\right) + \ldots$
$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$
$\lim_{T\to 0}\frac{1}{N k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$	$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$
$D(E) \sim \exp(N s_0)\sinh\left(\sqrt{2N\gamma E}\right)$	$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\sinh\left(\left[\sqrt{\pi}A_0^{3/2}\frac{c^3}{\hbar G}\frac{E}{\hbar c}\right]^{1/2}\right)$



$$D(E_0 + E) \sim 2\, e^{N s_0} \sinh(\sqrt{2N\gamma E})$$

$$\frac{S(T)}{k_B} = N(s_0 + \gamma T) - \frac{3}{2} \ln\left(\frac{U}{T}\right)$$

SYK model
or
charged black hole



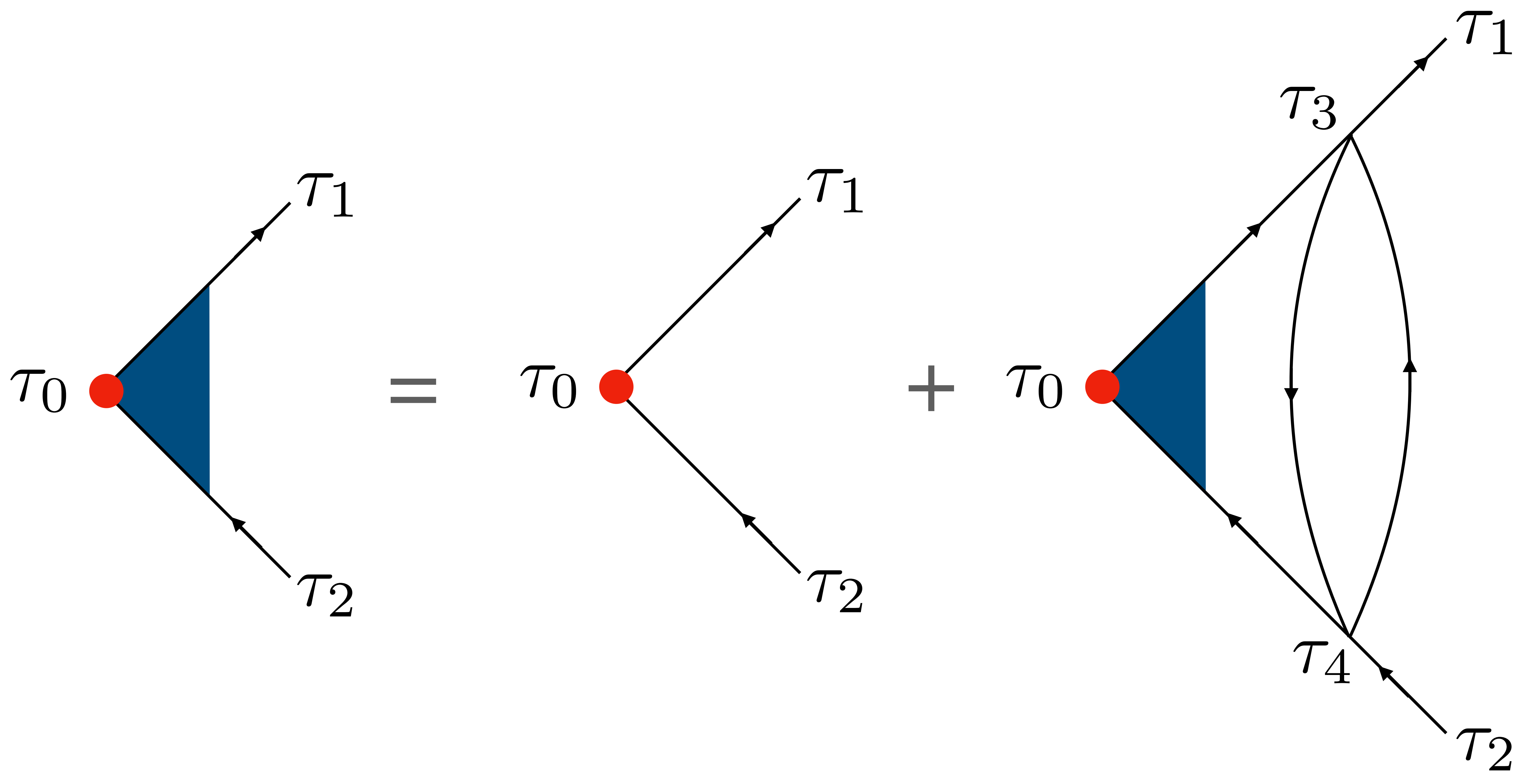
charged black hole
in string theory
or
supersymmetric SYK model
or
supersymmetric CFT

