Eigenstate Thermalization Hypothesis: from interacting qubits to QFT

Anatoly Dymarsky

University of Kentucky

Collège de France, November 22, 2022

Warm-up: quantum chaos

observation

- universality of energy level statistics in complex systems – Wigner surmise

precise formulation

- Bohigas-Giannoni-Schmit conjecture – level statistics is given by RMT

physical significance

- late times, exponential sensitivity

derivation

- for semiclassical limit of classically chaotic systems

What is ETH?

observation

- individual energy eigenstates are thermal

precise formulation(?)

- matrix elements of observables can be described statistically

physical significance(?)

- universality of late-time behavior (thermalization)

derivation(?)

- for semiclassical limit of classically chaotic systems

Eigenstate Thermalization Hypothesis

- individual energy eigenstates are thermal

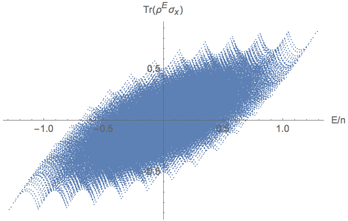
$$\langle E_n|A|E_m\rangle = A_{nm} = A^{\mathrm{eth}}(E_n)\delta_{nm} + e^{-S/2} f(\bar{E},\omega)\, R_{nm}$$

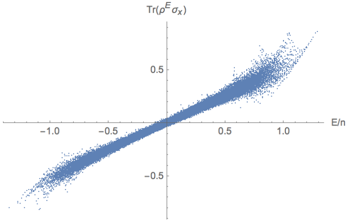
Deutsch'91 Srednicki'94; 99 Rigol, Dunjko, Olshanii'08

- A^{eth} is a smooth function of its argument
- f is a smooth function of $\bar{E} = (E_n + E_m)/2$ and $\omega = E_n - E_m$
- S is entropy, e^S is density of states
- R_{nm} are pseudo-random fluctuations of zero mean and unit variance

Integrability vs “chaoticity” (ETH)

diagonal matrix elements $A_{nn} = \langle E_n | A | E_n \rangle$ for integrable and non-integrable spin-chains





- $A_{nn} = A^{\mathrm{eth}}(E_n/V) + \varepsilon_n$, where $\langle \varepsilon^2 \rangle \sim \mathrm{Vol}^{-\#}$ or $e^{-O(\mathrm{Vol})}$
- max $\varepsilon_n \sim O(1)$ or $e^{-O(\mathrm{Vol})}$ (strong ETH)

systems “in the middle” – quantum scars

Thermalization of Isolated Quantum System

- unitary evolution $\Psi(t) = U(t)\Psi_0$ precludes emergence of thermal state $\rho(t) \to \rho^{\text{th}} = e^{-\beta H}/Z$

- ETH guarantees (eventual) thermalization of *all* initial states

$$\langle\Psi|A(t)|\Psi\rangle = \sum_n |c_n|^2 A_{nn} + \sum_{n\neq m} e^{-i(E_n-E_m)t} c_n^* c_m A_{nm}$$

universal (thermal) value of the "diagonal" ensemble

$$\overline{A(t)} = \overline{\langle\Psi|A(t)|\Psi\rangle} = \sum |c_n|^2 A_{nn} \approx \left(\sum |c_n|^2\right) A^{\text{eth}} \approx A^{\text{th}}$$

- ETH = quantum ergodicity (independence of initial state)

no promise regarding dynamics, except

$$\overline{A^2(t)} - \overline{A(t)}^2 \leq \max_{n\neq m} |A_{nm}|^2 \sim e^{-S}$$

Different notions of "quantum ergodicity"

- in this talk: quantum ergodicity is universality of $\overline{A(t)}$ for any initial state

- single particle context: statistically equal distribution of $|\langle i|E_n\rangle|^2 \approx \mathrm{const}$ in the local basis $|i\rangle$

  standard indicator of ergodicity vs localization is inverse participating ratio

$$\left(\sum_i |\langle i|E_n\rangle|^4\right)^{-1} \approx N_{\mathrm{eff}}$$

- relation between different definitions in the many-body case is not clear

Relation between different ensembles

- statistical mechanics defines thermal quantities with help of an ensemble
- $A^{\mathrm{eth}}(E)$ is the thermal expectation of A in the "eigenensemble"
- thermal expectations of A in the microcanonical and canonical ensembles

$$\langle A \rangle_{\mathrm{mic}} = \int_{E}^{E+\Delta E} \frac{dE\, A^{\mathrm{eth}}(E)}{\Delta E}, \quad \langle A \rangle_{\beta} = \int \frac{dE\, e^{S-\beta E} A^{\mathrm{eth}}(E)}{Z}$$

are only polynomially close to $A^{\mathrm{eth}}(E)$

$$\langle A \rangle_{\beta} = A^{\mathrm{eth}}(E) + O(1/\mathrm{Vol})$$

Subsystem ETH

- which operators A satisfy ETH?
- universal form of the reduced density matrix
  of the subsystem $\mathcal{A}$

$$\left|\left|\rho^E - \rho^{\rm eth}(E_n)\right|\right| = O(e^{S_{\mathcal{A}}-S/2}), \quad \rho^E = {\rm Tr}_{\bar{\mathcal{A}}}|E_n\rangle\langle E_n|$$

