ENTANGLEMENT IN DISORDERED SYSTEMS

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FOREWORD

Quantum theory provides an extremely accurate description of fundamental processes in physics.

Quantum Entanglement is a complex and delicate (spooky) quantum phenomenon (Einstein et al 1935, Schrödinger 1935, Bell 1960') and a widely believed resource of quantum information science (Deutsch, Feynman, Manin 1980').

It is the Q-analog of the statistical dependence in probability theory implemented in a special kind of correlations that quantum systems can have even when no forces or other influence links them.

Entanglement plays an important role in quantum information science (coding, computation, etc.), quantum statistical mechanics (quantum phase transitions, thermalization, etc.), cosmology (black holes), etc...

PROGRAM of the COURSE

1. Introduction: basic notions of quantum mechanics, bipartite systems, entanglement, reduced density matrix, entanglement quantifiers. A toy model of black hole radiation.
2. Dynamics of two qubits in random environment: general setting, models of environment, basic approximations, random matrix environment (analytical and numerical results).
3. Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities (dedicated to the Nobel Prize in Physics 2022)
4. Entanglement in extended systems: setting and basic facts for translation invariant systems.
5. Entanglement for free disordered fermions: Anderson localization, area law and its violations.

- Quantum Mechanics I
- Notation and Comments
- Quantum Mechanics II
- Entanglement Entropy of the Black Hole Radiation
  - Model
  - Previous Results
  - Modern Form

Consider a mathematical scheme pertinent to any theoretical description of physical systems (although not always in explicit form).

1. **States.** A state is a complete description of a physical system.
2. **Observables.** An observable is a property (a characteristic) of a physical system.
3. **Dynamics.** A dynamics is given by an equation that describes the time evolution of observables and/or states.
4. **Measurements.** A measurement is a procedure that provides a numerical value of a given observable in a given state of a physical system.

Here are two important implementation of the scheme

Theory $\Rightarrow$ Elements $\Downarrow$	Classical Mechanics	Quantum Mechanics
States	**Pure** (Newton) : points of phase space $\mathcal{M}$ (a manifold) ; **Mixed** (Liouville): probability distributions (measures) on $\mathcal{M}$.	**Pure** (Schrödinger): vectors of state space $\mathcal{H}$ (a Hilbert space); **Mixed** (von Neumann) density matrices (p.d.o. of trace 1) in $\mathcal{H}$.
Observables	**Functions** on $\mathcal{M}$.	**Hermitian operators** in $\mathcal{H}$.
Dynamics (equations of motion)	**Observables** (Hamilton); **States:** (Liouville).	**Observables** Heisenberg) ; **States** (Schrödinger, von Neumann).
Measurements: observables + states	Integrals of product of functions and a measure of total mass 1.	Traces of product of hermitian operators and a p.d.o. of trace 1.

Dirac Notations

Hilbert space of dimension n: $\mathcal{H}$, $\dim \mathcal{H} = n$: the collections of n-tuples

$$"ket" : |x\rangle = (x_1, \ldots x_n) =: \{x_j\}_{j=1}^n, \ "bra" : \langle x| = (\overline{x_1}, \ldots \overline{x_n}) =: \{\overline{x_j}\}_{j=1}^n,$$

equipped with the operations of superposition

$$\alpha|x\rangle + \beta|y\rangle = |\alpha x + \beta y\rangle = \{\alpha x_j + \beta y_j\}_{j=1}^n$$

and the inner (scalar) product

$$\langle x|y\rangle = \sum_{j=1}^n \overline{x_j} y_j, \ \overline{\langle x|y\rangle} = \langle y|x\rangle,$$

$$\langle x|\beta' y' + \beta'' y''\rangle = \langle x|\beta' y'\rangle + \langle x|\beta'' y''\rangle = \beta'\langle x|y'\rangle + \beta''\langle x|y''\rangle.$$

$$\langle x|x\rangle = \sum_{j=1}^n |x_j|^2 =: ||x||^2,$$

Linear Operators (Matrices) in $\mathcal{H}$: linear maps $A : \mathcal{H} \to \mathcal{H}$

