Rank-uniform local law and quantum unique ergodicity for Wigner matrices

László Erdős (IST Austria)

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Joint with Giorgio Cipolloni, PCTS (Princeton University) and Dominik Schröder (ITS-ETH)

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Collége de France, Paris
Quantum Unique Ergodicity
Quantization of classical systems: $p \rightarrow -i\hbar \nabla_x$

Motto:

Eigenfunctions of the quantization of a chaotic classical dynamics are uniformly distributed.
Quantum (Unique) Ergodicity

Most prominent example:

$\psi_i$: efn’s of Laplace-Beltrami operator on a surface with ergodic geodesic flow, then

$$\langle \psi_i, A\psi_j \rangle \to \delta_{ij} \int_{S^*} \sigma(A), \quad i, j \to \infty$$

holds for any appropriate pseudo-differential operator $A$ with symbol $\sigma(A)$ (defined on the unit tangent bundle).
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Proven for most index pairs Quantum Ergodicity (Šnirel’man 1974), (Zelditch 1987), (Colin de Verdière 1985).

Analogous discrete version on large regular graphs (Anantharaman, Le Masson 2015)
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Physics prediction for generic systems (Feingold, Peres 1986), (Eckhardt et al. 1995)

\[ \text{Var}[\langle \psi_i, A\psi_i \rangle] \sim (\text{local ev. spacing}) \cdot \int_{S^*} \sigma(|A|^2). \]
E. Wigner’s vision: energy levels of large quantum systems can be modelled by eigenvalues of large random matrices (e.g. by Wigner matrices).
**Wigner matrices**

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**Definition [Wigner matrix]:** $N \times N$ Hermitian random matrix $W = W^*$

- Independent identically distributed entries up to Hermitian symmetry $w_{ab} = \overline{w_{ba}}$
- Normalization: $E w_{ab} = 0$  $E |w_{ab}|^2 = \frac{1}{N}$
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\[
\rho(x) = \frac{\sqrt{4 - x^2}}{2\pi}
\]

Semicircular density of states $\rho$; Bulk level spacing $\sim N^{-\frac{1}{2}}$

Histogram of rescaled gaps and Wigner surmise
E. Wigner’s vision: energy levels of large quantum systems can be modelled by eigenvalues of large random matrices (e.g. by Wigner matrices)

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![Semicircular density of states](image1)

- Semicircular density of states $\rho$; Bulk level spacing $\sim N^{-1}$

Wigner’s revolutionary observation: the gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric), independent of the distribution.

Formulated as the Wigner-Dyson-Mehta conjecture in 60’s, ground-breaking step by Johansson in 1998 (add Gaussian component); finally proven by DBM around 2010.
Extension of Wigner’s vision to Quantum Chaos: Random matrices model chaotic quantum systems, hence QUE is expected to hold for Wigner matrices with optimal speed. Formulated as the Eigenstate Thermalisation Hypothesis by (Deutsch 1991).
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**Theorem [Cipolloni, E., Schröder 2022]**

For the orthonormal eigenvectors \( u_i \) of an \( N \times N \) Wigner matrix \( W \) and for any deterministic observable (matrix) \( A \)

\[
\max_{i,j \in \text{bulk}} \left| \langle u_i, Au_j \rangle - \delta_{ij} \langle A \rangle \right| \lesssim \frac{N \epsilon \langle |\tilde{A}|^2 \rangle^{1/2}}{\sqrt{N}}
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with very high probability, where \( \langle A \rangle := \frac{1}{N} \text{Tr} A \) and \( \tilde{A} := A - \langle A \rangle \) is the traceless part of \( A \).
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Eigenbasis $\{u_i\}$ is asymptotically orthogonal to $\{A u_j\}$ for $\langle A \rangle = 0$.

As if $u_i$ and $Au_j$ were independently distributed $\ell^2$-bounded $N$-vectors.

Two basic methods:

- Resolvent method
- Dyson Brownian Motion (DBM)
Comparison with previous results

\[
\max_{i,j \in \text{bulk}} |\langle u_i, \hat{A} u_j \rangle| \leq \frac{N^\epsilon \langle |\hat{A}|^2 \rangle^{1/2}}{\sqrt{N}}, \quad \hat{A} := A - \langle A \rangle, \quad \text{with high prob.}
\]

Previous results:

- \( A = |q \rangle \langle q | \) rank-1 observable = delocalization of eigenvectors, \( |\langle u_i, q \rangle| \lesssim N^{-1/2} + \epsilon \) [E. et al. (2009), Knowles-Yin (2011)] [Resolvent method]
- \( \langle u_i, A u_i \rangle \rightarrow \langle A \rangle \) in probability for each \( u_i \) [Bourgade-Yau (2013)] [DBM]
- Simultaneously in \( i \) and \( j \) [in the bulk] — proven only for Wigner matrices with large (almost \( \mathcal{O}(1) \)) Gaussian component [Bourgade-Yau-Yin (2018)] [DBM]
- Uniformly in the spectrum if \( \langle |\hat{A}|^2 \rangle^{1/2} \) replaced by \( ||\hat{A}|| \) [Cipolloni, E, Schröder (2020)] [Resolvent method].
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Novelties of our results: [Resolvent method]

