Entanglement Entropy in Disordered Systems

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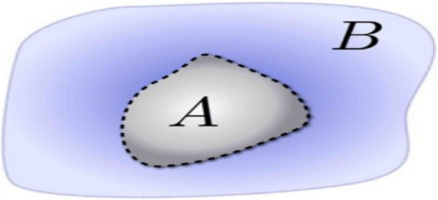
PROGRAM OF THE COURSE

1. Introduction: basic notions of quantum mechanics, bipartite systems, entanglement, reduced density matrix, entanglement quantifiers. A toy model of black hole radiation.
2. Dynamics of two qubits in random environment: general setting, models of environment, basic approximations, random matrix environment (analytical and numerical results).
3. Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities (dedicated to the Nobel Prize in Physics 2022)
4. Entanglement entropy in extended systems: setting and basic facts for translation invariant systems.
5. Entanglement Entropy of Disordered Fermions: setting, Anderson localization, area law and its violations.

ENTANGLEMENT ENTROPY FOR DISORDERED FREE FERMIONS



Outline

- Free Fermions
- Large Block Behavior of Entanglement Entropy in Translation Invariant Case
- Large Block Behavior of Entanglement Entropy in Extended Disordered Case

A bipartite system $\mathcal{S}_{AB}$:

- $\mathcal{S}_{AB}$ occupies a cube $\Omega \in \mathbb{Z}^d$ of side length N, $|\Omega| = N^d$,
- $\mathcal{S}_A$ (block) occupies a subcube $\Lambda \subset \Omega \in \mathbb{Z}^d$ of side length L, $|\Lambda| = L^d$
- $\mathcal{S}_B$ occupies $\Omega \setminus \Lambda \subset \Omega \subset \mathbb{Z}^d$

and we assume

$$1 \ll L \ll N. \qquad (*)$$

PROBLEM: the asymptotic behavior of the block entanglement entropy in asymptotic regime $(*)$:

cosmology (black holes, holographic principle), statistical mechanics (quantum phase transitions, thermalization).

The most widely used implementation of heuristic inequalities $(*)$ is:

$$\boxed{\text{(1) macroscopic limit } N \to \infty, \quad \text{(2) asymptotics as } L \to \infty \quad (**)}$$

Change of notation: $\rho_{AB},\ \rho_A,\ S_{AB},\ S_A,\ \Rightarrow\ \rho_\Omega,\ \rho_{\Lambda\Omega},\ S_\Omega,\ S_{\Lambda\Omega}.$

We assume (not too hard to prove usually)

$$S_\Lambda := \lim_{\Omega \nearrow \mathbb{Z}^d} S_{\Lambda\Omega}$$

and we are about the asymptotic behavior S_Λ as

$$\Lambda = [-M, M]^d,\ |\Lambda| = (2M+1)^d = L^d \to \infty.$$

We have a many body quantum systems of free (non-interacting fermions) described by the Hamiltonian

$$\widehat{H}_{\Omega} = \sum_{j,k\in\Omega} H_{jk} c_j^{+} c_k, \ \{c_j, c_k^{+}\} = \delta_{jk}$$

acting in the Hilbert space of dimension $2^{|\Omega|}$, where

$$H_{\Omega} = \{H_{jk}\}_{j,k\in\Omega}$$

is the **one body Hamiltonian** of free fermions, acting in the Hilbert space of $|\Omega|$ ($|\Omega| \times |\Omega|$ hermitian matrix).

Denoting $H = \lim_{\Omega \nearrow \mathbb{Z}^d} H_\Omega$ (strong limit), we have

$$S_\Lambda(T) = \mathrm{Tr}_\Lambda\, h(n_\Lambda(T)),$$

with $\quad h(x) = -x\log_2 x - (1-x)\log_2(1-x),\ 0 \le x \le 1$, and

$$\begin{aligned}
n_\Lambda(T) &= n(T)|_\Lambda = \{(n_{jk}(T)\}_{j,k\in\Lambda},\\
n(T) &= \{n_{jk}(T)\}_{j,k\in\mathbb{Z}^d} = n_T(H - E_F),\\
n_T(E) &= \left(e^{E/T} + 1\right)^{-1},\ T \ge 0,\ \text{Fermi distribution},
\end{aligned}$$

where E_F is the Fermi energy and T is the temperature. In particular

$$n_0(E) = \theta(E) \Longrightarrow n(0) := P = \theta(E_F - H),$$

with θ being the Heaviside function. P is the **Fermi projection** of H.

