Fractal Uncertainty Principle

Semyon Dyatlov (MIT)

December 13, 2022

- The topic of this talk is a Fractal Uncertainty Principle (FUP):

  No function can be localized
  in both position and frequency
  near a fractal set

- Applications include lower bound on mass of eigenfunctions on compact surfaces and spectral gaps on noncompact surfaces
- I will discuss the general proof a bit but focus on the simpler case of discrete Cantor sets
- We know FUP for subsets of $\mathbb{R}$; in higher dimensions it is largely an open problem

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Unitary semiclassical Fourier transform on $L^2(\mathbb{R})$:

$$\mathcal{F}_h f(x) = (2\pi h)^{-\frac{1}{2}} \widehat{f}\Big(\frac{x}{h}\Big) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{i}{h}xy} f(y)\, dy$$

Here $h \to 0$ is the semiclassical parameter. For applications to Laplacian eigenfunctions, $h \sim \lambda^{-1}$ where λ^2 is the eigenvalue.

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Here $\nu > 0$ should be independent of h. For instance we can take $X(h) = X_0 + [-\varepsilon h, \varepsilon h]$ where X_0 is porous up to scale 0.

Example: mid-third Cantor set $\mathcal{C} \subset [0, 1]$ is $\frac{1}{6}$-porous up to scale 0

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Statement of Fractal Uncertainty Principle (FUP)

Theorem 1 [Bourgain–D '18]

Let X, Y be ν-porous up to scale h. Then there exist $\beta > 0$, C depending only on ν such that

$$\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq Ch^\beta$$

where $\mathbb{1}_X$ is the multiplication operator by the indicator function of X etc.

Theorem 1' (a restatement of Theorem 1)

Let X, Y be ν-porous up to scale h. Then there exist $\beta > 0$, C depending only on ν such that for all $f \in L^2(\mathbb{R})$

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Basic uncertainty principles

- Looking for

$$\|\mathbb{1}_X\,\mathcal{F}_h\,\mathbb{1}_Y\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as} \quad h\to 0$$

- Trivial bound: $\beta = 0$ as $\|\mathbb{1}_X\mathcal{F}_h\mathbb{1}_Y\|_{L^2\to L^2} \leq 1$
- Volume bound: if $|X|,|Y| = \mathcal{O}(h^{1-\delta})$ then get $\beta = \frac{1}{2}-\delta$:

$$\begin{aligned}\|\mathbb{1}_X\mathcal{F}_h\mathbb{1}_Y\|_{L^2\to L^2} &\leq \|\,\mathbb{1}_X\,\|_{L^\infty\to L^2}\|\mathcal{F}_h\|_{L^1\to L^\infty}\|\,\mathbb{1}_Y\,\|_{L^2\to L^1}\\ &\leq \sqrt{\frac{|X|\cdot|Y|}{2\pi h}} = \mathcal{O}(h^{\frac{1}{2}-\delta})\end{aligned}$$

- Cannot be improved if we only know the volume, e.g.

$$X = Y = [-\sqrt{h},\sqrt{h}] \quad\Longrightarrow\quad \text{cannot get} \quad \beta > 0$$

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On the proof of FUP I

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- Write $X \subset \bigcap_j X_j$ where each $X_j \subset X_{j-1}$ has holes on scale $2^{-j} \geq h$
- Will show: for each j, $\|1_{X_j} f\|_{L^2} \leq (1-c)\|1_{X_{j-1}} f\|_{L^2}$
- This requires a lower bound on the mass of f on the 'holes' in $\mathbb{R} \setminus X_j$, reducing FUP to the following

Key Lemma

Assume that for each $\ell \in \mathbb{Z}$, $I_\ell \subset [\ell, \ell+1]$ is an interval of length $\alpha > 0$. Then there exists $c > 0$ depending only on ν, α such that

$$\operatorname{supp} \widehat{f} \subset h^{-1} Y \quad \Longrightarrow \quad \|\mathbb{1}_{\bigsqcup_\ell I_\ell} f\|_{L^2(\mathbb{R})} \geq c\|f\|_{L^2(\mathbb{R})}.$$

