Entanglement Entropy in Disordered Systems

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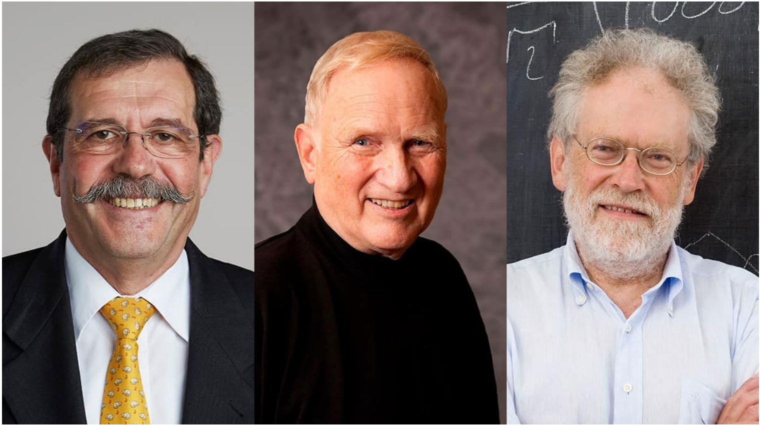
PROGRAM OF THE COURSE

1. Introduction: basic notions of quantum mechanics, bipartite systems, entanglement, reduced density matrix, entanglement quantifiers. A toy model of black hole radiation.
2. Dynamics of two qubits in random environment: general setting, models of environment, basic approximations, random matrix environment (analytical and numerical results).
3. Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities (dedicated to the Nobel Prize in Physics 2022)
4. Entanglement entropy in extended systems: setting and basic facts for translation invariant systems.
5. Entanglement Entropy of Disordered Fermions: setting, Anderson localization, area law and its violations.

- QM Reminder
- Einstein-Podolsky-Rosen (EPR) Paradox
- Bell Inequality
- Comments

Alain Aspect (1982, University Paris-Sud France), John Clauser (1972, University of California at Berkeley) and Anton Zeilinger (1998 University of Innsbruck) have won the 2022 Nobel Prize for Physics "for their experiments with entangled photons, establishing the violation of Bell's inequalities and pioneering quantum information science".



QM Reminder

If a quantum system is in a pure state ψ and O is an observable, hence, by spectral theorem

$$O = \sum_s o_s |\psi_s\rangle\langle\psi_s| = \sum_t \omega_t P_t, \;\; P_t = \sum_{s:\omega_t = o_s} |\psi_s\rangle\langle\psi_s|$$

then, according to the Born rule (*M. Born, J. von Neumann*):

$$\mathbf{P}_\psi\{O = \omega_t\} = \langle\psi|P_t|\psi\rangle = \sum_{s:\omega_t = o_s} |\langle\psi|\psi_s\rangle|^2$$

EPR Paradox (Bohm's version, EPRB)

Let $\mathcal{Q}_A \cup \mathcal{Q}_B$ be two qubits in the (**premeasured**) Bell state

$$\Psi_{pre} = \Phi^+ := 2^{-1/2}(|00\rangle + |11\rangle),$$

(singlet of parallel (aligned) spins), where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are eigenvectors of

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; O|\alpha\rangle = \alpha|\alpha\rangle, \; \alpha = 0, 1$$

and $|\alpha\beta\rangle := |\alpha_A\rangle \otimes \beta_B\rangle, \; \alpha_A, \beta_B = 0, 1.$

If A wants to find (measure) $\mathbf{P}\{\mathcal{Q}_A$ in the state $|0\rangle\}$, i.e.,

$$\mathbf{P}\{O^{AB}=0\},\ O^{AB}=O^A\otimes\mathbf{1}^B,$$

hence,

$$O^A\otimes\mathbf{1}^B|0\beta\rangle=0\cdot|0\beta\rangle,\ \beta=0,1;\ \ O^A\otimes\mathbf{1}^B|1\beta\rangle=1\cdot|1\beta\rangle,\ \beta=0,1,$$

then, by Born rule,

$$\mathbf{P}\{O^{AB}=0\}=|\langle\Phi^+|00\rangle|^2+|\langle\Phi^+|01\rangle|^2=1/2.$$

The **postmeasured** state of $\mathcal{Q}_A\cup\mathcal{Q}_B$ is, according to the Copenhagen interpretation (collapse of wave function, reduction of wave packet, projection postulate, etc.):

$$\Psi_{pos}=P_0\Phi^+(||P_0\Phi^+||)^{-1}=|00\rangle,$$

a pure state.

