Rank-uniform local law and quantum unique ergodicity for Wigner matrices

László Erdős (IST Austria)

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Joint with Giorgio Cipolloni, PCTS (Princeton University) *and* Dominik Schröder (ITS-ETH)

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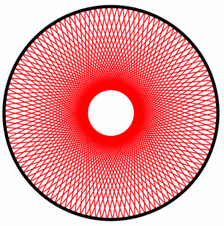
Giorgio Cipolloni

Dominik Schröder

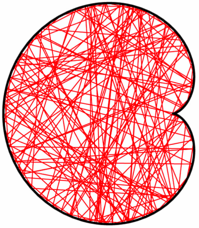
Quantum Unique Ergodicity

Motto:

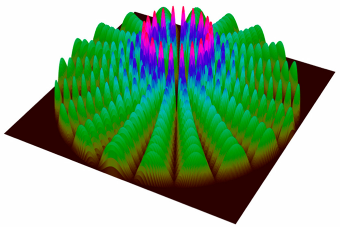
Eigenfunctions of the quantization of a chaotic classical dynamics are uniformly distributed.



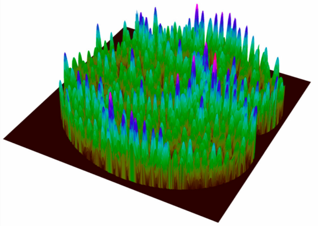
Regular (integrable) billiard



Chaotic billiard



Wavefunctions with symmetries



Chaotic wavefunctions

Quantum (Unique) Ergodicity

Most prominent example:

ψ_i: efn's of Laplace-Beltrami operator on a surface with ergodic geodesic flow, then

$$\langle \psi_i, A\psi_j \rangle \to \delta_{ij} \int_{S^*} \sigma(A), \qquad i,j \to \infty$$

holds for any appropriate pseudo-differential operator A with symbol $\sigma(A)$ (defined on the unit tangent bundle).

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Analogous discrete version on large regular graphs (Anantharaman, Le Masson 2015)

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Physics prediction for generic systems (Feingold, Peres 1986), (Eckhardt et al. 1995)

$$\mathrm{Var}\left[\langle \psi_i, A\psi_i \rangle\right] \sim \text{(local ev. spacing)} \cdot \int_{S^*} \sigma(|A|^2).$$

Wigner matrices

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Definition [Wigner matrix]: $N \times N$ Hermitian random matrix $W = W^*$

- Independent identically distributed entries up to Hermitian symmetry $w_{ab} = \overline{w}_{ba}$
- normalization: $\mathbf{E}\, w_{ab} = 0 \quad \mathbf{E}\, |w_{ab}|^2 = \frac{1}{N}$

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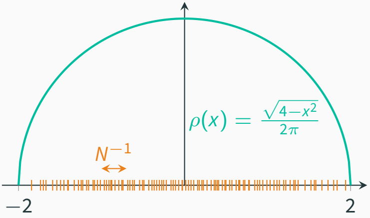

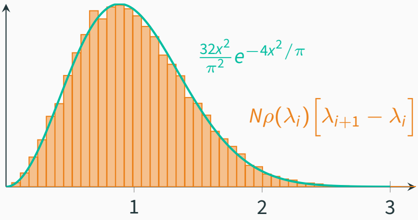

semicircular density of states ρ; Bulk level spacing $\sim N^{-1}$

Histogram of rescaled gaps and Wigner surmise

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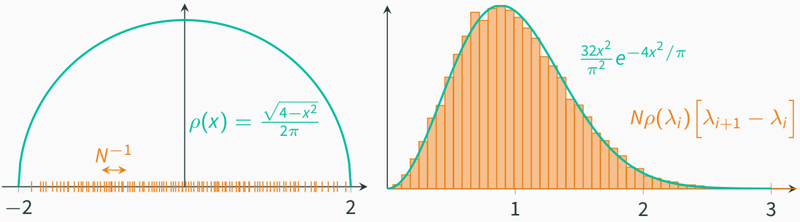

semicircular density of states ρ; Bulk level spacing $\sim N^{-1}$ Histogram of rescaled gaps and Wigner surmise

Wigner's revolutionary observation: the gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric), independent of the distribution.