$$\begin{aligned} A \quad &: \quad \mathcal{H} \to \mathcal{H}, \ A|x\rangle := |Ax\rangle \\ A(\alpha|x\rangle + \beta|y\rangle) \quad &= \quad \alpha|Ax\rangle + \beta|Ay\rangle, \ (\alpha A + \beta B)|x\rangle = \alpha A|x\rangle + \beta B|x\rangle, \end{aligned}$$

or the collection of $n \times n$ "tables"

$$A = \{A_{jk}\}_{j,k=1}^{d} = \begin{pmatrix} A_{11} & A_{12} & . & . & . & A_{1n} \\ A_{21} & A_{21} & . & . & . & A_{21} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{n1} & A_{n2} & . & . & . & A_{nn} \end{pmatrix}, \ (A|x\rangle)_j = \sum_{k=1}^{n} A_{jk} x_k,$$

equipped with the operation of multiplication ("row by column")

$$AB = \{(AB)_{jk}\}_{j,k=1}^{n}, \ (AB)_{jk} = \sum_{l=1}^{n} A_{jl} B_{lk}, \ j,k = 1, \dots n.$$

Examples:

- Diagonal matrix $\{d_j \delta_{jk}\}$, unit matrix $\mathbf{1}_n := \{\delta_{jk}\}_{j,k=1}^n$
- Pauli matrices

$$\sigma_0 = \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma_1\sigma_3,$$

hence, matrix multiplication is noncommutative,

$$\text{commutator } [A, B] := AB - BA \neq 0 \ \text{ in general.}$$

The set of linear operators in $\mathcal{H}$ is denoted $B(\mathcal{H})$ and is an noncommutative algebra (operations of addition and noncommutative multiplication), while in a Hilbert space we have only the operation (addition).

- Hermitian operators: $A = A^*,\ A = \{A_{jk}\}_{j,k=1}^n,\ A^* = \{\overline{A_{kj}}\}_{j,k=1}^n$.
- Positive definite operators (p.d.o.): $\langle x|Ax\rangle = \langle A^* x|x\rangle =: \langle x|A|x\rangle \geq 0$.
- Rank-one operators: $|u\rangle\langle v| = \{u_j \overline{v_k}\},\ |u\rangle\langle v| \cdot |x\rangle = \langle v|x\rangle|u\rangle$.
- Orthogonal projection: $P_u = |u\rangle\langle u|,\ ||u|| = 1,\ P_u^2 = P_u$, since $|u\rangle\langle u||x\rangle = \langle u|x\rangle|u\rangle$.

- Spectral theorem for hermitian operators

$$A = \sum_{t=1}^n a_t|\psi_t\rangle\langle\psi_t|,\ f(A) = \sum_{t=1}^n f(a_t)|\psi_t\rangle\langle\psi_t|,\ f : \mathbb{R} \to \mathbb{C}$$

eigenvalues $\{a_t \in \mathbb{R}\}_{t=1}^n$, eigenvectors $\{|\psi_t\rangle\}_{t=1}^n,\ \langle\psi_{t'}|\psi_{t''}\rangle = \delta_{t't''}$,

$\rho = \sum_{t=1}^n p_t|\psi_t\rangle\langle\psi_t|,\ 0 \leq p_i \leq 1$ is a mixture of pure states.

Trace of operator:

$$\mathrm{Tr}\, A = \sum_{j=1}^n A_{jj},\ \mathrm{Tr}\, f(A) = \sum_{j=1}^n f(a_j),$$

Tensor product of Hilbert spaces: given $\mathcal{H}', \ \dim \mathcal{H}' = n'$ and $\mathcal{H}'', \ \dim \mathcal{H}'' = n''$, define $\mathcal{H}' \otimes \mathcal{H}''$ as the collection of $n'n''$ tuples

$$x = \{x_{jk}\}_{j,k=1}^{n',n''},$$

a discrete analog of functions of two variables.