If then B wants (immediately!) to find

$$\mathbf{P}\{\mathcal{Q}_B \text{ in the state } |0\rangle\},$$

then $O^{BA} = \mathbf{1}^A \otimes O^B$ has to be used implying:

$$P\{O^{BA} = 0\} = |\langle 00|00\rangle|^2 + |\langle 00|01\rangle|^2 = 1 \ !$$

(with probability 1) even if A and B are space-like.

Thus, if A is found to be in the state $|0\rangle$ (with probability 1/2), B must be always in the same state and vice versa, i.e., the measurements are instantaneously correlated whether the distance between A and B is 1 meters or 1 light year.

This is the EPR paradox.

Note that EPR had no doubts that quantum mechanics is *correct*. They only claimed that it is an *incomplete description* of physical reality: The wave function $|\psi\rangle$ is not the whole story and some other quantity, λ, is needed, in addition to $|\psi\rangle$, to characterize the state of a system completely.

It was called the hidden variable because, at this stage, we have no idea how to calculate or measure it, hence, random. It could be a single number or more, perhaps will be calculated in some future theory, or maybe it is for some reason in principle incalculable.

All that EPR claimed is that there must be something (if only a list of the outcomes of every possible experiment) associated with the system prior to a measurement.

It took almost 30 years before J. S. Bell proved that (Bell's thm):

any hidden variable theory is incompatible with quantum mechanics.

Bell Inequality

It can be shown that if one replaces in the above the aligned singlet

$$\Psi_{pre} = \Phi^+ := 2^{-1/2}(|00\rangle + |11\rangle),$$

by the anti-aligned singlet

$$\Psi_{pre} = \Psi^- := 2^{-1/2}(|01\rangle - |10\rangle),$$

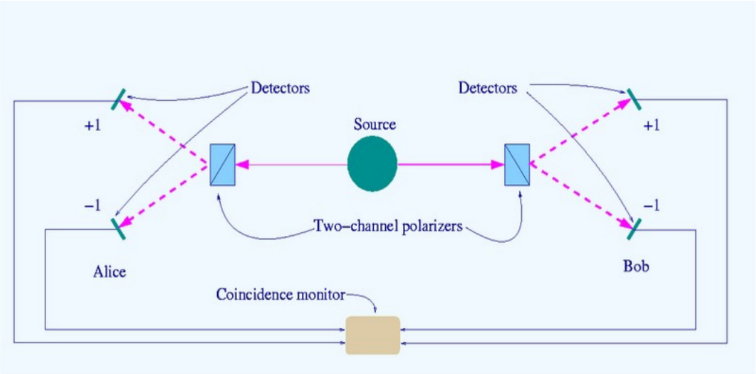
and

$$O^{A,B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

by

$$O^{A,B} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that the possible values (eigenvalues of O^A, O^B) will be ± 1 (spin up and spin down), then the result of the B-measurement will be always anti-parallel to that of the A-measurement.



Scheme of a "two-channel" Bell test The source S produces pairs of "photons", sent in opposite directions. Each photon encounters a two-channel polariser with orientation (a or b) to be set by the experimenter. Emerging signals from each channel are detected and coincidences of four types (++, −−, +− and −+) counted by the coincidence monitor.

Bell suggested a generalization of the above experiment: instead of orienting the A and the B detectors (Stern-Gerlach devices) along the same z-direction, allow them to be rotated independently. Hence, it is necessary to measure the spin components

$$\sigma_{\mathbf{a}}^A = (\sigma^A, \mathbf{a}) := \sigma_1^A a_1 + \sigma_2^A a_2 + \sigma_3^A a_3,\ \sigma_{\mathbf{b}}^B = (\sigma^B, \mathbf{b}),$$

in the directions $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3,\ ||a|| = 1$,
$\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3,\ ||\mathbf{b}|| = 1$ for A and B respectively.

This requires the observables

$$\sigma_{\mathbf{a}}^A \otimes 1^B \text{ and } 1^A \otimes \sigma_{\mathbf{b}}^B.$$

Since $\sigma_{\mathbf{a}}^2 = 1,\ a = A, B$, the eigenvalues of the both are again ± 1.