- Optimal \( N^{-1/2} \) speed of convergence. In physics: Eigenstate Thermalisation Hypothesis
- Limit is controlled in very high probability, and thus simultaneous in \( i, j \).
- Optimal dependence of the error on \( A \) (HS is the correct norm – our newest result).
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\max_{i,j \in \text{bulk}} \left| \langle u_i, \hat{A} u_j \rangle \right| \leq \frac{N^c \langle |\hat{A}|^2 \rangle^{1/2}}{\sqrt{N}}, \quad \hat{A} := A - \langle A \rangle, \quad \text{with high prob.}
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These are LLN-type results. Next: What about CLT for \( [\langle u_i, A u_i \rangle - \langle A \rangle] \)?
Averaged CLT for overlaps

CLT (central limit theorem) for \([\langle u_i, Au_i \rangle - \langle A \rangle]\) can be asked in two senses.

First, we proved CLT after averaging in the index \(i\):

**Theorem (Cipolloni, E., Schröder 2020)**

For any bounded deterministic matrix \(A\), \(i_0 \in [\delta N, (1 - \delta)N]\) (i.e. bulk) and for any \(K \geq N^\epsilon\)

\[
\frac{1}{\sqrt{2K}} \sum_{|i - i_0| \leq K} \sqrt{N} \left[ \langle u_i, Au_i \rangle - \langle A \rangle \right] \overset{m}{\approx} \mathcal{N}(0, \langle |\hat{A}|^2 \rangle) + \mathcal{O}(N^{-\epsilon'} \|\hat{A}\|)
\]

in the sense of moments, where \(\hat{A} := A - \langle A \rangle\) is the traceless part of \(A\).

Similar result holds at the edge with a variance \(\frac{\sqrt{2}}{3} \langle |\hat{A}|^2 \rangle\).
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Similar result holds at the edge with a variance \(\frac{\sqrt{2}}{3} \langle |\hat{A}|^2 \rangle\).

\(\implies\) Indication that \(\langle u_i, \hat{A}u_i \rangle, \langle u_j, \hat{A}u_j \rangle\) are asymptotically independent for \(i \neq j\).

This CLT is a special case of our general functional CLT: \(\langle f(W)A \rangle \approx \mathcal{N}\) for any fn. of the Wigner matrix \(W\); unlike usual tracial CLT in random matrices, this involves eigenvectors as well!

Averaged CLT uses resolvent method.
Averaged CLT for overlaps

CLT (central limit theorem) for $[\langle u_i, Au_i \rangle - \langle A \rangle]$ can be asked in two senses.

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Averaged CLT uses resolvent method.

Second, CLT for each $\langle u_i, Au_i \rangle - \langle A \rangle$ without averaging?
Theorem (Cipolloni, E, Schröder (2022))

For the bulk eigenvectors $u_i$ of an $N \times N$ Wigner matrix $W$ and for any deterministic hermitian observable (matrix) $A$ with $1 \gtrsim \langle \hat{A}^2 \rangle \geq N^{-1+\delta} \|\hat{A}\|^2$ it holds:

$$\sqrt{\frac{N}{2 \langle \hat{A}^2 \rangle}} \left[ \langle u_i, Au_i \rangle - \langle A \rangle \right] \to \mathcal{N}(0, 1)$$  \hspace{1cm} (1)

in the sense of moments, where $\hat{A} := A - \langle A \rangle$ is the traceless part of $A$. 

Remark: $\langle \hat{A}^2 \rangle \geq N^{-1+\delta} \|\hat{A}\|^2$ implies $A$ not finiterank. (1) is not true for finiterank: for $A = \hat{A} = |e_x\rangle\langle e_x| - |e_y\rangle\langle e_y|$ we have $\langle u_i, \hat{A} u_i \rangle = |u_i(x)|^2 - |u_i(y)|^2$ (difference of independent $\chi^2$) [Bourgade-Yau (2017)].