Translation Invariant Free Fermions: Results

It was found in the last decade that for free translation invariant fermions, where $H = \{H_{j-k}\}_{j,k \in \mathbb{Z}^d}$ with sufficiently fast decaying $H_{j-k},\ |j-k| \to \infty$ and spectrum σ_H, the leading term of asymptotic formula for $S_\Lambda(T)$ can be:

(i) if E is in a gap of σ_H: area law

$$S_\Lambda(0) = C'_d\, L^{d-1}(1 + o(1)),\ L \to \infty;$$

(ii) if $E \in \sigma_H$: enhanced (violation of) area law

$$\boxed{S_\Lambda(0) = C''_d\, L^{d-1} \log_2 L(1 + o(1)),\ L \to \infty;}$$

(iii) if $T > 0$: volume law

$$\boxed{S_\Lambda(T) = C'''_d\, L^{d}(1 + o(1)),\ T > 0,\ L \to \infty.}$$

H. Widom 1980 (d=1), H. Leschke et al 2013 - 2021 ($d \geq 1$).

Volume Law and Corrections. Write

$$\mathcal{N}_{n_\Lambda(T)}(\nu) = \sum_\alpha \delta(\nu - \nu_\alpha) = \mathrm{Tr}_\Lambda\, \delta(\widehat{n}_\Lambda(T) - \nu),$$

$$S_\Lambda(T) = \int_0^1 h(\nu)\mathcal{N}_{n_\Lambda(T)}(\nu)d\nu.$$

We have (recall statistical mechanics)

$$\mathcal{N}_{n_\Lambda(T)} = L^d \mathcal{D}_{n(T)} + L^{d-1}\mathcal{C}_{n(T)} + o(L^{d-1}) = bulk + surface + \ldots,\ L \to \infty$$

$$S_\Lambda(T) = L^d s_0(T) + L^{d-1} s_1(T) + o(L^{d-1}),\ L \to \infty,$$

$$s_0(T) = \int_0^1 h(\nu)|\mathcal{D}_{n(T)}(\nu)d\nu = 0 \Rightarrow \mathcal{D}_{n(T)} = \alpha\delta_0 + (1-\alpha)\delta_1,$$

e.g. $n(T)$ is an orthogonal projection, i.e., $n(0) = \theta(H - E_F)$ $= P,\ \sigma_P = \{0, 1\}$ and the area law is due the "surface" term of the limiting DOS of $P_\Lambda = P|_\Lambda,\ L \to \infty$.

Exact asymptotic formula for $\mathcal{N}_{n_\Lambda(0)}$, $d = 1$, "singular case"

(*H. Landau, H. Widom 1980, H. Leschke et al 2014*)

$$\mathcal{N}_{\widehat{n}_\Lambda(0)}(\nu) = \left(\frac{\pi - k_F}{\pi}\delta(\nu) + \frac{k_F}{\pi}\delta(\nu - 1) \right) L$$
$$+ \frac{1}{\pi^2 \nu(1-\nu)} \log_2 L + o(\log_2 L), \; L \to \infty$$

hence $s_0(0) = 0$ and we have instead of $|L|^{d-1} s_1(T)|_{d=1} = s_1(T)$:

$$\boxed{S_\Lambda(0) = \frac{1}{3}\log_2 L, \; L \to \infty.}$$

Bounds for the Entanglement Entropy

The bounds for $h(x) := -x\log_2 x - (1-x)\log_2(1-x),\ x \in [0,1]$:

$$2(x(1-x))^{1/2} \geq h(x) \geq 4x(1-x).$$

imply

$$2\,\mathrm{Tr}_\Lambda(P_\Lambda(1-P_\Lambda))^{1/2} = S_\Lambda(0) = \mathrm{Tr}_\Lambda\, h(P_\Lambda) \geq 4\,\mathrm{Tr}_\Lambda\, P_\Lambda(1-P_\Lambda),$$
$$P_\Lambda = \{P_{jk}\}_{j,k\in\Lambda},\ \{P_{jk}\}_{j,k\in\mathbb{Z}} = P.$$