Bell proposed to calculate the expectations

$$E(\mathbf{a}, \mathbf{b}) := \langle \Psi^- | \sigma_{\mathbf{a}}^A \sigma_{\mathbf{b}}^B | \Psi^- \rangle,\ \Psi^- = 2^{-1/2}(|01\rangle - |10\rangle)$$

of the product of the spins, for a given $\mathbf{a}, \mathbf{b}$.

The EPR-like argument yields

$$E(\mathbf{a},\mathbf{b}) = -(\mathbf{a},\mathbf{b}),$$

in particular,

$$E(\mathbf{a},\mathbf{a}) = -1,\ E(\mathbf{a},-\mathbf{a}) = 1$$

for parallel and anti-parallel detectors respectively (in fact, the above formulas are valid with probability 1).

According to Bell, this result is *incompatible* with any *local hidden variable* theory.

Indeed, suppose that

(i) the "complete" state of the system is characterized by the **hidden variable** λ which varies in a way that we neither understand nor control, hence, is random with a probability distribution μ;

(ii) the outcome of the A-measurement is independent of the orientation **b** of the B-detector which may, after all, be chosen by B experimenter just before the A-measurement is made (**locality**).

Then there exists random functions

$$A(\mathbf{a}, \lambda) = \pm 1, \;\; B(\mathbf{b}, \lambda) = \pm 1$$

that determine the results of A- and B-measurements and are such that if the detectors are aligned, the results are perfectly anti-correlated:

$$A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda)$$

for any λ (c.f. EPR).

We have:

$$E(\mathbf{a},\mathbf{b}) := \int A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)d\mu(\lambda)$$

$$\underset{A(\mathbf{a},\lambda)=-B(\mathbf{a},\lambda)}{=} -\int A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)d\mu(\lambda),$$

and for any $\mathbf{c}$, $||\mathbf{c}|| = 1$

$$E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c}) = \int \big(A(\mathbf{a},\lambda)A(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)A(\mathbf{c},\lambda)\big)d\mu(\lambda)$$

$$\underset{A^2(\mathbf{b},\lambda)=1}{=} \int \big(1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\big)A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)d\mu(\lambda)$$

$$\underset{|A(\mathbf{b},\lambda|=1}{=} \int \big(1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\big)d\mu(\lambda)$$

$$\leq 1 + E(\mathbf{b},\mathbf{c}).$$

Summary

Bell Inequality		
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	Inequality	Result
$\mathbf{a} = \mathbf{b} = \mathbf{c}$	$0 \leq 0$	valid!
$\mathbf{a} \perp \mathbf{b},\ \mathbf{a} = \mathbf{c}$	$1 \leq 2$	valid!
$\mathbf{a} \perp \mathbf{b},\ \mathbf{b} = \mathbf{c}$	$0 \leq 2$	valid!
$\mathbf{a} \perp \mathbf{b},\ \widehat{\mathbf{ac}} = \widehat{\mathbf{bc}} = \pi/4$	$\sqrt{2} = 1,414 \not\leq 1$	not valid!!!

Comments

(1) *Experiment*. To exclude the remote possibility that the A-detector might somehow "sense" the orientation of the B-detector (a loophole), in the experiment of *Aspect et al 1982* both orientations were set quasi-randomly after the photons were already in flight.

(2) *Conclusions*:

(i) any local hidden variable theory is incompatible with quantum mechanics,

(ii) nature itself is nonlocal, e.g., via the instantaneous collapse of the wave function (and, by the way, in the symmetrization requirement for identical particles).

(3) *Causality* (spooky action, superluminal propagation).

A careful analysis shows that it is necessary to distinguish two types of influence: the causal, which produce actual changes in some physical property of the receiver, detectable by measurements on that subsystem alone, and the ethereal, which do not transmit energy or information, and for which the only evidence is a correlation in the data taken on the two separate subsystems – a correlation which by its nature cannot be detected by examining either list alone. Causal influences cannot propagate faster than light, but there is no compelling reason why ethereal ones should not. The influences associated with the collapse of the wave function are of the latter type, and the fact that they "travel" faster than light may be surprising, but it is not, after all, catastrophic.