To prove this theorem we need DBM method on top of the resolvent method.
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CLT for individual overlaps

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To prove this theorem we need DBM method on top of the resolvent method.
Comparison with previous results

\[ \sqrt{\frac{N}{2\langle A^2 \rangle}} \langle u_i, \hat{A} u_i \rangle \to \mathcal{N}(0, 1), \quad \hat{A} = A - \langle A \rangle \]

Previous results:

- Joint (squared) Gaussianity for finitely many \( N|u_{\alpha l}(i_k)|^2 \) under the additional four moment matching assumption for bulk eigenvectors (only two moments at the edge). [Knowles-Yin (2011)] [Resolvent method]

- Rank 1: \( N|\langle u_i, q \rangle|^2 \) is asymptotically (squared) Gaussian [Bourgade-Yau (2013)] [DBM]

- Finite rank: Joint (squared) Gaussianity for finitely many \( u \)'s and \( q \)'s [Marcinek-Yau (2020)] [DBM]

- (Almost) full rank: Gaussianity for \( 1 \gtrsim \langle \hat{A}^2 \rangle \gtrsim \delta \|\hat{A}\|^2 \) [Cipolloni, E, Schröder (2021)] [DBM]
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Related independent result:

Gaussianity of \( \langle u_i, \hat{A}u_i \rangle \) for the special case \( A = \sum_{j \in I} |e_j\rangle \langle e_j | \) with \( N^\epsilon \leq |I| \leq N^{1-\epsilon} \), i.e. low rank (also at the edge) [Benigni-Lopatto (2021)] [DBM]
Proof of Gaussian fluctuation (via DBM)
**GOAL:** Let $u_i$ be the eigenvectors of a Wigner matrix $W$, then

$$E \left[ \sqrt{\frac{N}{2 \langle \hat{A}^2 \rangle}} \langle u_i, \hat{A} u_i \rangle \right]^n \rightarrow (n - 1)!! \mathbf{1}(n \text{ even}), \quad \hat{A} = A - \langle A \rangle.$$
GOAL: Let $u_i$ be the eigenvectors of a Wigner matrix $W$, then

$$E \left[ \sqrt{\frac{N}{2}} \left\langle u_i, \hat{A} u_i \right\rangle \right]^n \rightarrow (n - 1)!! 1 (n \text{ even}), \quad \hat{A} = A - \left\langle A \right\rangle.$$ 

We do it dynamically:

$$dW_t = \frac{dB_t}{\sqrt{N}}, \quad W_0 = W. \quad (2)$$

The flow (2) adds a Gaussian component of size $\sqrt{t}$ to $W_0$. 

$\text{DBM for eigenvectors}$
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We do it **dynamically**:

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dW_t = \frac{d\hat{B}_t}{\sqrt{N}}, \quad W_0 = W. \quad (2)
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The flow (2) adds a Gaussian component of size $\sqrt{t}$ to $W_0$.

**Need only** $t \sim N^{-1+\epsilon}$. This Gaussian component can later be removed by simple perturbation theory known as Green function comparison theorem (GFT).
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The flow (2) induces the Dyson Brownian Motion (DBM) for eigenvalues and eigenvectors:

$$d\lambda_i(t) = \frac{d\hat{B}_{ii}(t)}{\sqrt{N}} + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} \, dt$$

$$d\mathbf{u}_i(t) = \frac{1}{\sqrt{N}} \sum_{j \neq i} \frac{dB_{ij}(t)}{\lambda_i(t) - \lambda_j(t)} \mathbf{u}_j(t) - \frac{1}{2N} \sum_{j \neq i} \frac{\mathbf{u}_i(t)}{(\lambda_i(t) - \lambda_j(t))^2} \, dt.$$ 

(3)
GOAL: Let $u_i$ be the eigenvectors of a Wigner matrix $W$, then

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Remark: The BMs $B_{ij}(t)$, for $i \neq j$, and $B_{ii}(t)$ are independent!
Example for $n = 2$

By Ito’s formula (from now on $\langle A \rangle = 0$, i.e. $A = \hat{A}$):

$$
\text{d} E \left[ |\langle u_i, A u_i \rangle|^2 | \lambda \right] = \sum_{k \neq i} \frac{E \left[ |\langle u_k, A u_i \rangle|^2 | \lambda \right] - E \left[ |\langle u_i, A u_i \rangle|^2 | \lambda \right]}{N(\lambda_k - \lambda_i)^2} + \ldots
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**Problem:** The flow for diagonal overlaps $\langle u_i, Au_i \rangle$ depends on off-diagonal overlaps $\langle u_i, Au_j \rangle$!
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Problem: The flow for diagonal overlaps $\langle u_i, Au_i \rangle$ depends on off-diagonal overlaps $\langle u_i, Au_j \rangle$!