In particular, if $|x'\rangle = \{x_j'\}_{j=1}^{n'} \in \mathcal{H}', \ |x''\rangle = \{x_k''\}_{k=1}^{n''} \in \mathcal{H}''$, then

$$|x'\rangle \otimes |x''\rangle := |x'x''\rangle = \{x_j'x_k''\}_{j,k=1}^{n',n''}$$

is the *tensor product of vectors.*

Operators in $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$:

$$\begin{aligned} A &= \{A_{j'j'',k'k'',}\}_{j'k'=1,j''k''=1}^{n'n''}, \\ (A|x\rangle)_{j'j''} &= \sum_{k'k''=1}^{n''} A_{j'j'',k'k''}x_{k'k''}, \ |x\rangle = \{x_{k'k''}\}_{k',k''=1}^{n'n''} \in \mathcal{H}' \otimes \mathcal{H}''. \end{aligned}$$

In particular, if $A' \in \mathcal{H}', \; A'' \in \mathcal{H}''$, then

$$(A' \otimes A'')_{j'j'',k'k''} = A'_{j'k'}A''_{j''k''}, \; A' \otimes A''|x'\rangle \otimes |x''\rangle = A'|x'\rangle \otimes A''|x''\rangle.$$

is the *tensor product of operators*, in particular

$$A' \otimes \mathbf{1}''|x'\rangle \otimes |x''\rangle = A'|x'\rangle \otimes |x''\rangle (cf. \text{ partial derivative}).$$

Partial trace: if $A = \{A_{j'j'',k'k'',}\}_{j'k'=1,j''k''=1}^{n'n''} \in \mathcal{H}' \otimes \mathcal{H}''$, then

$$(\mathrm{Tr}_{\mathcal{H}'} A)_{j''k''} = \sum_{j'=k'=1}^{n'} A_{j'j'',k'k''},$$

$$\mathrm{Tr}_{\mathcal{H}'}(A' \otimes A'') = (\mathrm{Tr}_{\mathcal{H}'} A')\, A'', \; \mathrm{Tr}_{\mathcal{H}'} |u^{'}\rangle\langle v^{'}| \otimes |u''\rangle\langle v''| = \langle v'|u'\rangle |u''\rangle\langle v''|.$$

Dirac notation is rather convenient, because virtually anything can put inside a "bra" and a "ket" as long as its meaning is not ambiguous.

Comments

Scheme again

Theory $\Rightarrow$ Elements $\Downarrow$	Classical Mechanics	Quantum Mechanics
States	**Pure**: $x \in \mathcal{M}$ - phase space; **Mixed**: $\mu : \mathcal{M} \to \mathbb{R}_+$, $\int_{\mathcal{M}} \mu(x)dx = 1$.	**Pure**: $\psi \in \mathcal{H}$ - state space; **Mixed**: p.d.o. $\rho \in B(\mathcal{H})$, $\mathrm{Tr}\, \rho = 1$.
Observables	Real-valued $f : \mathcal{M} \to \mathbb{R}$.	Hermitian $A \in$B$(\mathcal{H})$.
Dynamics	**Observables**: $\frac{\partial f}{\partial t} = \{H, f\}$; **States** $\frac{\partial \mu}{\partial t} = -\{H, \mu\}$.	**Observables**: $\hbar \frac{\partial A}{\partial t} = i[H, A]$; **States**: $\hbar \frac{\partial \rho}{\partial t} = -i[H, \rho]$.
Measurements	$\overline{f} = \int_{\mathcal{M}} f(x)\mu(x)dx$	$\overline{A} = \mathrm{Tr}\, A\rho$

Comments

- Born rule: Given an observable $O = \sum_t o_t \langle \psi_t |$ and a pure state ψ,

$$\mathbf{P}_\psi\{O \doteq \ o_t\} := \mathbf{P}\{O = o_t |\psi_t\rangle\langle\psi_t|\} = |\langle\psi_t|\psi\rangle|^2 = \mathrm{Tr}\, |\psi_t\rangle\langle\psi_t| \cdot |\psi\rangle\langle\psi|.$$

- Classical observables commute while quantum observables do not commute, manifesting "quantumness" via the uncertainty principle.

- Quantum pure states obey the superposition principle, manifesting quantumness via constructive and destructive superposition.