Consider 1d case with the discrete Schrodinger operator as H:

$$(Hu)_j = -u_{j+1} - u_{j-1} + 2u_j + V_j u_j,\ j \in \mathbb{Z}.$$

We have for $\Lambda = [-M, M] \subset \mathbb{Z},\ L = |\Lambda| = 2M + 1 \to \infty$:

$$S_\Lambda(0) \geq C_-\log_2 L,\ E_F \in \sigma_H,\ V = \text{ const. (=0), enh. area law,}$$
$$S_\Lambda(0) \leq C_+,\ E_F \text{ is in gap of } \sigma_H,\ V \text{ is periodic, area law.}$$

Indeed, for $V = 0$

$$u_\kappa = \{(2\pi)^{-1/2} e^{i\kappa j}\}_{j\in\mathbb{Z}},\ E(\kappa) = 4\sin^2\kappa/2,\ |\kappa| \le \pi,$$

hence, for $|\kappa| < \pi,\ E_F = 4\sin^2\kappa_F \in \sigma_H$ =[0,4]

$$P_{jk} = \frac{1}{2\pi}\int_{|\kappa|\le\kappa_F} e^{i\kappa(j-k)} d\kappa = \frac{\sin\kappa_F(j-k)}{\pi(j-k)},$$

$$S_\Lambda \ge \frac{4}{\pi^2}\sum_{|j|\le M}\sum_{|k|\ge M+1}\frac{\sin^2\kappa_F(j-k)}{(j-k)^2} = \frac{4}{\pi^2}\sum_{t=1}^{\infty}\min(t,L)\frac{1-\cos 2\kappa_F t}{t^2}$$
$$= C\log_2 L + O(1),\ L\to\infty,\quad C = (4\log 2)/\pi^2\,,$$

hence, the first above bound for S_Λ that excludes the area law.

Note that $C \simeq 85\%$ of the exact coefficient 1/3 known since the 80s in different context.

The "mechanism": the weak decay of P_{jk} as $|j-k| \to \infty$ because of non-decaying (plane wave) eigenfunctions of H.

For a periodic V there exists the general bound

$$|P_{jk} \leq | \leq C(E_F)e^{-\gamma(E_F)|j-k|}, E_F \text{ in a gap},$$

implying the second above bound for $S_\Lambda(0)$ that excludes the enhanced area law because of the fast decay of P_{jk} as $|j-k| \to \infty$.

Disordered Free Fermions

We will show that analogous asymptotic formulas, although under different conditions, hold in the disordered case as well both for the expectation of S_Λ and for typical realizations.

Consider as H the discrete Schrodinger operator (Anderson model.)

$$(H\psi)_j = -\sum_{|j-k|=1} \psi_k + V_j\psi_j,\ j \in \mathbb{Z}^d$$

with random potential $V = \{V_j\}_{j\in\mathbb{Z}^d}$ and i.i.d. (short correlated) V_j's,

and the continuous Schrodinger operator with random potential

$$(H\psi)(x) = -(\Delta\psi)(x) + V(x)\psi(x),\ x \in \mathbb{R}^d,\ V(x) = \sum_j u(x - x_j)$$

with $|u(x)| = 0,\ |x| > a$ and $\{x_j\}_j$ uniformly distributed over $\mathbb{R}$ with density c.

Anderson Localization

(i) **Density of States** of H:

$$\mathcal{D}_{H_\Omega}(E) = |\Omega|^{-1} \sum_\alpha \delta(E - E_\alpha) \underset{\Omega \to \mathbb{Z}^d, \mathbb{R}^d}{\to} D_H(E),$$

and the limit is valid for all typical realizations (with probability 1), selfaveraging property.

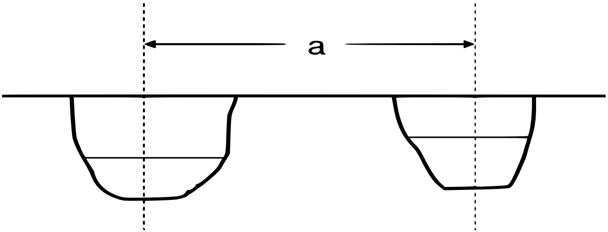
Recall the selfaveraging of the free energy in statistical mechanics and the same property of conductivity in the solid state theory of disordered media (bulk properties!).

Next, we have for $u \geq 0$ (attractive impurities), $\sigma_H = [0, \infty)$ and for the bottom of the spectrum $E \searrow 0$:

$$\mathcal{D}_H(E) = \begin{cases} E^{d/2-1}, & u = 0, \text{ classical}, \\ \sim \exp\{-ca_d/E^{d/2}\}, & u \geq 0, \text{ Lifshitz tail}. \end{cases}$$

$a_1 = \pi$, by a spectral version of large deviation techniques.