However, for a special combination

$$f_t(i, j) = \mathbb{E} \left[ 2|\langle u_i, Au_j \rangle|^2 + \langle u_i, Au_i \rangle \langle u_j, Au_j \rangle | \lambda \right]$$

we have

$$\partial_t f_t(i, j) = \sum_{k \neq i} \frac{f_t(k, j) - f(i, j)}{N(\lambda_k - \lambda_i)^2} + \sum_{k \neq i} \frac{f_t(i, k) - f(i, j)}{N(\lambda_k - \lambda_j)^2}.$$
Example for \( n = 2 \)

By Ito’s formula (from now on \( \langle A \rangle = 0 \), i.e. \( A = \hat{A} \)):

\[
d E \left[ |\langle \mathbf{u}_i, A\mathbf{u}_i \rangle|^2 | \chi \right] = \sum_{k \neq i} \frac{E \left[ |\langle \mathbf{u}_k, A\mathbf{u}_i \rangle|^2 | \chi \right] - E \left[ |\langle \mathbf{u}_i, A\mathbf{u}_i \rangle|^2 | \chi \right]}{N(\lambda_k - \lambda_i)^2} + \ldots
\]

Problem: The flow for diagonal overlaps \( \langle \mathbf{u}_i, A\mathbf{u}_i \rangle \) depends on off-diagonal overlaps \( \langle \mathbf{u}_i, A\mathbf{u}_j \rangle \)!

However, for a special combination

\[
f_t(i, j) = E \left[ 2|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2 + \langle \mathbf{u}_i, A\mathbf{u}_i \rangle \langle \mathbf{u}_j, A\mathbf{u}_j \rangle | \chi \right]
\]

we have

\[
\partial_t f_t(i, j) = \sum_{k \neq i} \frac{f_t(k, j) - f(i, j)}{N(\lambda_k - \lambda_i)^2} + \sum_{k \neq i} \frac{f_t(i, k) - f(i, j)}{N(\lambda_k - \lambda_j)^2}.
\]

For general \( n \): Closed equation for a certain lin. combination of overlaps \( f_t \) [Bourgade-Yau-Yin (2018)]

\[
\partial_t f_t = \mathcal{L}(t)f_t,
\]

for a certain operator \( \mathcal{L}(t) \).

We explain the basic construction from [Bourgade-Yau-Yin (2018)]
**Particle representation 1**

**Representation:** We can think of $f_t = f_t(i_1, \ldots, i_n)$ as a function of "n-particle configurations" on $N$. 

$\begin{align*}
f_t(i,j) &= |\langle u_i, A u_j \rangle|^2 + |\langle u_i, A u_i \rangle \langle u_j, A u_j \rangle|^2 \\
\end{align*}$
Representation: We can think of \( f_t = f_t(i_1, \ldots, i_n) \) as a function of ”n-particle configurations” on \( N \).

Example for \( n = 2 \):

\[
\begin{align*}
\bullet & \quad \bullet \\
\_ & \quad _
\end{align*}
\]

\( i \quad j \)

**Figure 3:** Particle configuration for \( n = 2, f_t = f_t(i, j) \).
**Particle representation 1**

**Representation:** We can think of $f_t = f_t(i_1, \ldots, i_n)$ as a function of "n-particle configurations" on $N$.

**Example for $n = 2$:**

![Figure 3: Particle configuration for $n = 2, f_t = f_t(i, j)$.](image)

$f_t(i, j) = 2|\langle u_i, Au_j \rangle|^2 + \langle u_i, Au_i \rangle \langle u_j, Au_j \rangle$ takes value in the "doubled space":

![Figure 4: $|\langle u_i, Au_j \rangle|^2 + |\langle u_i, Au_j \rangle|^2$](image)

![Figure 5: $\langle u_i, Au_i \rangle \langle u_j, Au_j \rangle$](image)
**Particle representation 2**

*Representation*: We can think of \( f_t = f_t(i_1, \ldots, i_n) \) as a function of ”n-particle configurations” on \( N \).
Particle representation 2

**Representation:** We can think of \( f_t = f_t(i_1, \ldots, i_n) \) as a function of "n-particle configurations" on \( \mathbb{N} \).

**Example for \( n = 7 \):**

**Figure 6:** Argument of \( f(i, j, k, k, l, l) \)

**Figure 7:**

\[
E\left[ |\langle u_i, Au_j \rangle|^2 \langle u_k, Au_k \rangle |\langle u_k, Au_l \rangle|^4 \lambda \right]
\]
Particle representation 2

**Representation:** We can think of \( f_t = f_t(i_1, \ldots, i_n) \) as a function of "n-particle configurations" on \( N \).