Classical mechanics (probability theory):

$$\mu_1, \mu_2, \text{ states } \Longrightarrow \mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2, \text{ a state.}$$

Quantum mechanics:

$$\psi_1, \psi_2, \text{ pure orthogonal states } \Longrightarrow \psi = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2, \text{ a pure state.}$$

For instance, for a classical particle

$$\mathbf{P}_{\mu}\{\text{particle} \in dx\} = \frac{1}{2}\mathbf{P}_{\mu_1}\{\text{particle} \in dx\} + \frac{1}{2}\mathbf{P}_{\mu_2}\{\text{particle} \in dx\},$$

while, for a quantum particle (*Born rule*)

$$\begin{aligned} \mathbf{P}_{\psi}\{\text{particle} &\in& dx\} = |\psi(x)|^2, \ (\psi(x) \text{ is the probability amplitude}) \\ &=& \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \Re\psi_1(x)\overline{\psi_2(x)} = \\ &=& \frac{1}{2}\mathbf{P}_{\psi_1}\{\text{particle} \in dx\} + \frac{1}{2}\mathbf{P}_{\psi_2}\{\text{particle} \in dx\} + \mathbf{T}_Q(x) \end{aligned}$$

and for $\psi_a = |\psi_a|e^{i\alpha_a} = |\mathbf{P}_a|^{1/2}e^{i\alpha_a}, \ a = 1, 2$

$$\mathbf{P}_{\psi}\{\text{particle} \in dx\} = \begin{cases} 2^{-1}(\mathbf{P}_1 + \mathbf{P}_2), & \alpha_1 - \alpha_2 = 0, \pm 2\pi, \ \text{cl. mech.,} \\ 2^{-1}|\mathbf{P}_1 - \mathbf{P}_2|, & \alpha_1 - \alpha_2 = \pm\pi, \ \text{q. mech..} \end{cases}$$

$\mathbf{T}_Q$ is due to the "interference" of $\psi_a, \ a = 1, 2$, recall wave mechanics by Schrödinger.

Bipartite System: $\mathcal{S}_{AB} = \mathcal{S}_A \cup \mathcal{S}_B$ with the state space $\mathcal{H}_{AB}$ $= \mathcal{H}_A \otimes \mathcal{H}_B$.

Given the **Density Matrix** ρ_{AB} of $\mathcal{S}_{AB}$ (Q-analog of joint probability distribution of probability theory), introduce the **Reduced Density Matrix (RDM)** of $\mathcal{S}_A$

$$\rho_A := \mathrm{Tr}_B\, \rho_{AB}$$

(Q-analog of a marginal in probability theory).

A pure state $\Psi_{AB} \in \mathcal{H}_{AB}$ is **entangled** if it is not a **product** state

$$\Psi_A \otimes \Psi_B, \ \Psi_a \in \mathcal{H}_a, \ a = A, B$$

(Q-analog of independence, but do not forget superposition principle!).

A mixed state ρ_{AB} of $\mathcal{S}_{AB}$ is **entangled** if it is not a convex linear combination of **product (separable)** states $\rho_A^j \otimes \rho_B^j, \ j = 1, 2, \dots$:

$$\rho_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j, \ p_j \geq 0, \ \sum_j p_j = 1.$$

"**Entanglement** is the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought" *Schrödinger 1935*.

Decoherence: various processes of entanglement destruction (by **environment**).

Qubit: A basic entity of quantum information science (two-level atom, spin, etc.), $\mathcal{H} = \mathbb{C}^2, \ \dim \mathcal{H} = 2$, with the canonical basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In its simplest form entanglement causes two qubits to share a common pure state but each of them does not have pure state of its own.

How to quantify the entanglement?

Entanglement Entropy

von Neumann Entropy: Given a state ρ of a quantum system with the state space $\mathcal{H}$, introduce

$$S[\rho] := -\operatorname{Tr} \rho \log_2 \rho = -\sum_{t=1}^{n} p_t \log_2 p_t \geq 0.$$

We have:

(i) $\rho = |\Psi\rangle\langle\Psi|$ is pure $\Longleftrightarrow$ $S[\rho] = 0$ (faithfulness)

(ii) if $d_{\mathcal{H}} = \dim \mathcal{H}$, then

$$\max_{\rho} S[\rho] = S[\mathbf{1}_{\mathcal{H}}/d_{\mathcal{H}}] = \log_2 d_{\mathcal{H}}$$

and $\rho = \mathbf{1}_{d_{\mathcal{H}}}/d_{\mathcal{H}}$ is the maximum entangled state.

