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Entanglement Entropy in Disordered Systems

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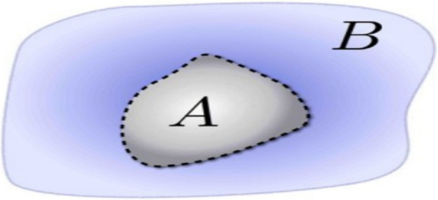
PROGRAM OF THE COURSE

1. Introduction: basic notions of quantum mechanics, bipartite systems, entanglement, reduced density matrix, entanglement quantifiers. A toy model of black hole radiation.
2. Dynamics of two qubits in random environment: general setting, models of environment, basic approximations, random matrix environment (analytical and numerical results).
3. Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities (dedicated to the Nobel Prize in Physics 2022)
4. Entanglement entropy in extended systems: setting and basic facts for translation invariant systems.
5. Entanglement Entropy of Disordered Fermions: setting, Anderson localization, area law and its violations.

ENTANGLEMENT ENTROPY IN EXTENDED SYSTEM



Outline

- Entanglement Entropy
- Extended Systems (Setting)
- Large Block Behavior of Entanglement Entropy in Translation Invariant Systems (Outline)
- Free Fermions
  - Generalities
  - Entanglement Entropy of Translation Invariant Free Fermions

Entanglement Entropy

Entropy

(i) Classical physics: a measure of the lack of knowledge, e.g., of microstates corresponding a given macrostate, hence related to classical randomness (Boltzmann, Gibbs, Shannon).

(ii) Quantum physics: a measure (a **quantifier**) of quantum correlations due to the intrinsic quantum "randomness" (von Neumann).

- Bipartite systems $\mathcal{S}_{AB} = \mathcal{S}_A \cup \mathcal{S}_B$;
- Density matrices:
  - Density matrix ρ_{AB} of $\mathcal{S}_{AB}$ (q-analog of the joint probability distribution)
  - Reduced Density Matrix ρ_A (q-analog of a marginal distribution) of A
- **Entanglement Entropy** $S_A = -\operatorname{Tr} \rho_A \log_2 \rho_A$.

We have:

- $\mathcal{S}_{AB}$ occupies a cube $\Omega \in \mathbb{Z}^d$ of side length N, $|\Omega| = N^d$,
- $\mathcal{S}_A$ (block) occupies a subcube $\Lambda \subset \Omega \in \mathbb{Z}^d$ of side length L, $|\Lambda| = L^d$

and we assume

$$1 \ll L \ll N. \qquad (*)$$

PROBLEM: the asymptotic behavior of S_A in asymptotic regime $(*)$, e.g. cosmology (black holes, holographic principle), statistical mechanics (quantum phase transitions, thermalization).

There exists a wide variety of works dealing with a number of models of quantum gravity, quantum field theory, quantum statistical mechanics and quantum information science.

Bekenstein 1973, Hawking 1974 (black holes physics), *Bombelli et al. 1986, Srednicki 1993, Wilczek et al, 1994* (QFT), *Cardy et al 2004 Korepin et al 2004* (CFT, quantum spin chains).

The most widely used implementation of heuristic inequalities $(*)$ is:

(1) macroscopic limit $N \to \infty$, (2) asymptotics as $L \to \infty$.

Change of notation: $\rho_{AB},\ \rho_A,\ S_{AB},\ S_A,\ \Rightarrow\ \rho_\Omega,\ \rho_{\Lambda\Omega},\ S_\Omega,\ S_{\Lambda\Omega}.$

It was found on the various levels of rigor that in the case of translation invariant systems with *short range interaction* the leading term of the large L asymptotic form of the macroscopic limit (recall $(*)$)

$$S_\Lambda := \lim_{\Omega \to \mathbb{Z}^d} S_{\Lambda\Omega}$$

can be:

(i) Area law

$$S_\Lambda = C_d' \, L^{d-1}(1 + o(1)), \; L \to \infty,$$

if the whole system $\mathcal{S}_\Omega$ is in a non-critical ground state (no quantum phase transition) or/and if there is a spectral gap in its spectrum.

ii) Enhanced (violated) area law

$$S_\Lambda = C_d'' \, L^{d-1} \mathbf{log\ L} \, (1 + o(1)), \; L \to \infty,$$

if the whole system $\mathcal{S}_\Omega$ is a critical ground state (quantum phase transition is present for a given values of model parameters);

(iii) Volume law

$$S_\Lambda = C_d''' \, L^d (1 + o(1)), \; L \to \infty,$$

if the whole system $\mathcal{S}_\Omega$ is either in a mixed state, say, the Gibbs state, or in a pure but sufficiently strongly excited state, the latter case is important for the fundamental Entanglement Thermalization Hypothesis (ETH).