**Example for \( n = 7 \):**

\[
\begin{align*}
\text{Figure 6: Argument of } f(i, j, k, k, k, l, l)
\end{align*}
\]

\[
\begin{align*}
\text{Action of } L(t):
\end{align*}
\]

\[
\begin{align*}
\text{Figure 8: Argument of } Lf_t:
\end{align*}
\]

\[
\begin{align*}
\text{Figure 7: } E \left[ | \langle u_i, Au_j \rangle |^2 \langle u_k, Au_k \rangle | \langle u_k, Au_l \rangle |^2 \lambda \right]
\end{align*}
\]

\[
\begin{align*}
\text{Figure 9: }
\end{align*}
\]

\[
\begin{align*}
E \left[ | \langle u_i, Au_j \rangle |^2 \langle u_k, Au_k \rangle | \langle u_k, Au_l \rangle |^2 | \langle u_k, Au_{l_0} \rangle |^2 \lambda \right]
\end{align*}
\]
Recall: We have $\partial_t f_t = \mathcal{L}(t)f_t$. For $n = 2$:

$$\partial_t f_t(i,j) = \sum_{k \neq i} \frac{f_t(k,j) - f(i,j)}{N(\lambda_k - \lambda_i)^2} + \sum_{k \neq i} \frac{f_t(i,k) - f(i,j)}{N(\lambda_k - \lambda_j)^2}.$$
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• Here

$$\mathcal{L}(t) = \sum_{r=1}^{n} \mathcal{L}_r(t), \quad \mathcal{L}_r(t)f_t(i_1,\ldots,i_n) = \sum_{k \neq i_r} \frac{f(i_r \rightarrow k) - f_t(i_1,\ldots,i_n)}{N(\lambda_k - \lambda_{i_r})^2}.$$
Sketch of the proof II

Recall: We have $\partial_t f_t = \mathcal{L}(t)f_t$. For $n = 2$:

$$\partial_t f_t(i, j) = \sum_{k \neq i} \frac{f_t(k, j) - f(i, j)}{N(\lambda_k - \lambda_i)^2} + \sum_{k \neq i} \frac{f_t(i, k) - f(i, j)}{N(\lambda_k - \lambda_j)^2}.$$ 

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$\mathcal{L}_r$ acts on the location index of the $r$-th particle; it has a kernel

$$\frac{1}{N(\lambda_k - \lambda_{i_r})^2} \sim \frac{N}{|i_r - k|^2}.$$
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$$

• Here

$$
\mathcal{L}(t) = \sum_{r=1}^{n} \mathcal{L}_r(t), \quad \mathcal{L}_r(t)f_t(i_1, \ldots, i_n) = \sum_{k \neq i_r} \frac{f(i_r \rightarrow k) - f_t(i_1, \ldots, i_n)}{N(\lambda_k - \lambda_{i_r})^2}.
$$

$\mathcal{L}_r$ acts on the location index of the $r$-th particle; it has a kernel

$$
\frac{1}{N(\lambda_k - \lambda_{i_r})^2} \sim \frac{N}{|i_r - k|^2}.
$$

Note that this is the discretisation of the $\sqrt{-\Delta} = |\rho|$ operator in 1d

$$
\implies \partial_t f_t = \mathcal{L}(t)f_t \text{ is a (discrete) heat equation with fractional Laplacian}
$$
Sketch of the proof III

- For $\partial_t f_t = \mathcal{L}(t)f_t$, want: heat kernel $e^{-\mathcal{L}(t)}$ averages in all directions.

Have (recall $f_t(k,j) = 2|\langle u_k, Au_j \rangle|^2 + \langle u_k, Au_k \rangle \langle u_j, Au_j \rangle$):

$$\sum_{k \neq i} \frac{f_t(k,j)}{N(\lambda_k - \lambda_i)^2} \approx \sum_k \frac{f_t(k,j)\eta}{(\lambda_k - \lambda_i)^2 + \eta^2} = \langle u_j, A\mathcal{G}(\lambda_i + i\eta)Au_j \rangle + \ldots,$$

with $\eta \sim N^{-1}$ and $G(z) := (W - z)^{-1}$.

Want: average also in the $j$ index!
Sketch of the proof III

• For $\partial_t f_t = \mathcal{L}(t)f_t$, want: heat kernel $''e^{-\mathcal{L}(t)}''$ averages in all directions.

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$$\sum_{k \neq i} \frac{f_t(k, j)}{N(\lambda_k - \lambda_i)^2} \approx \sum_k \frac{f_t(k, j)t}{(\lambda_k - \lambda_i)^2 + \eta^2} = \langle u_j, A\mathcal{G}(\lambda_i + i\eta)Au_j \rangle + \ldots,$$

with $\eta \sim N^{-1}$ and $G(z) := (W - z)^{-1}$.

Want: average also in the $j$ index!

Why averages? Because they can be understood by local laws!
For $\partial_t f_t = \mathcal{L}(t)f_t$, want: heat kernel $e^{-\mathcal{L}(t)}$ averages in all directions.

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with $\eta \sim N^{-1}$ and $G(z) := (W - z)^{-1}$.

Want: average also in the $j$ index!

Why averages? Because they can be understood by local laws!