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- This is a unique continuation estimate: need $f|_{\bigsqcup_\ell I_\ell} = 0 \;\Rightarrow\; f = 0$
- This is known if f has Fourier transform decaying fast enough, e.g.

$$|\widehat{f}(\xi)| = \mathcal{O}(e^{-|\xi|/(\log|\xi|)^s}) \quad \text{for some } s < 1 \tag{1}$$

- Using porosity of Y and the Beurling–Malliavin theorem, can construct a compactly supported multiplier, $g \in C_c^\infty((-\frac{\alpha}{10}, \frac{\alpha}{10}))$, where $\widehat{g}$ has decay (1) but only on $h^{-1} \cdot Y$
- Now use that the convolution $f * g$ satisfies (1) everywhere

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Hyperbolic FUP

For applications to hyperbolic surfaces, we replace the phase xy in $\mathcal{F}_h$ by $2\log|x-y|$ and introduce a cutoff $\chi \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi \subset \{x \neq y\}$:

$$\mathcal{B}_{\chi,h} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\, dy$$

The operator $\mathcal{B}_{\chi,h}$ appears in the composition $B_-^{-1}B_+$ where $B_\pm$ are FIOs locally straightening out stable/unstable foliations

One can deduce from FUP for $\mathcal{F}_h$ a similar statement for $\mathcal{B}_{\chi,h}$:

Theorem 2 (Hyperbolic FUP)

Assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale h. Then there exist $\beta = \beta(\nu) > 0$ and $C = C(\nu, \chi)$ such that

$$\|\mathbb{1}_X \, \mathcal{B}_{\chi,h} \, \mathbb{1}_Y\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le Ch^\beta.$$

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A bit on reducing hyperbolic FUP to Fourier FUP

- Replace Y by its $h^{1/2-}$-neighborhood $\widetilde{Y}$: $\|\mathbb{1}_X\mathcal{B}_h\mathbb{1}_Y\| \le \|\mathbb{1}_X\mathcal{B}_h\mathbb{1}_{\widetilde{Y}}\|$
- Split $X = \bigsqcup_j X_j$, each X_j lies in an $h^{1/2}$-sized interval $[x_j, x_j + h^{1/2}]$
- Show $B_j := \mathbb{1}_{X_j}\mathcal{B}_h\mathbb{1}_{\widetilde{Y}}$ almost orthogonal: for $|j-\ell| \gg 1$

$$\|B_j^* B_\ell\| = \mathcal{O}(h^\infty), \quad \|B_j B_\ell^*\| = \mathcal{O}(h^\infty)$$

  so by Cotlar–Stein $\|\mathbb{1}_X\mathcal{B}_h\mathbb{1}_{\widetilde{Y}}\| \lesssim \max_j \|\mathbb{1}_{X_j}\mathcal{B}_h\mathbb{1}_{\widetilde{Y}}\|$
- Use a change of variables to bound $\|\mathbb{1}_{X_j}\mathcal{B}_h\mathbb{1}_{\widetilde{Y}}\|$ using the Fourier FUP: if $\Phi(x,y) = -2\log|x-y|$ and $|x-x_j| \le h^{1/2}$ then on $\operatorname{supp}\chi$

$$e^{\frac{i}{h}\Phi(x,y)} \approx e^{\frac{i}{h}\Phi(x_j,y)} e^{\frac{i}{h}(x-x_j)\kappa_j(y)}, \quad \kappa_j(y) := \partial_x\Phi(x_j,y)$$

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Discrete Cantor sets

We now present a proof of FUP in the special setting of Cantor sets.
This is much simpler than the general case but keeps some key features.
We follow D–Jin '17, with the exposition from [arXiv:1903.02599]

- Discrete unitary Fourier transform $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-\frac{2\pi i j \ell}{N}} u(\ell)$$

- Fix $M \geq 3$, $\mathscr{A} \subset \{0, \dots, M-1\}$. Put $N := M^k$, $k \gg 1$ and define

$$\mathcal{C}_k := \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathscr{A}\}$$