More precisely, we have

(i) $d = 1$: mostly 1d quantum spin chains by various methods;

(ii) $d > 1$: mostly conjectured, established rigorously only for the toy model of free translation invariant fermions.

A many body quantum systems of free (non-interacting fermions) described by the many body quantum Hamiltonian

$$\widehat{H}_\Omega = \sum_{j,k\in\Omega} H_{jk} c_j^+ c_k,$$

acting in the $2^{|\Omega|}$ dimension Hilbert space, where

$$\{c_j, c_k^+\} = \delta_{jk}$$

are the Fermi annihilation and creation operators and

$$H_\Omega := \{H_{jk}\}_{j,k\in\Omega}$$

is the **one body Hamiltonian** of free fermions, acting in $|\Omega|$ dimension space (an $|\Omega| \times |\Omega|$ hermitian matrix).

It can be shown (e.g., by the Bogolyubov transformation) that the entanglement entropy at temperature $T \geq 0$ is:

$$S_{\Lambda,\Omega}(T) = \mathrm{Tr}_\Lambda\, h(n_{\Lambda\Omega}(T)),$$

with $\quad h(x) = -x \log_2 x - (1-x) \log_2(1-x),\ 0 \leq x \leq 1,$
and with the restriction

$$n_{\Lambda\Omega}(T) = \{(n_\Omega)_{jk}(T)\}_{j,k \in \Lambda},$$

of

$$n_\Omega(T) = \{(n_\Omega)_{jk}\}_{j,k\in\Omega} = \left(e^{(H_\Omega - E_F)/T} + 1\right)^{-1},\ T \geq 0$$

to Λ where E_F (the Fermi energy) is a parameter.

In particular, the operator

$$n_\Omega(0) := P_\Omega = \theta(E_F - H_\Omega),$$

with θ being the Heaviside function, is the **Fermi projection** of H_Ω.

We obtain then

$$\boxed{S_{\Lambda\Omega}(0) = \mathrm{Tr}_{\Lambda}\, h(P_{\Lambda\Omega})},$$

where for an $|\Omega| \times |\Omega|$ hermitian matrix

$$P_{\Omega} = \{(P_{\Omega})_{jk}\}_{j,k\in\Omega}$$

the $|\Lambda| \times |\Lambda|$ matrix

$$P_{\Lambda\Omega} = \{(P_{\Omega})_{jk}\}_{j,k\in\Lambda}$$

is the restriction of P_{Ω} to Λ.

The initial many body problem reduces to the one body problem.

Under minimal assumptions on the decay of $\{H_{jk}\}$ there exist an appropriately defined "limiting" operator

$$H := \lim_{\Omega \to \mathbb{Z}^d} H_\Omega,$$

acting in $\mathbb{Z}^d$, and we obtain from the above

$$\boxed{S_\Lambda(T) := \lim_{\Omega \to \mathbb{Z}^d} S_{\Lambda\Omega}(T) = \mathrm{Tr}_\Lambda\, h(n(T)|_\Lambda)},$$

where $n(T)|_\Lambda$ is the restriction to Λ of the "Fermi distribution" operator

$$n(T) = \left(e^{(H-E_F)/T} + 1\right)^{-1}$$

We are interested in asymptotic behavior of $S_\Lambda(T),\ T \geq 0$ for $|\Lambda| >> 1$.

Translation Invariant Free Fermions: Results

It has been shown rigorously in the last decade that in the case of free translation invariant fermions

$$H = \{H_{j-k}\}_{j,k\in\mathbb{Z}^d}$$

with sufficiently fast decaying $H_{j-k},\ |j-k| \to \infty$ (the short range one body Hamiltonian), the leading term of asymptotic formula for the entanglement entropy S_Λ can be:

(i) the area law

$$S_\Lambda(0) = C'_d\, L^{d-1}(1 + o(1)),\ L \to \infty,$$

if E_F is in a gap of the spectrum of H;

(ii) the enhanced (violation of) area law

$$S_\Lambda(0) = C''_d\, L^{d-1} \log L(1 + o(1)),\ L \to \infty,$$

if E_F belongs to the spectrum of H;

(iii) the volume law

$$S_\Lambda(T) = C_d''' \, L^d(1 + o(1)), \; L \to \infty \tag{1}$$

if $T > 0$.

More generally, the volume law is the case whenever the role of the Fermi operator

$$\widehat{n}(T) = \left(e^{(H-E_F)/T} + 1\right)^{-1}$$

plays an operator

$$f(H)$$

with any sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}_+$ and this is related to the classical Szegö theorem of analysis.