$S[\rho]$ is the measure (quantifier) of the state mixedness.

It is algebraic, faithful and invariant respect to unitary transformations of the state space.

If ρ_{AB} is a state of $\mathcal{S}_{AB}$, $\rho_A := \mathrm{Tr}_B\, \rho_{AB}$ is the RDM of $\mathcal{S}_A$, then

Entanglement Entropy of $\mathcal{S}_A$ (with the rest $\mathcal{S}_B$ of $\mathcal{S}_{AB}$) is

$$S_A := S[\rho_A] = -\,\mathrm{Tr}_A\, \rho_A \log_2 \rho_A.$$

It is algebraic, invariant with respect to local (in $\mathcal{S}_A$ and $\mathcal{S}_B$ only) unitary transformations.

A Bell State: the pure state of two qubits, i.e., $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 2$

$$|\Phi\rangle = (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)/\sqrt{2}.$$

Pure, hence, not entangled and of zero entropy. The RDM $\rho_A = \mathbf{1}_A/2$ is maximum entangled and of maximum entropy $\log_2 2 = 1$ manifesting the maximum entanglement of qubit A (Alice) with qubit B (Bob).

Entanglement entropy is a measure (a quantifier) of entanglement: if $\mathcal{S}_{AB}$ is in a pure state, $\mathcal{S}_A$ generally is in a mixed state with nonzero entropy manifesting its entanglement with $\mathcal{S}_B$.

Entanglement Entropy of the Black Hole Radiation

A Toy Model

A black hole on the initial stage of its evaporation process viewed as bipartite quantum system $\mathcal{S}_{AB} = \mathcal{S}_A \cup \mathcal{S}_B$, where $\mathcal{S}_A$ is outgoing Hawking radiation and $\mathcal{S}_B$ is the black hole. The idea was that the generic evaporative dynamics of a black hole may be captured by the random sampling of subsystems from an initially pure state (*D. Page 1993*) .

We have $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and if we index the degrees of freedom of the radiation and the black hole by $l = 1, \dots, \mathrm{L}$ and $k = 1, \dots, \mathrm{K}$, then

$$\dim \mathcal{H}_A = \mathrm{L}, \ \dim \mathcal{H}_B = \mathrm{K}, \ \dim \mathcal{H}_{AB} = \mathrm{KL} =: \mathrm{N}.$$

Assuming the complete ignorance on the black hole, we write its state

$$|\Psi\rangle = \{\Psi_{kl}\}_{k,l=1}^{\mathrm{K},L}$$

as the random vector uniformly distributed over the surface of the unit sphere in the N-dimensional state space $\mathcal{H}_{AB}$ of the black hole, i.e.,

$$\sum_{k,l=1}^{K,L} |\Psi_{kl}|^2 = 1.$$

Recall the microcanonical ensemble of classical statistical mechanics.

The entries of density matrix ρ_{AB} of black hole $\mathcal{S}_{AB}$ and the RDM ρ_A of the radiation $\mathcal{S}_A$ are

$$\{(\rho_{AB})_{k_1 l_1; k_2 l_2}\}_{k_1,k_2=1,l_1,l_2=1}^{K,L} = \Psi_{k_1 l_1}\overline{\Psi_{k_2 l_2}}$$

and

$$(\rho_A)_{l_1 l_2} = \sum_{k=1}^{K} \Psi_{k l_1}\overline{\Psi_{k l_2}}.$$

It is of interest to find the typical behavior of the corresponding entanglement entropy $S_A = -\operatorname{Tr}\rho_A \log \rho_A$, a measure (quantifier) of quantum correlation between the radiation and the black hole.