Corrections to scaling at the SYK saddle point

Conformal Perturbation theory

$$S = S_{\mathrm{CFT}} + \sum_h g_h \int_0^\beta d\tau O_h(\tau)$$

where $G_{\mathrm{CFT}} = G_* \sim \mathrm{sgn}(\tau)/\sqrt{|\tau|}$ and $\langle O_h(\tau) O_h(0)\rangle \sim 1/|\tau|^{2h}$



Gross, Rosenhaus (2017)
Klebanov, Tarnopolsky (2017)

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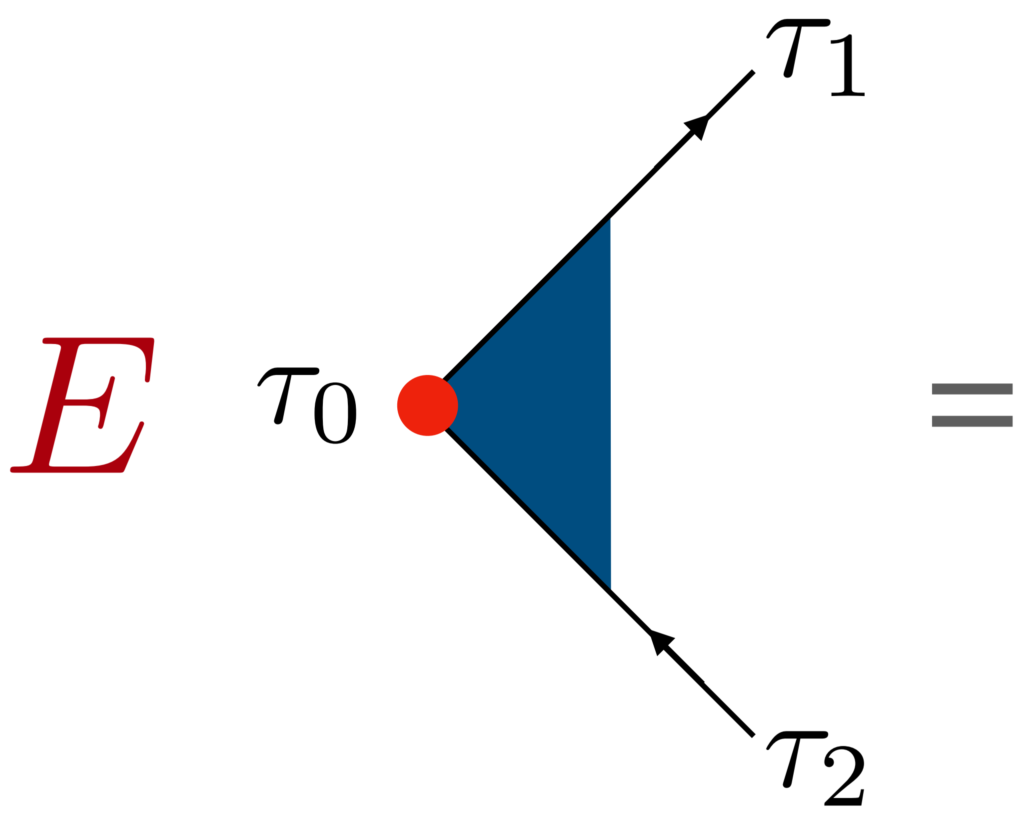
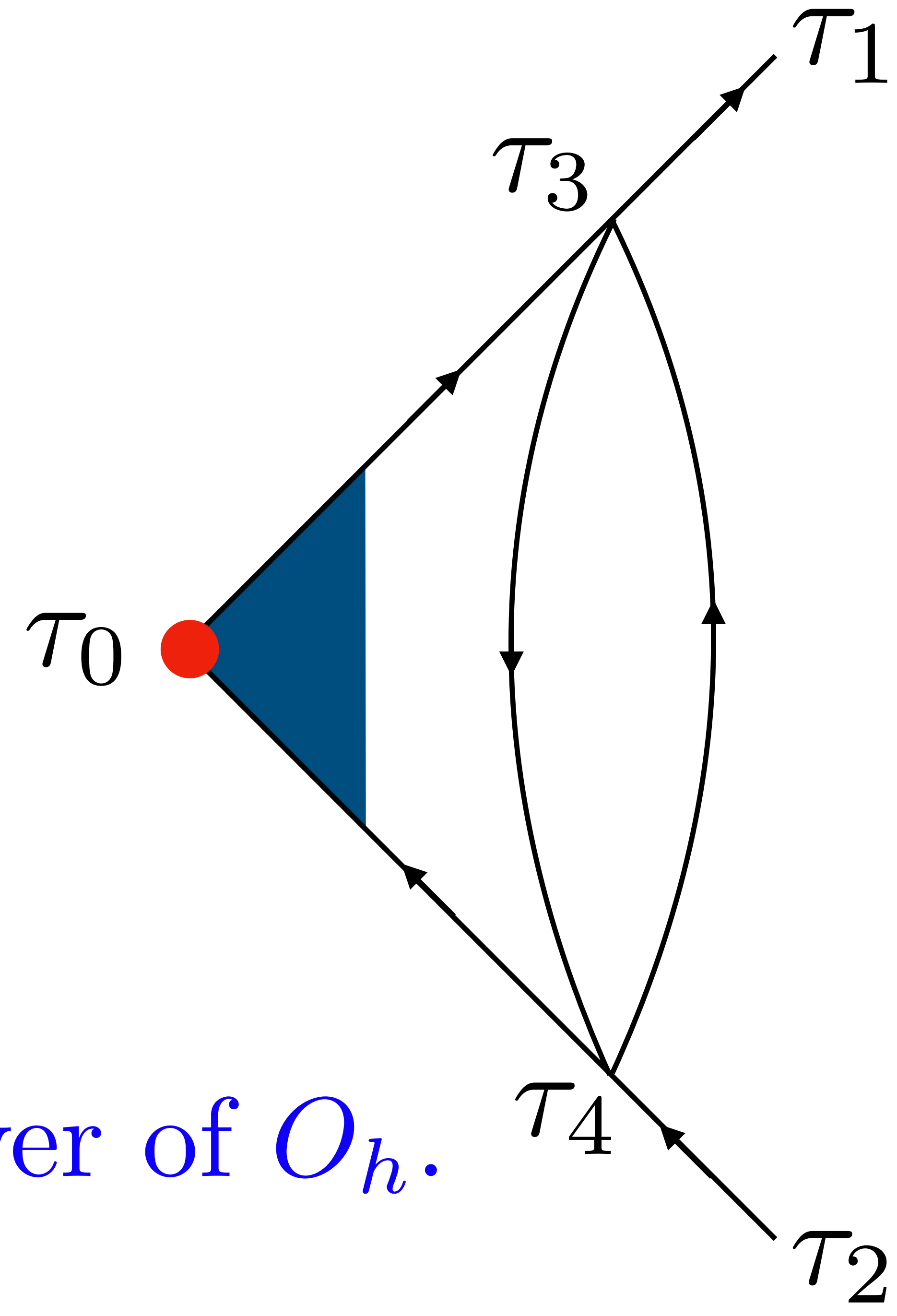