- Example: if $M = 3$, $\mathscr{A} = \{0, 2\}$, then $\mathcal{C}_k \subset \{0, \dots, N-1\}$, $N = 3^k$, is the discrete mid-3rd Cantor set $\{0, 2, 6, 8, 18, 20, 24, 26, \dots\}$
- The number of elements of $\mathcal{C}_k$ is $|\mathcal{C}_k| = N^\delta$ where $\delta = \log_M |\mathscr{A}|$

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We now present a proof of FUP in the special setting of Cantor sets. This is much simpler than the general case but keeps some key features. We follow D–Jin '17, with the exposition from [arXiv:1903.02599]

- Discrete unitary Fourier transform $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-\frac{2\pi i j \ell}{N}} u(\ell)$$

- Fix $M \geq 3$, $\mathscr{A} \subset \{0, \dots, M-1\}$. Put $N := M^k$, $k \gg 1$ and define

$$\mathcal{C}_k := \{a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathscr{A}\}$$

- Example: if $M = 3$, $\mathscr{A} = \{0, 2\}$, then $\mathcal{C}_k \subset \{0, \dots, N-1\}$, $N = 3^k$, is the discrete mid-3rd Cantor set $\{0, 2, 6, 8, 18, 20, 24, 26, \dots\}$
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Uncertainty principle for discrete Cantor sets

Theorem 3

Assume that $0 < \delta < 1$, i.e. $1 < |\mathscr{A}| < M$. Then there exists $\beta = \beta(M, \mathscr{A}) > \max(0, \frac{1}{2} - \delta)$ such that as $N = M^k \to \infty$,

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} = \mathcal{O}(N^{-\beta}).$$

- Trivial bound $\beta = 0$: since $\mathcal{F}_N$ is unitary, $\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq 1$
- Volume bound $\beta = \frac{1}{2} - \delta$: defining the Hilbert–Schmidt norm

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |a_{jk}|^2 \quad \text{where} \quad A = (a_{jk})_{j,k=1}^N$$

we have

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathrm{HS}} = N^{\delta - \frac{1}{2}}.$$

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Submultiplicativity

The proof of FUP for Cantor sets is greatly simplified by the

Submultiplicativity Lemma

Define $r_k := \| \mathbb{1}_{\mathcal{C}^k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$. Then $r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}$ for all k_1, k_2.

To prove it, we employ the following decomposition also used in FFT:

- Write $k = k_1 + k_2, \quad N = M^k = N_1 \cdot N_2, \quad N_j := M^{k_j}$
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An example of the 'Fast Fourier Transform' decomposition

Let's say $N = 4 = N_1 N_2$ where $N_1 = N_2 = 2$.

Take $u = (u_0, u_1, u_2, u_3) \in \mathbb{C}^4$. Follow the instructions on the last slide:

- Take $U = \begin{pmatrix} u_0 & u_2 \\ u_1 & u_3 \end{pmatrix}$, $\mathcal{F}_2$ each row to get $\frac{1}{\sqrt{2}} \begin{pmatrix} u_0 + u_2 & u_0 - u_2 \\ u_1 + u_3 & u_1 - u_3 \end{pmatrix}$
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FUP with $\beta > 0$

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- Assume for simplicity that $M - 1 \notin \mathscr{A}$, then the degree of P satisfies

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- On the other hand, $P(\omega^j) = 0$ for all $j \in \{0, \dots, N-1\} \setminus \mathcal{C}_k$, so P has at least $N - |\mathcal{C}_k| \geq M^k(1 - (1 - \frac{1}{M})^k)$ roots
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- For k large, $M^k (1 - (1 - \frac{1}{M})^k) > M^k (1 - \frac{1}{M})$, so $r_k < 1$ as needed

FUP with $\beta > \frac{1}{2} - \delta$ ('baby Dolgopyat')