Formulated as the Wigner-Dyson-Mehta conjecture in 60's, ground-breaking step by Johansson in 1998 (add Gaussian component); finally proven by DBM around 2010.

Eigenstate Thermalisation Hypothesis for Wigner matrices

Extension of Wigner's vision to Quantum Chaos: Random matrices model chaotic quantum systems, hence QUE is expected to hold for Wigner matrices with optimal speed.

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Theorem [Cipolloni, E., Schröder 2022]

For the orthonormal eigenvectors $\mathbf{u}_i$ of an $N \times N$ Wigner matrix W and for any deterministic observable (matrix) A

$$\max_{i,j \in \text{bulk}} \left| \langle \mathbf{u}_i, A\mathbf{u}_j \rangle - \delta_{ij} \langle A \rangle \right| \lesssim \frac{N^{\epsilon} \left\langle |\mathring{A}|^2 \right\rangle^{1/2}}{\sqrt{N}}$$

with very high probability, where $\langle A \rangle := \frac{1}{N} \operatorname{Tr} A$ and $\mathring{A} := A - \langle A \rangle$ is the traceless part of A.

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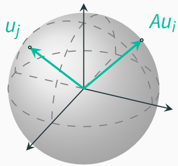
Eigenbasis $\{\mathbf{u}_i\}$ is asymptotically orthogonal to $\{A\mathbf{u}_j\}$ for $\langle A \rangle = 0$.

As if $\mathbf{u}_i$ and $A\mathbf{u}_j$ were independently distributed ℓ^2-bounded N-vectors.



Two basic methods:

Resolvent method and Dyson Brownian Motion (DBM)

Comparison with previous results

$$\max_{i,j\in\text{bulk}} \left|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_j\rangle\right| \leq \frac{N^\epsilon \langle|\mathring{A}|^2\rangle^{1/2}}{\sqrt{N}}, \quad \mathring{A} := A - \langle A\rangle, \qquad \text{with high prob.}$$

Previous results:

- $A = |\mathbf{q}\rangle\langle\mathbf{q}|$ rank-1 observable = delocalization of evectors, $|\langle\mathbf{u_i}, \mathbf{q}\rangle| \lesssim N^{-1/2+\epsilon}$ [E. et al. (2009), Knowles-Yin (2011)][Resolvent method]
- $\langle\mathbf{u}_i, A\mathbf{u}_i\rangle \to \langle A\rangle$ in probability for each $\mathbf{u}_i$ [Bourgade-Yau (2013)] [DBM]
- Simultaneously in i and j [in the bulk] — proven only for Wigner matrices with large (almost $\mathcal{O}(1)$) Gaussian component [Bourgade-Yau-Yin (2018)] [DBM]
- Uniformly in the spectrum if $\langle|\mathring{A}|^2\rangle^{1/2}$ replaced by $\|\mathring{A}\|$ [Cipolloni, E, Schröder (2020)] [Resolvent method].

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Novelties of our results: [Resolvent method]

- Optimal $N^{-1/2}$ speed of convergence. In physics: Eigenstate Thermalisation Hypothesis
- Limit is controlled in very high probability, and thus simultaneous in i, j.
- Optimal dependence of the error on A (HS is the correct norm – our newest result).

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These are LLN-type results. Next: What about CLT for $[\langle\mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle]$?

Averaged CLT for overlaps

CLT (central limit theorem) for $\left[\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle\right]$ can be asked in two senses.

First, we proved CLT after averaging in the index i:

Theorem (Cipolloni, E., Schröder 2020)

For any bounded deterministic matrix A, $i_0 \in [\delta N, (1-\delta)N]$ (i.e. bulk) and for any $K \geq N^{\epsilon}$

$$\frac{1}{\sqrt{2K}} \sum_{|i-i_0|\leq K} \sqrt{N}\left[\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle\right] \stackrel{m}{=} \mathcal{N}\left(0, \langle|\mathring{A}|^2\rangle\right) + \mathcal{O}(N^{-\epsilon'}\|\mathring{A}\|)$$

in the sense of moments, where $\mathring{A} := A - \langle A\rangle$ is the traceless part of A.

Similar result holds at the edge with a variance $\frac{\sqrt{2}}{3}\langle|\mathring{A}|^2\rangle$.