---

**Local law:** Approximation of the resolvent $G(z)$ by a deterministic object, i.e.

$$\langle G(z) \rangle = m(z) + O\left(\frac{1}{N\Im z}\right), \quad m(z) := \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{x - z} \, dx = O(1).$$
For $\frac{\partial}{\partial t} f_t = \mathcal{L}(t) f_t$, want: heat kernel $e^{-\mathcal{L}(t)}$ averages in all directions.

Have (recall $f_t(k, j) = 2|\langle u_k, Au_j \rangle|^2 + \langle u_k, Au_k \rangle \langle u_j, Au_j \rangle$):

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Local law: Approximation of the resolvent $G(z)$ by a deterministic object, i.e.

$$\langle G(z) \rangle = m(z) + \mathcal{O} \left( \frac{1}{N \Im z} \right), \quad m(z) := \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{x - z} \ dx = O(1).$$

By spectral decomposition

$$\langle \Im G(E + i\eta) A \Im G(E' + i\eta') A \rangle = \frac{1}{N^2} \sum_{ij} N|\langle u_i, Au_j \rangle|^2 \frac{\eta}{|\lambda_i - E|^2 + \eta^2} \frac{\eta'}{|\lambda_j - E'|^2 + (\eta')^2}.$$

$$\implies \text{Av}_{ij} [N|\langle u_i, Au_j \rangle|^2] \approx \langle \Im G(E + i\eta) A \Im G(E' + i\eta') A \rangle.$$
Sketch of the proof III

- For $\partial_t f_t = \mathcal{L}(t)f_t$, want: heat kernel $e^{-\mathcal{L}(t)}$ averages in all directions.

  Have (recall $f_t(k,j) = 2|\langle u_k, Au_j \rangle|^2 + \langle u_k, Au_k \rangle \langle u_j, Au_j \rangle$):

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  with $\eta \sim N^{-1}$ and $G(z) := (W - z)^{-1}$.

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$$\Rightarrow \text{Av}_{ij} \left[ N|\langle u_i, Au_j \rangle|^2 \right] \approx \langle \Im G(E + i\eta)A\Im G(E' + i\eta')A \rangle.$$

$$\Rightarrow \text{Need local law } \langle \Im G A\Im G A \rangle \approx \Im m \Im m' \langle A^2 \rangle \text{ with error in terms of } \langle A^2 \rangle!$$
Sketch of the proof III

- For $\partial_t f_t = \mathcal{L}(t)f_t$, want: heat kernel $e^{-\mathcal{L}(t)}$ averages in all directions.

Have (recall $f_t(k, j) = 2|\langle u_k, Au_j \rangle|^2 + \langle u_k, Au_k \rangle \langle u_j, Au_j \rangle$):

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with $\eta \sim N^{-1}$ and $G(z) := (W - z)^{-1}$.

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**Local law**: Approximation of the resolvent $G(z)$ by a deterministic object, i.e.

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By spectral decomposition

$$\langle \mathcal{G}(E + i\eta) A \mathcal{G}(E' + i\eta') A \rangle = \frac{1}{N^2} \sum_{ij} N|\langle u_i, Au_j \rangle|^2 \frac{\eta}{|\lambda_i - E|^2 + \eta^2} \frac{\eta'}{|\lambda_j - E'|^2 + (\eta')^2}. $$

$$\Rightarrow Av_{ij}[N|\langle u_i, Au_j \rangle|^2] \approx \langle \mathcal{G}(E + i\eta) A \mathcal{G}(E' + i\eta') A \rangle.$$

$$\Rightarrow \text{Need local law } \langle \mathcal{G} A \mathcal{G}' A \rangle \approx \mathcal{G}m \mathcal{G}' \langle A^2 \rangle \text{ with error in terms of } \langle A^2 \rangle!$$

**Gain**: $\langle \mathcal{G} A \mathcal{G}' A \rangle$ much easier to understand than $\langle u_j, A \mathcal{G}(\lambda_i)Au_j \rangle$!
Recall: We consider $\partial_t f_t = \mathcal{L}(t)f_t$ (discrete heat equation with fraction Laplacian on $N$).

Want heat kernel $" e^{-\mathcal{L}(t)}"$ averages in all directions.

**Why averages?** Because only they can be understood by local laws!

- **Heuristically:***

  $$\mathcal{L}(t) = \sum_{r=1}^{n} \mathcal{L}_r(t), \quad \mathcal{L}_r(t) \approx |p_r| := \sqrt{-\Delta_r},$$

  i.e. $\mathcal{L}(t)$ (=infinitesimally the heat kernel) averages only in one coordinate direction. One direction is not enough, local laws require averaging in ALL directions.
Recall: We consider $\partial_t f_t = \mathcal{L}(t)f_t$ (discrete heat equation with fraction Laplacian on $\mathbb{N}$).