Two Wells



Eigenelements of wells: $a = \infty : \ (\varepsilon_t, \psi_t), \ t = 1, 2$ and $a < \infty : \ (E_t, \Psi_t), \ t = 1, 2$. Denote $\delta = |\varepsilon_1 - \varepsilon_2|$,

$$I(a) = \int \psi_1(x)\psi_2(x+a)dx =, \ a \gg r.$$

We have $E_{1,2} = \varepsilon_{1,2} + O(e^{-a/r})$ but

(i) $\delta \gg I(a)$: $\Psi_{1,2} = \psi_{1,2} + O(-a/r)$, no resonance tunneling;

(ii) $\delta \ll I(a)$: $\Psi_{1,2} = (\psi_{1,2} \pm \psi_2)2^{-1/2} + O(-a/r)$, resonance tunneling.

(ii) **Anderson Localization.** The analysis of tunneling between two wells ($u(x) \leq 0$, attractive impurities) suggests that the eigenfunctions of the random Schrodinger operator are localized, i.e.,

$$\psi(x) \sim e^{-|x-x_0|/r_{\text{loc}}}.$$

The localization is rigorously established in dimension $d = 1$ for the whole spectrum and in dimension $d \geq 2$ for the bottom of the spectrum (more generally for the neighborhoods of spectrum edges).

A quite general quantitative result (implying many others)

$$\mathbf{E}\{|P_{jk}|\} \leq Ce^{-|j-k|/r(E_F)},\ E_F \in \sigma_H,\ j,k \in \mathbb{Z}^d$$

is valid for
- any potential if the Fermi energy is in a spectral gap;
- random potential, for all $E_F,\ d = 1$ and for the bottom of $\sigma_H,\ d \geq 2$;
- certain incommensurate (quasiperiodic) potentials and $d = 1$.

Asymptotics of Entanglement Entropy

Assume the above exponential bound for the Fermi projection.

(i) **Area Law in the Mean** (discrete case)

$$\boxed{\mathbf{E}\{S_\Lambda(0)\} = c_d L^{d-1}(1+o(1)),\ L \to \infty},$$

$$\boxed{c_d = 2d\,\mathbf{E}\{\mathrm{Tr}_{\mathbb{Z}^d_+}\, h(P_{\mathbb{Z}^d_+})\},\ P_{\mathbb{Z}^d_+} = P|_{\mathbb{Z}^d_+}}$$

and $\mathbb{Z}^d_+$ is the lattice half-space.

Unlike the translation invariant case with ballistically moving carriers and the violation of the area law, the disordered case with localized carriers obeys the area law, although in the mean for $E_F \in \sigma_H$.

On the other hand, for translation invariant free fermions the area law is the case only if the Fermi energy is in the gap of the spectrum of H.

(ii) **Entanglement Entropy is Selfaveraging for** $d \geq 2$.

$$\boxed{\mathbf{Var}\{L^{-(d-1)} S_\Lambda(0)\} = O(1/L^{\alpha_d}),\ \alpha_d = 2(d-1)/(d+1),}$$

It is believed that $\alpha_d = (d-1)$ and that

$$L^{-(d-1)/2}(S_\Lambda(0) - \mathbf{E}\{S_\Lambda(0)\})$$

is Gaussian as $L \to \infty$, i.e., the CLT for the appropriately normalized entanglement entropy (recall the Central Limit Theorem).

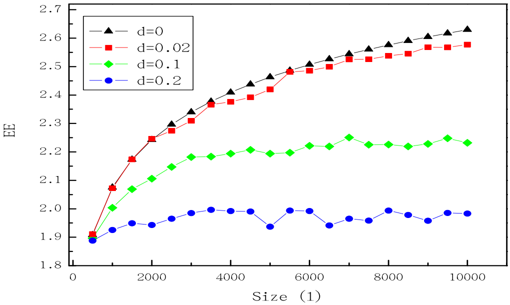
The area law (the absence of logarithmic corrections) is the case for $d \geq 2$, all typical realizations of random potential and E_F belonging to the part of the spectrum where Fermi projection decays exponentially (e.g. because of the localization of states).