Solution of eigenvalue equation with $E = 1$ yields a tower of O_h.

$$G(\tau) \sim \frac{\mathrm{sgn}(\tau)}{\sqrt{|\tau|}} \left(1 + \sum_h \frac{g_h}{|\tau|^{h-1}} + \ldots \right) \quad , \quad S(T) = N \left[s_0 + \sum_h s_h T^{h-1} + \ldots \right]$$

Gross, Rosenhaus (2017)
Klebanov, Tarnopolsky (2017)

Conformal Perturbation theory

We define the three point function

$$v_h(\tau_1, \tau_2, \tau_0) = \langle c(\tau_1) c^\dagger(\tau_2) O_h(\tau_0) \rangle \,.$$

In the long time scaling limit, we can drop the bare first time on the right hand side, and obtain the eigenvalue equation

$$Ev(\tau_1, \tau_2, \tau_0) = \int d\tau_3 d\tau_4 K(\tau_1, \tau_2; \tau_3, \tau_4) v_h(\tau_3, \tau_4, \tau_0) \,,$$

where the kernel K is

$$K(\tau_1, \tau_2; \tau_3, \tau_4) = -3U^2 G_*(\tau_{13}) G_*(\tau_{24}) G_*(\tau_{34})^2 \,,$$

with $\tau_{ij} \equiv \tau_i - \tau_j$, and we are interested in the eigenvalue $E = 1$. We can use the limit $\tau_0 \to \infty$, where we can assume $v \sim \text{sgn}(\tau_{12})/|\tau_{12}|^{1/2-h}$; then the eigenvalue equation is

$$E = -\frac{3 \tan(\pi h/2 - \pi/4)}{2h - 1} = 1 \,.$$

There are an infinite number of solutions, and the lowest values are $h = 2$, $3.77354\ldots$, $5.567946\ldots$, $7.63197\ldots$, $\ldots$. Consequently, the low T behavior of the entropy is

$$S(T) = N \left[s_0 + \gamma \, T + \gamma_2 \, T^{2.77354\ldots} + \ldots \right] \,.$$

We will have a particular interest in the $h = 2$ operator in the remaining discussion.

Random *t-J* model

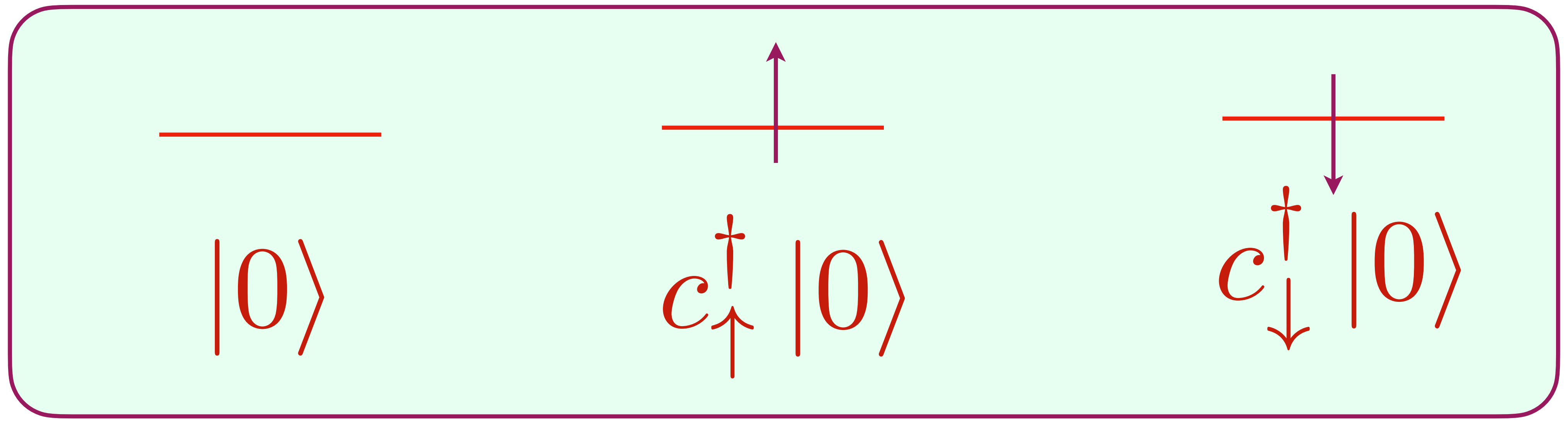
$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij}\, c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$$

We consider the hole-doped case, with no double occupancy.

$$\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij}\delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0$$

$$\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N}\sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

$$J_{ij} \text{ random}, \ \overline{J_{ij}} = 0, \ \overline{J_{ij}^2} = J^2$$
$$t_{ij} \text{ random}, \ \overline{t_{ij}} = 0, \ \overline{t_{ij}^2} = t^2$$