It was found, as a first step of the program, that the expectation $\mathbf{E}\{S_A\}$ is

$$\mathbf{E}\{S_A\} = \sum_{l=\mathrm{K}+1}^{\mathrm{KL}} \frac{1}{l} - \frac{\mathrm{L}-1}{2\mathrm{K}}, \ \mathrm{L} \leq \mathrm{K}.$$

In particular, the two term asymptotic formula for large K and any L is

$$\mathbf{E}\{S_A\} = \log \mathrm{L} - \frac{\mathrm{L}}{2\mathrm{K}} + O(1/\mathrm{K}), \ \mathrm{K} \to \infty. \qquad (*)$$

Recall that if ρ is $\mathrm{L} \times \mathrm{L}$, then

$$\max_{\rho} S[\rho] = S[\mathbf{1}_L/\mathrm{L}] = \log_2 \mathrm{L}.$$

Conclusion: the random state of radiation is close in the mean to the maximally entangled state with the "deficit" $\mathrm{L}/2\mathrm{K}$ (2nd term in $(*)$).

Moreover:

(i) If $\mathsf{K} \to \infty$, L is fixed (early stage of evaporation), then the deficit is zero in the mean ($O(1/\mathsf{K})$):

$$\mathbf{E}\{S_A\} = \log \mathsf{L} + o(1), \ \mathsf{K} \to \infty.$$

(ii) If $\mathsf{K} \to \infty, \mathsf{L} \to \infty, \ \mathsf{L}/\mathsf{K} \to \lambda \in (0, \infty)$ (a later, but not too late stage of evaporation, the dimension of $\mathcal{H}_A$ is a nonzero fraction of that of $\mathcal{H}_B$), then the deficit is non zero in the mean

$$\mathbf{E}\{S_A\} = \log \mathsf{L} - \frac{\lambda}{2} + o(1), \ \mathsf{K} \to \infty.$$

Let us show that the standard facts of random matrix theory, that date back to the late 1960', imply the validity of the r.h.s. of the above formulas for a rather wide class of random vectors and not only for the expectation $\mathbf{E}\{S_A$ of S_A, but also for all its typical realization, i.e., with probability 1.

Let

$$\{X_{lk}\}_{l,k=1}^{\infty}, \; \mathbf{E}\{X_{lk}\} = \mathbf{E}\{X_{lk}^2\} = 0, \; \mathbf{E}\{|X_{lk}|^2\} = 1$$

be an infinite collection of i.i.d. complex random variables,

$$X_{AB} = \{X_{lk}\}_{k,l=1}^{\mathrm{K,L}} \in \mathcal{H}_{AB}$$

be the $\mathrm{N} = \mathrm{KL}$ dimensional random vector and

$$Z_{AB} := \sum_{l=1}^{\mathrm{L}} \sum_{k=1}^{\mathrm{K}} |X_{lk}|^2 = ||X_{AB}||^2$$

be its squared norm.

Set

$$\Psi_{AB} = X_{AB}/(Z_{AB})^{1/2}.$$

If $\{X_{kl}\}_{k,l=1}^{\infty}$ are the complex Gaussian random variables with zero mean and unit variance, then Ψ_N is uniformly distributed over the N-dimensional unit sphere (*D. Page 1993*).

View now X_{AB} as a L $\times$ K rectangular matrix. Then the RDM ρ_A is

$$\rho_A = X_{AB}X_{AB}^*/Z_{AB} = W_A/Y_A$$

with the L $\times$ L hermitian

$$W_A = X_{AB}X_{AB}^*/\mathrm{K}, \;\; Y_A = Z_{AB}/\mathrm{K}.$$

Then with $\log_2 \to \log_e =: \log$

$$\begin{aligned} S_A &= \log Y_A - \frac{1}{Y_A} \operatorname{Tr} W_A \log W_A \\ &= \log Y_A - \frac{1}{Y_A} \int_0^{\infty} w \log w \, D_{W_A}(w) dw, \end{aligned}$$

where D_{W_A} is its Density of State of W_A

$$D_{W_A}(w) = \mathsf{L}^{-1} \sum_{t=1}^{\mathsf{L}} \delta(w - w_t)$$

and $\{w_t\}_{t=1}^{\mathsf{L}}$ are its eigenvalues.