-strong(est) formulation: list of operators follows

-volume-dependence of the pre-factor

AD, Lashkari, Liu, Phys. Rev. E 97, 012140

- at late times state of a subsystem becomes thermal

$${\rm Tr}_{\bar{\mathcal{A}}}\, \rho(t) \to \rho^{\rm th}_{\mathcal{A}} = {\rm Tr}_{\bar{\mathcal{A}}}\, e^{-\beta H}/Z \approx e^{-\beta H_{\mathcal{A}}}/Z_{\mathcal{A}}$$

rest of the system acts as a “heat bath”

Weak ETH

- fluctuations of diagonal matrix elements ε_n are always small on average
$$\langle \varepsilon_n^2 \rangle \sim \mathrm{Vol}^{-1}$$
- for translationally-invariant systems

$$A_{nn} \equiv \hat{A}_{nn}, \qquad \hat{A} = \sum_x A(x)/\mathrm{Vol}$$

quantity of interest is a minimum with respect to A^{eth}

$$\langle \varepsilon_n^2 \rangle \equiv \langle (A_{nn} - A^{\mathrm{eth}}(E_n))^2 \rangle \leq \langle A_{nn}^2 \rangle - \langle A_{nn} \rangle^2$$

this can be bounded using $\hat{A}_{nn}^2 \leq \sum_m |\hat{A}_{nm}|^2 = (\hat{A}^2)_{nn}$

$$\#\mathrm{Vol}^{-1} \leq \langle A_{nn}^2 \rangle - \langle A_{nn} \rangle^2 \leq \langle (\hat{A}^2)_{nn} \rangle - \langle \hat{A}_{nn} \rangle^2 \leq \#\mathrm{Vol}^{-1}$$

Weak ETH

- in full generality

$$
\begin{aligned}
\langle (\hat{A}^2)_{nn} \rangle - \langle \hat{A}_{nn} \rangle^2 &= \frac{1}{\mathrm{Vol}^2} \sum_{x,y} \sum_{n} \langle E_n | A(x) A(y) | E_n \rangle^c \frac{e^{-\beta E_n}}{Z} \\
&= \frac{1}{\mathrm{Vol}} \sum_{x} \langle A(x) A(0) \rangle_\beta^c \sim \mathrm{Vol}^{-1}
\end{aligned}
$$

  extension for continuous systems, not exactly tr. invariant, etc.

- variance of ε_n is never too big, altough individual ε_n could be large, $O(1)$

Off-diagonal ETH

- factor $e^{-S/2}$ is kinematic *assuming* all A_{nm} are of the same order

$$O(1) \sim A^2_{nn} = \sum_m |A^2_{nm}| \sim e^S |A_{nm}|^2$$

- the same applies to reduced density matrix

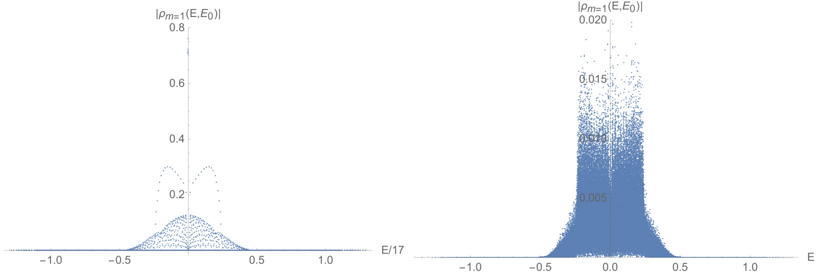
$$||\rho_{nm}|| = O(e^{S_{\mathcal{A}}-S/2}), \quad \rho_{nm} = \mathrm{Tr}_{\bar{\mathcal{A}}} |E_n\rangle\langle E_m|$$

AD, Lashkari, Liu, Phys. Rev. E 97, 012140

- for some (simple) integrable models A_{nm} is distinctively different, for interacting models, e.g. XXZ, the structure is similar

Off-diagonal matrix elements

off-diagonal matrix elements $|\rho_{nm}|$ for integrable and non-integrable spin-chains



- in the non-integrable case A_{nm} can be described statistically

ETH recap

- ETH – anzats for the matrix elements
- diagonal ETH – quantum ergodicity (independence of initial state)
  stark difference between integrable and generic case at the level of $\langle \varepsilon_n^2 \rangle$ and $\max |\varepsilon_n|$
- off-diagonal ETH – statistical nature of A_{nm}

questions to explore

- ETH in semiclassical regime for classically ergodic systems
  Berry'72-89; Hortikar, Srednicki, 9711020, 9908009; Eckhardt, Main, Physc. Rev. Lett. 75, 2300
- off-diagonal ETH for single particle problem (continuous and discrete)
  extension of Shnirelman theorem
- systems with symmetry sectors, e.g. tr. invariant