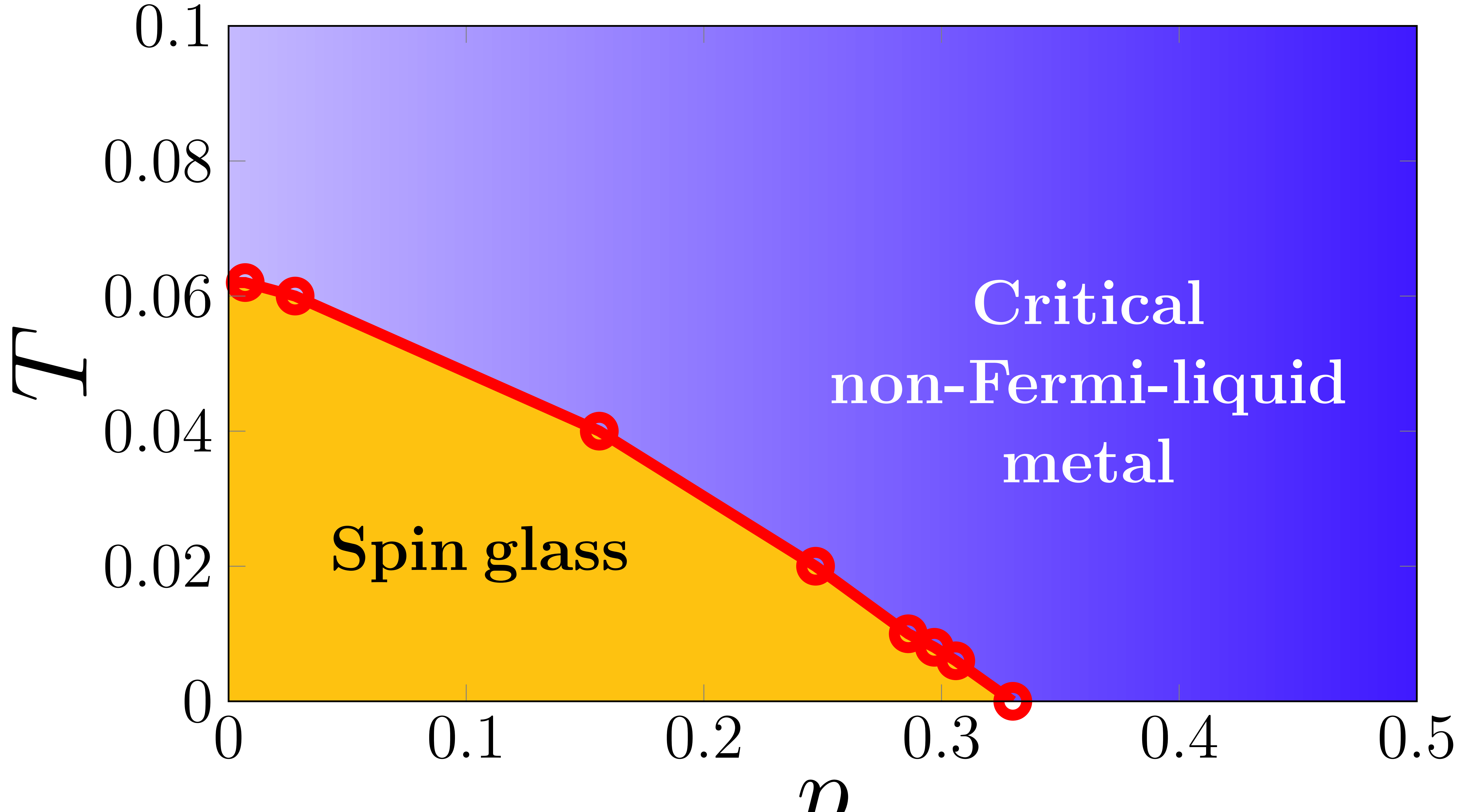


Random *t-J* model

Solvable in a SYK-like large M limit after fractionalizing $c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons

$$G_b(i\omega_n) = \frac{1}{i\omega_n + \mu_b - \Sigma_b(i\omega_n)}, \quad G_f(i\omega_n) = \frac{1}{i\omega_n + \mu_f - \Sigma_f(i\omega_n)}$$

$$\Sigma_b(\tau) = -t^2 G_f(\tau) G_f(-\tau) G_b(\tau), \quad \Sigma_f(\tau) = -J^2 G_f(\tau)^2 G_f(-\tau) + k t^2 G_f(\tau) G_b(\tau) G_b(-\tau)$$





M. Christos,
D. G. Joshi,
S. S. and
M. Tikhanovskaya,

arXiv:2203.16548

Random *t-J* model

Solvable in a SYK-like large M limit after fractionalizing
$c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons
or $c_\alpha = \mathfrak{b}_\alpha \mathfrak{f}^\dagger$ into bosonic spinons and fermionic holons

Metallic spin glass

Condense spinon $\mathfrak{b}_\alpha$.