By the Strong Law of Large Numbers (SLLN) for the collection $\{X_{kl}\}_{k,l=1}^{\mathsf{KL}}$ with probability 1 for any L and K $\to \infty$:

$$\begin{aligned} Z_{AB}/\mathsf{KL} &= \frac{1}{\mathsf{KL}} \sum_{k,l=1}^{\mathsf{K,L}} |X_{kl}|^2 = \frac{1}{\mathsf{KL}} \sum_{k,l=1}^{\mathsf{K,L}} \mathbf{E}\{|X_{kl}|^2\} + o(1) \\ &= 1 + o(1),\ \mathsf{N} := \mathsf{KL} \to \infty \end{aligned}$$

and the first term on the right of S_{AB} is

$$\log Y_A = \log \mathsf{L} + o(1),\ \mathsf{K} \to \infty.$$

i.e., the first term on the right of its asymptotic (just, due to the normalization factor Y_A).

Consider now the second term on the right of S_A and write

$$(W_A)_{l_1 l_2} = \mathrm{K}^{-1} \sum_{k=1}^{\mathrm{K}} X_{l_1 k} X^*_{l_2 k}, \ \ l_1, l_2 = 1, \dots, \mathrm{L}.$$

(i) L is fixed, $\mathrm{K} \to \infty$, the very early stage of the evaporation process. It follows, again from the SLLN with probability 1

$$(W_A)_{l_1 l_2} \to \delta_{l_1 l_2}, \ \ l_1, l_2 = 1, \dots, \mathrm{L} \Longleftrightarrow W_A \to \mathbf{1}_{\mathrm{L}},$$

hence,

$$\frac{1}{Y_A} \operatorname{Tr} W_A \log W_A = \frac{1}{\mathrm{L}} \operatorname{Tr} \mathbf{1}_L \log \mathbf{1}_L + o(1) = 1 \log 1 + o(1) \to 0,$$

i.e., in this case with probability 1

$$\lim_{\mathrm{K} \to \infty} S_A = \log \mathrm{L},$$

and we have the maximum entangled state with zero deficit, valid with probability 1 for any collection of i.i.d. random variables $\{X_{kl}\}_{k,l=1}^{\infty}$.

(ii) $\mathsf{K} \to \infty,\ \mathsf{L} \to \infty,\ \mathsf{L}/\mathsf{K} \to \lambda \in (0, \infty)$, the later stage of the evaporation process.

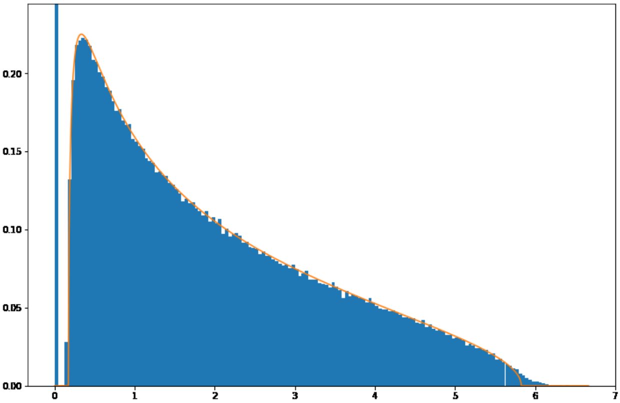
According to RMT, we have with probability 1

$$\lim_{\mathsf{K}\to\infty,\ \mathsf{L}\to\infty,\ \mathsf{L}/\mathsf{K}\to\lambda} D_{W_A}(w) =: D_W = \max\{0, 1-\lambda\}\delta(w)$$
$$+ \frac{\sqrt{(w_+ - w)(w - w_-)}}{2\pi\lambda w}\mathbf{1}_{[w_-, w_+]},\ w_\pm = (1 \pm \sqrt{\lambda})^2,$$

hence, by calculating the integral,

$$S_A = \log \mathsf{L} - \frac{\lambda}{2} + o(1),\ \mathsf{K} \to \infty,\ \mathsf{L} \to \infty,\ \mathsf{L}/\mathsf{K} \to \lambda,$$

i.e., a close to the maximum entangled state with a finite deficit, valid again with probability 1 and for any collection of i.i.d. $\{X_{kl}\}_{k,l=1}^{\infty}$ (recall the macroscopic universality of random matrix theory).



The plot of limiting density of states D_W of matrix W_A for $\lambda = 0.2$ (red) and its histogram for 60×60 matrices (blue).