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  i.e. $\mathcal{L}(t)$ (=infinitesimally the heat kernel) averages only in one coordinate direction. One direction is not enough, local laws require averaging in ALL directions.

- **To get more averaging:** Replace $\mathcal{L}(t) = \sum_r |p_r|$ by the regularised product
  $$\mathcal{A}(t) := \frac{1}{\eta} \prod_{r=1}^{n} (1 - e^{-\eta|p_r|}) \left( \sim \eta^{n-1} \prod_{r=1}^{n} |p_r| \text{ morally} \right)$$
  with $\eta \sim N^{-1} \implies$ Average in any direction.
  The replacement is possible on the level of Dirichlet form, $D(f) := \langle f, \mathcal{L}f \rangle$. 

Sketch of the proof V

Replace $\mathcal{L}(t) \approx \sum_r |p_r| \ (\text{average in one direction})$ by the regularised product

$$\mathcal{A}(t) := \frac{1}{\eta} \prod_{r=1}^{n} \left( 1 - e^{-\eta |p_r|} \right) \left( \sim \eta^{n-1} \prod_{r=1}^{n} |p_r| \text{ morally} \right)$$

with $\eta \sim N^{-1} \implies \text{Average in any direction.}$

• Indeed heuristically:

$$\mathcal{L}(t)f_t = \sum_{r=1}^{n} \sum_{k \neq i_r} f(i_r \rightarrow k) - f_t(i_1, \ldots, i_n) \frac{N(\lambda_k - \lambda_{i_r})^2}{N},$$

$$\mathcal{A}(t)f_t = \frac{1}{N^{n-1}} \sum_{k_1 \neq i_1} \cdots \sum_{k_n \neq i_n} \frac{f(i_r \rightarrow k_r \ \forall r) - f_t(i_1, \ldots, i_n)}{N^n(\lambda_{k_1} - \lambda_{i_1})^2 \cdots (\lambda_{k_n} - \lambda_{i_n})^2}.$$
Replace $\mathcal{L}(t) \approx \sum_r |p_r|$ (average in one direction) by the regularised product

$$
\mathcal{A}(t) := \frac{1}{\eta} \prod_{r=1}^n \left( 1 - e^{-\eta |p_r|} \right)
\begin{pmatrix}
\sim \eta^{n-1} \prod_{r=1}^n |p_r| & \text{morally}
\end{pmatrix}
$$

with $\eta \sim N^{-1} \implies$ Average in any direction.

- Indeed heuristically:

$$
\mathcal{L}(t)f_t = \sum_{r=1}^n \sum_{k \neq i_r} \frac{f(i_r \to k) - f_t(i_1, \ldots, i_n)}{N(\lambda_k - \lambda_{i_r})^2},
$$

$$
\mathcal{A}(t)f_t = \frac{1}{N^{n-1}} \sum_{k_1 \neq i_1} \cdots \sum_{k_n \neq i_n} \frac{f(i_r \to k_r \, \forall r) - f_t(i_1, \ldots, i_n)}{N^n(\lambda_{k_1} - \lambda_{i_1})^2 \cdots (\lambda_{k_n} - \lambda_{i_n})^2}.
$$

The replacement is possible on the level of Dirichlet form, $D(f) := \langle f, \mathcal{L}f \rangle$.

- Main technical steps:
  (i) the energy method for DBM [Marcinek-Yau (2020)] analysing

$$
\partial_t \|f_t\|_2^2 = -2D_t(f_t) \leq 0.
$$

(ii) local laws for $\langle GAGA \ldots \rangle$ with $\langle A \rangle = 0$ and $\langle A^2 \rangle$ errors [Cipolloni, E, Schröder (2022)].
Multi-resolvents local laws with $\langle A^2 \rangle$ errors

**Theorem (Cipolloni, E, Schröder 2022)**

For any deterministic observable $A$, with $\langle A \rangle = 0$, it holds:

$$\left| \langle G(z_1)AG(z_2)A \rangle - m(z_1)m(z_2) \langle A^2 \rangle \right| \leq \frac{\langle A^2 \rangle}{\sqrt{N_\eta}} \ll \langle A^2 \rangle$$

(5)

for any $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$ such that $\eta := \mathbb{S}z_1 \sim \mathbb{S}z_2 \geq N^{-1+\epsilon}$. 

Remark: (6) is a special case of a general multi-resolvents local laws for $G_1A_1 \ldots G_kA_k$ with optimal dependence on $A$ – we call it rank uniformity.
**Theorem (Cipolloni, E, Schröder 2022)**

For any deterministic observable $A$, with $\langle A \rangle = 0$, it holds:

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$$

(5)

for any $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$ such that $\eta := \Im z_1 \sim \Im z_2 \geq N^{-1+\epsilon}$.