$$\langle \boldsymbol{S}(\tau) \cdot \boldsymbol{S}(0) \rangle \sim \text{constant}$$

Critical metal

Holon: $\left\langle b(\tau) b^\dagger(0) \right\rangle \sim \dfrac{1}{\tau^{2\Delta_b}}$

Spinon: $\left\langle f_\alpha(\tau) f_\alpha^\dagger(0) \right\rangle \sim \dfrac{1}{\tau^{2\Delta_f}}$

$$\Delta_b + \Delta_f = 1/2, \quad 0 < \Delta_b < 1/4.$$

$$\langle \boldsymbol{S}(\tau) \cdot \boldsymbol{S}(0) \rangle \sim \frac{1}{\tau^{4\Delta_f}}$$

$$\left\langle c_\alpha(\tau) c_\alpha^\dagger(0) \right\rangle \sim \frac{1}{\tau}$$

p_c $\qquad$ p

M. Christos,
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arXiv:2203.16548

Random *t-J* model

Solvable in a SYK-like large M limit after fractionalizing $c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons

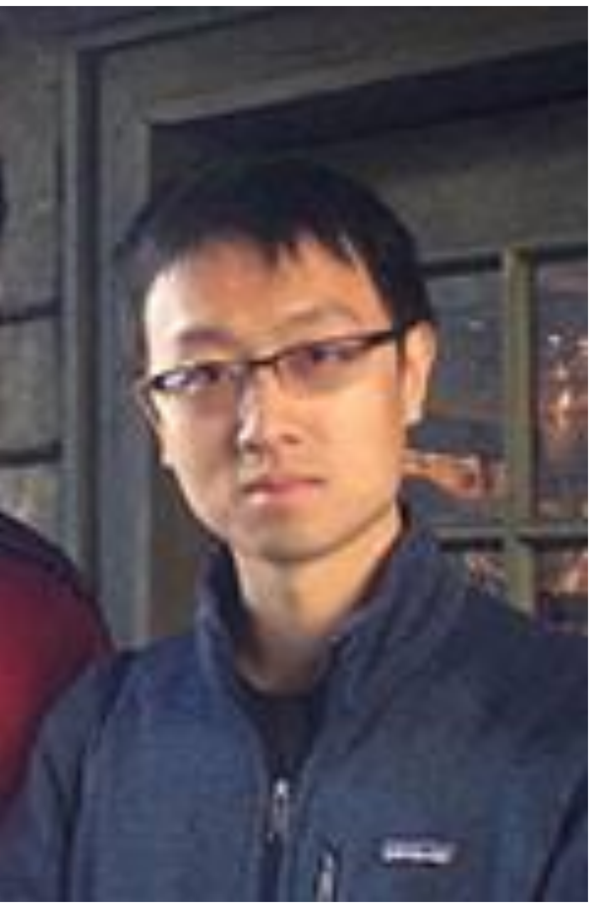
The $h = 2$ operator now leads to corrections to the Green's functions of the partons

$$G_{f,b}(\tau) \sim \frac{\pm 1}{\sqrt{|\tau|}} \left(1 + \frac{\alpha_{f,b}}{|\tau|} + \ldots \right)$$

We can compute the resistivity from this in a large-d model, and find as $T \to 0$ that

$$\rho(T) = \rho(0) \left(1 + \alpha_\rho \frac{T}{J} + \ldots \right) .$$

The linear-T term arises from the $h = 2$ operator, which we will see is a 'time reparameterization soft-mode', and a 'boundary graviton' in the charged black hole.

Haoyu Guo,
Yingfei Guo,
S. Sachdev,
Annals of Physics **418**, 168202 (2020)



M. Christos,
D. G. Joshi,
S. S. and
M. Tikhanovskaya,
arXiv:2203.16548

The Schwarzian theory: accounting for the $h=2$ operator exactly in the SYK model as a time reparameterization soft-mode

G-Σ path integral

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp\left[-\sum_{ia}\int_0^\beta d\tau\, c_{ia}^\dagger \left(\frac{\partial}{\partial\tau} - \mu\right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3}\sum_{ab}\int_0^\beta d\tau d\tau' \left|\sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau')\right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2) \exp\left[-N\int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1,\tau_2)\left(G(\tau_2,\tau_1) \right.\right.$$
$$\left.\left. + \frac{1}{N}\sum_i c_i(\tau_2) c_i^\dagger(\tau_1)\right)\right] .$$