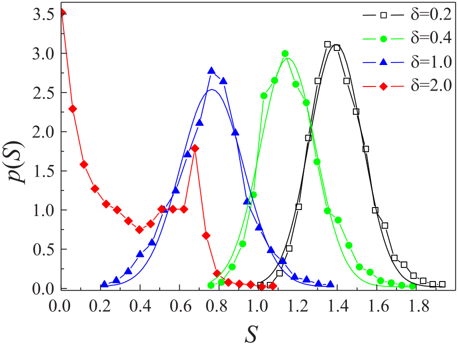
On the other hand, for translation invariant potentials the area law is valid in the gaps of the spectrum, a "less interesting case".

(iii) **Entanglement Entropy Is NOT Selfaveraging For $d = 1$.**

There exists a class of random potentials such that we have for the entanglement entropy and all sufficiently large $L = |\Lambda|$

$$\boxed{\mathbf{Var}\{S_\Lambda(0)\} := \mathbf{E}\{(S_\Lambda(0))^2\} - \mathbf{E}^2\{S_\Lambda\} \geq A[V] > 0.}$$





The probability density of $S_\Lambda(0)$ for different values of disorder parameter δ for H being the Anderson model (uniform distribution of V_j's: $p(V) = 1/2\delta, \ |V| < \delta$).

(iv) **Enhanced Area Law for $d = 1$.**

(1) *Transparency Energies.*

(a) $V(x) = \sum_{j\in\mathbb{Z}} v_j\delta(x - ja)$, $\{v_j > 0\}_{j\in\mathbb{Z}}$ are i.i.d., the random

Kronig-Penny, a model for substitutional alloys, e.g. CuZn.

Consider the functions

$$\psi_k(x) = C\sin\pi kx/a,\ k \in \mathbb{Z},\ x \in \mathbb{R}.$$

They are solutions (eigenfunctions) of the continuous Schrodinger equation corresponding to $E_k = (\pi k/a)^2$ and since for every k

$$\psi_k(ja) = 0,\ j, k \in \mathbb{Z},$$

the functions of the family $\{\psi_k\}_{k\in\mathbb{Z},}$ "do not feel" the random potential (perfect transmission), i.e., $E_k = (\pi k/a)^2$ are the transparency energies for all realizations of V.

(b) $V(x) = \sum_{x_j \in \mathbb{R}} u(x - x_j),\ y_j = x_{j+1} - x_j \geq 2a$ i.i.d

and u is the rectangular potential well of depth u_0 and width a.

By quantum mechanics the rectangular potential well has the transparency energies $E_k = -u_0 + (\pi k/a)^2$ (zero reflection coefficient for u). Hence, any realization of V has E_k as its transparency energies.

(2) *Enhanced Area Law in the Mean for $d = 1$*

It can be shown that there exist random potentials with transparency energies such that

$$\boxed{\mathbf{E}\{S_\Lambda(0)\} \geq C \log_2 L + o(\log_2 L),\ L \to \infty.}$$

For free disordered fermions the enhanced area law is "exceptional" while for the translation invariant free fermions it is "generic".

(v) **Volume Law and Corrections**

Assume that $T > 0$ or replace n_T by a smooth function. Then

$$S_\Lambda(T) = L^d s_{vol}(T) + \begin{cases} L^{d-1} s_{surf}(T) + o(L^{d-1}), & \text{trans. invariant,} \\ L^{d/2} s_{CLT} + o(L^{d/2}), & \text{disordered..} \end{cases}$$

In disordered case and for $d = 1$ the sub-leading term is random (unlike the leading one) and is $O(L^{1/2})$!

$$\boxed{(S_\Lambda(T) - L^d s_{vol}(T))/L^{1/2} \to s_{CLT} \ \text{Gaussian}, \mathbf{E}\{s_{CLT}\} = 0.}$$

Recall the Imry-Ma phenomenon in statistical mechanics.

Conclusion

Entanglement: the quantification by the entanglement entropy of the intuitively clear fact: disorder inhibits quantum correlations.

Cf. Transport: the quantification by the conductivity of the intuitively clear fact: disorder inhibits transport.

The above result on the disordered case (the dynamics of two qubits and the asymptotic behavior of the entanglement entropy of disordered free fermions) are obtained in the joint works by

E. Bratus, A.Elgart, P. Muller, L.P., R. Schulte, M. Shcherbina, V. Slavin, 2014 – 2020.

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