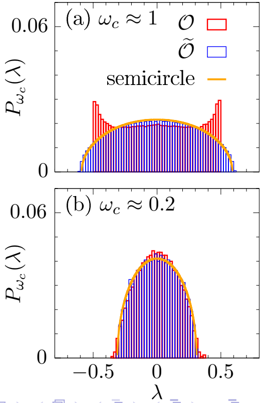
Beyond “standard” ETH:

Beyond "standard" ETH

- ETH and thermalization dynamics?
- RMT behavior of A_{nm}?

  within small energy window A_{nm} is a GUE (GOE)

- individual A_{nm} are distributed normally
- $f(\bar{E}, \omega)$ is approximately constant for $\omega \le \tau^{-1}$
- correct ratio of $\overline{A_{nn}^2}$ and $\overline{A_{nm}^2}$
- spectrum of A_{nm} approaches the Wigner's semicircle



Richter et al., Phys. Rev. E 102, 042127

When is the onset of GUE/GOE?

- energy scale marking the onset of GUE/GOE is parametrically smaller than τ^{-1}
- consider $a(t) = \langle\Psi|A(t)|\Psi\rangle$ in a special semiclassical initial state with known late time dynamics of $a(t)$

  for diffusive systems, Ψ describes "cos" distribution of energy density
- integral quantity $\int a(t)\sin(t/T)\,dt/t$ can be bounded from above by largest eigenvalue of A_{nm} within the band of size T^{-1}, for *any* Ψ,

$$T_{GUE} \geq \tau\, S$$

- interpretation of T_{GUE}: onset of late-time quantum fluctuations $a(t) \approx e^{-t/\tau} \sim e^{-S}$

AD, Phys. Rev. Lett. 128, 190601

RMT for A_{nm}?

- to what extent A_{nm} can be described by an RMT?

  possible non-Gaussian RMT description beyond T^{-1}_{GUE}

  Pappalardi, Foini, Kurchan, Phys. Rev. Lett. 129, 170603

  Jafferis et al, 2209.02131

- RMT implications for dynamics?

  2pt function is independent of the statistics of R_{nm}

$$\langle A(t)A(0)\rangle_{\bar{E}} = \int d\omega f^2(\bar{E},\omega)e^{-i\omega t}$$

- 2pt function fixes dynamics $\langle\Psi|A(t)|\Psi\rangle \sim \langle A(t)A(0)\rangle$
  Srednicki'99, Richter et al., Phys. Rev. E 99, 050104(R)
- (conjectural) bound on $\int a(t)\sin(t/T)dt/t$ in terms of max eigenvalues of A_{nm}
  AD, Phys. Rev. B 99, 224302

ETH for integrable systems?

Generalized ETH

- quantum systems with (infinitely) many conserved quantities Q_i

$$\langle E_n|A|E_n\rangle = A_{nn} = A^{\mathrm{eth}}(Q_i) + \varepsilon_n$$

- reduced density matrix is described by a GGE

$$\mathrm{Tr}_{\bar{\mathcal{A}}}\left(|E\rangle\langle E|\right) \approx \mathrm{Tr}_{\bar{\mathcal{A}}}\, e^{-\sum \mu_i Q_i}/Z$$

Rigol et al., Phys. Rev. Lett. 98, 050405 (2007)

- scaling of ε_n is not understood, as well as many other details

ETH in CFT

ETH in CFT

- conformal field theories (CFT) are QFTs without dimensional parameters

  rigid mathematical structure, theory is specified by the "CFT data" – dimensions Δ_i and OPE coefficients C_{ij}^k

- statements about matrix elements – statements about OPE coefficients

  diagonal ETH translates into

$$C_{HH}^{L} \sim {\Delta_H}^{\Delta_L/(d+1)}, \qquad \Delta_H \to \infty$$

Lashkari, AD, Liu, J. Stat. Mech. (2018) 033101

ETH in 2d CFT

- specifics of 2d case: vanishing thermal expectation value of (almost) all operators, infinite-dimensional Virasoro algebra
- conjecture:

$$C^{L}_{HH} \to 0, \quad \Delta_H \to 0$$

averaged value vanishes, $\overline{C^{L}_{HH}} \to 0$

- stress-energy tensor sector in an integrable system (quantum KdV), fixed by symmetry
  - infinite number of quantum KdV charges in involution
  - at leading $1/c$ order GETH with polynomial A^{geth}

$$\langle E|A|E\rangle = A^{\text{geth}}(Q) + O(1/c)$$

  - at finite c situation is more complicated

AD, Pavlenko, Phys. Rev. Lett. 123, 111602

Future directions

- ETH for single particle or semiclassical systems

  relation between ETH and level statistics
- ETH for systems with discrete translational invariance

  and other instances of symmetry sectors
- RMT description of A_{nm}
- ETH and approach to equilibrium
- formulation of generalized ETH for integrable systems

  which Q to include, scaling of ε_n, structure of off-diagonal matrix elements
- statistical approach to OPE coefficients in CFT
- a whole lot more...