G-Σ path integral

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

G-Σ path integral

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-E_0/T+Ns_0-NS_{\text{eff}}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Symmetries of the large N saddle point

Let us write the large N saddle point solutions of S as

$$G_s(\tau_1-\tau_2) \quad \sim \quad (\tau_1-\tau_2)^{-1/2}$$
$$\Sigma_s(\tau_1-\tau_2) \quad \sim \quad (\tau_1-\tau_2)^{-3/2}.$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1,\tau_2)=G_s(\tau_1-\tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1,\sigma_2)=G_s(\sigma_1-\sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau)=\frac{a\tau+b}{c\tau+d} \quad , \qquad ad-bc=1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

Symmetries of the large *N* saddle point

- The saddle-point

$$G(\tau_1 - \tau_2) = -A \frac{e^{-2\pi\mathcal{E}T(\tau_1-\tau_2)}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T(\tau_1 - \tau_2))} \right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a \dfrac{\tan(\pi T \tau)}{\pi T} + b}{c \dfrac{\tan(\pi T \tau)}{\pi T} + d} \quad , \quad ad - bc = 1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi\mathcal{E}T(\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

G-Σ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)),\tau\right\},$$

where $f(\tau)$ is a monotonic map from $[0,1/T]$ to $[0,1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T=0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau+b}{c\tau+d}, \phi(\tau)=0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Low temperature thermodynamics: for $k_B T \ll U$

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr} \exp\left(-\frac{\mathcal{H}}{k_B T}\right) \\ &= \exp\left(N \frac{S_0}{k_B}\right) \int \frac{\mathcal{D}f(\tau)\mathcal{D}\phi(\tau)}{\|\mathrm{SL(2,R)}\|} \exp\left(-\frac{1}{\hbar} S_{\text{eff}}\left[f(\tau), \phi(\tau)\right]\right) \end{aligned}$$

- Feynman path integral over $f(\tau)$, the reparameterization of the time of the SYK model, and $\phi(\tau)$ a phase conjugate to the total charge $\mathcal{Q}$.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
A. Kitaev (2015)
J. Maldacena and D. Stanford, Phys. Rev. D **94,** 106002 (2016)

The Schwarzian theory: boundary graviton (a time reparameterization soft-mode) in Einstein-Maxwell theory of charged black holes

Quantum path integral for charged black holes

1. Reduce the 4-spacetime dimensional theory in I_{EM} to a 1+1 dimensional theory $I_{EM,2}$ by taking all fields dependent only upon the radial co-ordinate r and imaginary time τ.

2. Take the low energy limit of $I_{EM,2}$ by mapping it to a near-horizon theory, I_{JT}, in a 1+1 dimensional spacetime with a boundary.

3. Compute fluctuations about the AdS_2 saddle point of I_{JT}. Einstein gravity in 1+1 dimensions has no graviton, and is 'pure gauge'. In the JT-gravity theory with boundary, there is a remnant degree of freedom which is a boundary graviton. The action for this boundary graviton is the Schwarzian theory. The partition function of this Schwarzian theory can be evaluated exactly.

Quantum path integral for charged black holes

1. Make the metric ansatz

$$ds^2 = \frac{ds_2^2}{\Phi(\zeta,\tau)} + [\Phi(\zeta,\tau)]^2 \, d\Omega_2^2$$

where ds_2^2 is an arbitrary metric in the (ζ,τ) spacetime, and Φ is a scalar field in the (ζ,τ) spacetime.

2. The low energy theory on the (ζ,τ) spacetime involves a metric h, and a scalar field Φ_1 given by $\lim_{\zeta\to\infty}[\Phi(\zeta,\tau)]^2 = R_h^2 + \Phi_1(\zeta,\tau)$, obeying the action

$$I_{JT} = -\frac{2\pi\mathcal{A}_0}{\kappa^2} + \int d^2x\sqrt{h}\left[-\frac{2\pi}{\kappa^2}\,\Phi_1\left(\mathcal{R}_2 + \frac{2}{R_h^3}\right)\right] - \frac{4\pi}{\kappa^2}\int_\partial dx\sqrt{h_b}\Phi_1\,\mathcal{K}_1$$

where $\mathcal{A}_0 = 4\pi R_h^2$ is the area of the horizon at $T=0$, and $\mathcal{K}_1$ is the extrinsic curvature of the one-dimensional boundary $\zeta\to 0$ where

$$h_{\tau\tau}(\zeta\to 0) = \frac{R_h^3}{\zeta^2} \quad , \quad \Phi_1(\zeta\to 0) = \frac{2R_h^3}{\zeta}$$