- Similarly to the previous slide, enough to show that $\exists k : r_k < N^{\delta - \frac{1}{2}}$ where $r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$, $N = M^k$
- We always have $r_k \leq \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathrm{HS}} = N^{\delta - \frac{1}{2}}$
- Assume $r_k = N^{\delta - \frac{1}{2}}$, then $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ has the same operator norm ($=$ max singular value σ_j) and H–S norm $\left(= \sqrt{\sigma_1^2 + \cdots + \sigma_N^2} \right)$
- This can only happen if $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is a rank 1 matrix, i.e. each of its 2×2 minors is equal to 0. This gives

$$(j - j')(\ell - \ell') \in N\mathbb{Z} \quad \text{for all} \quad j, j', \ell, \ell' \in \mathcal{C}_k$$

- This cannot happen already when $k = 2$ (and $|\mathscr{A}| > 1$): just take two different $a, b \in \mathscr{A}$ and put

$$j = \ell = Ma + a, \quad j' = \ell' = Ma + b$$

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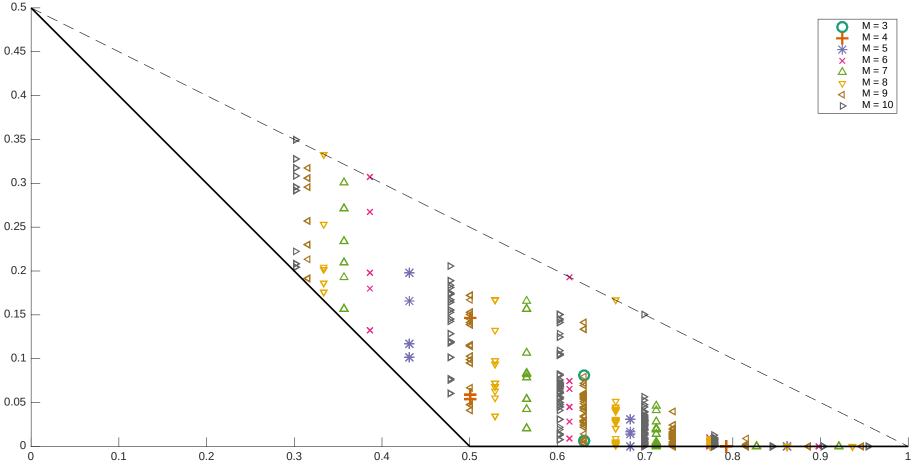
A picture of FUP exponents for all alphabets with $M \leq 10$



Horizontal axis: δ, vertical axis: β, solid line: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed line: $\beta = \frac{1-\delta}{2}$ (corresponding to the gap conjectured by Jakobson–Naud)

A higher dimensional FUP?

- Open problem: get FUP with $\beta > 0$ on $\mathbb{R}^n$, $n > 1$. Let's take $n = 2$
- $\mathcal{F}_h f(x) = (2\pi h)^{-1}\widehat{f}(\frac{x}{h})$ semiclassical Fourier transform
- Want $\| \mathbb{1}_X \, \mathcal{F}_h \, \mathbb{1}_Y \|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = \mathcal{O}(h^\beta)$ where $X, Y \subset \mathbb{R}^2$ are δ-regular up to scale h and $\delta < 2$
- This is false: take $\delta = 1$, $X = [0,h] \times [0,1]$, $Y = [0,1] \times [0,h]$
- Han–Schlag '20: FUP holds with $\beta > 0$ if one of X, Y is contained in the product of 2 fractal sets
- It could be that the hyperbolic FUP (with $e^{-\frac{i}{h}\langle x,y\rangle}$ replaced by $|x-y|^{-\frac{2i}{h}}$) still holds.
  Partial result by D–Zhang WIP, when one of X, Y is a curve

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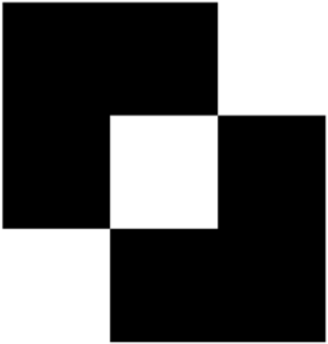
Two-dimensional FUP for discrete Cantor sets

Fix $M \geq 2$, $\mathscr{A} \subset \{0, \dots, M-1\}^2$. Put $N = M^k$, $k \gg 1$ and

$$\mathcal{C}_k = \{a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathscr{A}\}$$

which is a subset of $\{0, \dots, N-1\}^2$.