Significant improvement compared to

$$
\left| \langle G(z_1)IG(z_2)I \rangle - \frac{m(z_1)m(z_2)}{1 - m(z_1)m(z_2)} \right| \lesssim \frac{1}{N\eta^2} \sim N^{1-2\epsilon},
$$

for $\eta = \Im z_1 \sim \Im z_2 \sim N^{-1+\epsilon}$.
Multi-resolvents local laws with $\langle A^2 \rangle$ errors

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for $\eta = \Im z_1 \sim \Im z_2 \sim N^{-1+\epsilon}$. Gain almost an $N$-factor due to $\langle A \rangle = 0$!!
Multi-resolvents local laws with \( \langle A^2 \rangle \) errors

**Theorem (Cipolloni, E, Schröder 2022)**

For any deterministic observable \( A \), with \( \langle A \rangle = 0 \), it holds:

\[
\left| \langle G(z_1)AG(z_2)A \rangle - m(z_1)m(z_2) \langle A^2 \rangle \right| \leq \frac{\langle A^2 \rangle}{\sqrt{N\eta}} \ll \langle A^2 \rangle
\]  

(5)

for any \( z_1, z_2 \in \mathbb{C} \setminus \mathbb{R} \) such that \( \eta := \Im z_1 \sim \Im z_2 \geq N^{-1+\epsilon} \).

Significant improvement compared to

\[
\left| \langle G(z_1)IG(z_2)I \rangle - \frac{m(z_1)m(z_2)}{1 - m(z_1)m(z_2)} \right| \lesssim \frac{1}{N\eta^2} \sim N^{1-2\epsilon},
\]

for \( \eta = \Im z_1 \sim \Im z_2 \sim N^{-1+\epsilon} \). Gain almost an \( N \)-factor due to \( \langle A \rangle = 0 \)!!

**Remark:** (6) is a special case of a general multi-resolvents local laws for \( G_1A_1 \ldots G_kA_k \) with optimal dependence on \( A \) – we call it rank uniformity.
Previous local laws for the resolvent $G(z)$ of Wigner matrices

\[
\langle (G(z) - m(z))A \rangle \lesssim \frac{\|A\|}{N\eta}, \quad \eta := \Im z \quad \text{(averaged)}
\]

\[
\langle x, (G(z) - m(z))y \rangle \lesssim \sqrt{\frac{\rho}{N\eta}} \|x\| \|y\|, \quad \rho := \Im m \quad \text{(isotropic)}
\]

Note: $A = |y\rangle \langle x|$ for the averaged law gives an isotropic estimate off by a huge factor $\sqrt{N/\rho\eta}$ because $\|A\|$ is far from optimal for lower rank observables.
Rank uniform local law

Previous local laws for the resolvent $G(z)$ of Wigner matrices

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**Theorem [Rank-uniform local law (single resolvent)] (Cipolloni, E, Schröder 2022)**

For any deterministic observable $A$ and for any $N\eta\rho \gg 1$.

$$\left| \langle (G(z) - m(z))A \rangle \right| \lesssim \frac{|\langle A \rangle|}{N\eta} + \frac{\sqrt{\rho} \langle |\hat{A}|^2 \rangle^{1/2}}{N\sqrt{\eta}}, \quad \hat{\hat{A}} = A - \langle A \rangle.$$  \hspace{1cm} (6)

Unifies and extends the isotropic and averaged local laws. Multi-resolvent versions also hold.
Rank uniform local law

Previous local laws for the resolvent $G(z)$ of Wigner matrices

$$\langle (G(z) - m(z))A \rangle \lesssim \frac{||A||}{N\eta}, \quad \eta := \Im z \quad \text{(averaged)}$$

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**Method:** A nonlinear hierarchy of *master inequalities* for the quantities

$$\psi_k := \max \frac{N^{(3-k)/2}}{\sqrt{\rho}} \frac{1}{\prod_i \langle |A_i|^2 \rangle^{1/2}} \left| \langle G(z_1)A_1G(z_2) \ldots A_k - \prod_i m(z_i)A_i \rangle \right|,$$

where $\max$ runs over all $z_i$, $\Im z_i = \eta$ and deterministic $A_i$’s with $\langle A_i \rangle = 0$, AND a *reduction inequality* stating roughly $\psi_{2k} \ll (\psi_k)^2$ to close the hierarchy.
We proved:

- Eigenstate Thermalisation Hypothesis for Wigner matrices: eigenvector overlaps with deterministic $A$ are $\lesssim N^{-1/2}$.
- Gaussian fluctuations for eigenvector overlaps for all $A$. 
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Main technical steps:

- Energy estimates for multi indexed DBM.
- Dramatically improved local law for traceless observables.
- New hierarchy of master inequalities and its closure for $\psi_k$. 
THANK YOU VERY MUCH FOR YOUR ATTENTION!