Quantum path integral for charged black holes

3. Remarkably, the partition function of the $1+1$ dimensional JT gravity theory can be evaluated exactly (here we are ignoring the gauge field path integral, which is subdominant at fixed $\mathcal{Q}$)

$$\mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}h\mathcal{D}\Phi_1 \exp\left(-I_{JT}\right)$$

The action is linear in Φ_1, and the integral over Φ_1 yields a constraint $\mathcal{R}_2 = -2/R_h^3$ *i.e.* the metric h is rigidly AdS_2. The only dynamical degree of freedom in JT gravity is a time reparameterization along the boundary $\tau \to f(\tau)$. To ensure that the bulk metric obeys its boundary condition, we also have to make the spatial co-ordinate ζ a function of τ, so we map $(\tau, \zeta) \to (f(\tau), \zeta(\tau))$. Then the metric obeys its boundary condition provided $\zeta(\tau)$ is related to $f(\tau)$ by (here ζ_b is a small constant whose value cancels in the final result)

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \frac{\left[f''(\tau)\right]^2}{2f'(\tau)} + \mathcal{O}(\zeta_b^4)$$

Finally, we evaluate I_{GH} along this boundary curve. In this manner we obtain the action

$$I_{1,\text{eff}}[f] = -\frac{2\pi\mathcal{A}_0}{\kappa^2} - \frac{\gamma}{4\pi^2}\int d\tau \left\{f(\tau), \tau\right\} \quad , \quad \{f,\tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$ is precisely the linear-T co-efficient in the black hole entropy.

Quantum path integral for charged black holes

3. After a conformal map to finite temperature (and ignoring the contribution of the gauge field fluctuation), we can write the low energy partition function of a 3+1-dimensional black hole with charge $\mathcal{Q} = 4\pi R_h/(\kappa g_F)$, as a path integral over a single field $f(\tau)$ in one time dimension:

$$\mathcal{Z}_{\mathcal{Q}} = \exp\left(\frac{2\pi\mathcal{A}_0}{\kappa^2}\right) \int \frac{\mathcal{D}f}{||\text{SL(2,R)}||} \exp\left(\frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \, \{\tan(\pi T f(\tau)), \tau\}\right)$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$, $\mathcal{A}_0 = 4\pi R_h^2$, and $f(\tau)$ is a monotonic function of τ obeying

$$f(\tau + 1/T) = f(\tau) + 1/T\,.$$

We divide by the (infinite) volume of the SL(2,R) group because

$$\{f, \tau\} = \{\frac{af+b}{cf+d}, \tau\}$$

where a, b, c, d are constants with $ad - bc = 1$.

SYK model	Charged black holes
$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2} \ln \left(\frac{N^{1/3} U}{k_B T} \right) + \ldots$	$\frac{S(T)}{k_B} = \frac{1}{\hbar G} \left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2} \frac{k_B T}{\hbar} \right) - \frac{3}{2} \ln \left(\frac{c^2}{A_0^{1/6} (\hbar G)^{1/3} (k_B T/\hbar)} \right) + \ldots$
$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau} \left(\frac{T}{\sin(\pi T \tau)} \right)^{2\Delta}$
$\lim_{T \to 0} \frac{1}{N k_B} \frac{\partial S}{\partial Q} = 2\pi \mathcal{E}$	$\lim_{T \to 0} \frac{1}{k_B} \frac{\partial S}{\partial Q} = 2\pi \mathcal{E}$
$D(E) \sim \exp(N s_0) \sinh\left(\sqrt{2 N \gamma E} \right)$	$D(E) \sim \exp\left(\frac{A_0 c^3}{4 \hbar G} \right) \sinh \left(\left[\sqrt{\pi} A_0^{3/2} \frac{c^3}{\hbar G} \frac{E}{\hbar c} \right]^{1/2} \right)$

Large d t-J model with random J_{ij} and SU$(M \to \infty)$ symmetry: resistivity $\rho(T) = \rho(0) \left[1 + \alpha_\rho\, (T/J) + \ldots \right]$ in a critical metal phase as $T \to 0$.