Example ($M = 3$, $k = 1$):



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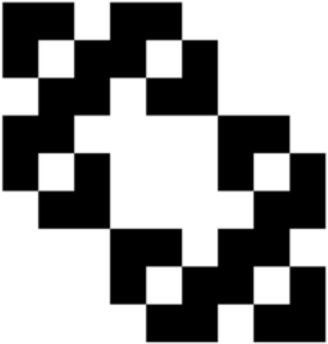
Example ($M = 3$, $k = 2$):



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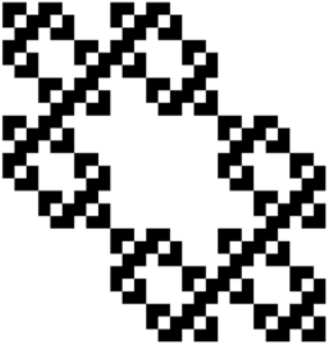
Example ($M = 3$, $k = 3$):



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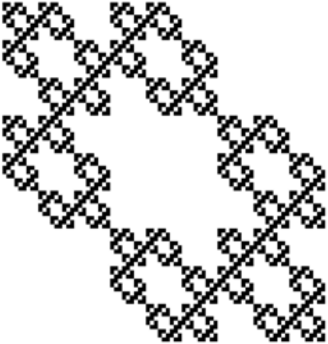
Example ($M = 3$, $k = 4$):



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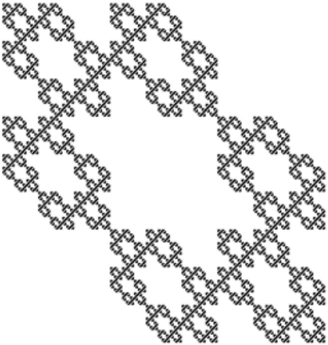
Example ($M = 3$, $k = 5$):



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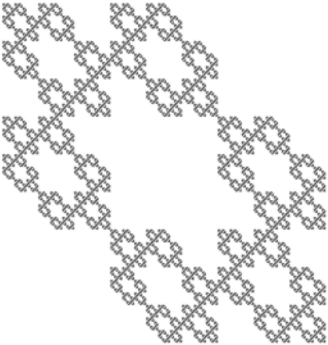
Example ($M = 3$, $k = 6$):



Two-dimensional FUP for discrete Cantor sets

Theorem 4 [Cohen '22]

Let $\mathcal{C}_k, \mathcal{D}_k \subset \{0, \dots, N-1\}^2$ be Cantor sets generated by two alphabets $\mathscr{A}, \mathscr{B} \subset \{0, \dots, M-1\}^2$. Then either

- FUP holds for $\mathcal{C}_k, \mathcal{D}_k$, namely $\exists \beta > 0$:

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{N \times N} \mathbb{1}_{\mathcal{D}_k} \|_{\mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}} = \mathcal{O}(N^{-\beta}),$$

- or the limiting Cantor sets $\mathcal{C}_\infty, \mathcal{D}_\infty \subset \mathbb{T}^2$ contain rational lines orthogonal to each other.

- Submultiplicativity still holds $\Rightarrow$ enough to show that $\exists k$: there is no

$$u \in \mathbb{C}^{N \times N} \setminus \{0\} : \quad u = \mathbb{1}_{\mathcal{D}_k} u, \quad \mathcal{F}_N u = \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N u$$

- The proof uses algebraic geometry and number theory; key ingredient is a quantitative version of Lang's conjecture on cyclotomic points on algebraic curves in $\mathbb{C}^2$ [Ruppert '93, Beukers–Smyth '02]

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Thank you for your attention!