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$\Longrightarrow$ Indication that $\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle$, $\langle \mathbf{u}_j, \mathring{A}\mathbf{u}_j\rangle$ are asymptotically independent for $i \neq j$.

This CLT is a special case of our general functional CLT: $\langle f(W)A\rangle \approx \mathcal{N}$ for any fn. of the Wigner matrix W; unlike usual tracial CLT in random matrices, this involves eigenvectors as well!

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Second, CLT for each $\langle \mathbf{u}_i, A\mathbf{u}_i\rangle - \langle A\rangle$ without averaging?

CLT for individual overlaps

Theorem (Cipolloni, E, Schröder (2022))

For the bulk eigenvectors $\mathbf{u}_i$ of an $N \times N$ Wigner matrix W and for any deterministic hermitian observable (matrix) A with $1 \gtrsim \langle \mathring{A}^2 \rangle \geq N^{-1+\delta} \|\mathring{A}\|^2$ it holds:

$$\sqrt{\frac{N}{2\langle \mathring{A}^2 \rangle}} \left[\langle \mathbf{u}_i, A\mathbf{u}_i \rangle - \langle A \rangle \right] \to \mathcal{N}(0,1) \tag{1}$$

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Remark: $\langle \mathring{A}^2 \rangle \geq N^{-1+\delta} ||\mathring{A}||^2$ implies A not finite rank.

(1) is not true for finite rank: for $A = \mathring{A} = |\boldsymbol{e}_x\rangle\langle \boldsymbol{e}_x| - |\boldsymbol{e}_y\rangle\langle \boldsymbol{e}_y|$ we have $\langle \boldsymbol{u}_i, \mathring{A}\boldsymbol{u}_i \rangle = |u_i(x)|^2 - |u_i(y)|^2$ (difference of independent χ^2) [Bourgade-Yau (2017)].

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To prove this theorem we need DBM method on top of the resolvent method.

Comparison with previous results

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Previous results:

- Joint (squared) Gaussianity for finitely many $N|\boldsymbol{u}_{\alpha_l}(i_k)|^2$ under the additional four moment matching assumption for bulk eigenvectors (only two moments at the edge). [Knowles-Yin (2011)] [Resolvent method]
- Rank 1: $N|\langle \mathbf{u}_i, \mathbf{q}\rangle|^2$ is asymptotically (squared) Gaussian [Bourgade-Yau (2013)] [DBM]
- Finite rank: Joint (squared) Gaussianity for finitely many $\mathbf{u}$'s and $\mathbf{q}$'s [Marcinek-Yau (2020)] [DBM]
- (Almost) full rank: Gaussianity for $1 \gtrsim \langle \mathring{A}^2\rangle \geq \delta\|\mathring{A}\|^2$ [Cipolloni, E, Schröder (2021)] [DBM]

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Related independent result:

Gaussianity of $\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle$ for the special case $A = \sum_{j\in I} |\mathbf{e}_j\rangle\langle \mathbf{e}_j|$ with $N^\epsilon \leq |I| \leq N^{1-\epsilon}$, i.e. low rank (also at the edge) [Benigni-Lopatto (2021)] [DBM]

Proof of Gaussian fluctuation (via DBM)

DBM for eigenvectors

GOAL: Let $\mathbf{u}_i$ be the eigenvectors of a Wigner matrix W, then

$$\mathbf{E}\left[\sqrt{\frac{N}{2\langle \mathring{A}^2\rangle}}\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle\right]^n \to (n-1)!!\mathbf{1}(n \text{ even}), \qquad \mathring{A} = A - \langle A\rangle\,.$$

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We do it dynamically:

$$\mathrm{d}W_t = \frac{\mathrm{d}\widehat{B}_t}{\sqrt{N}}, \qquad W_0 = W. \tag{2}$$

The flow (2) adds a Gaussian component of size $\sqrt{t}$ to W_0.

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The flow (2) induces the Dyson Brownian Motion (DBM) for eigenvalues and eigenvectors:

$$\begin{aligned} \mathrm{d}\lambda_i(t) &= \frac{dB_{ii}(t)}{\sqrt{N}} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\,\mathrm{d}t \\ \mathrm{d}\mathbf{u}_i(t) &= \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{\mathrm{d}B_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}\mathbf{u}_j(t) - \frac{1}{2N}\sum_{j\neq i}\frac{\mathbf{u}_i(t)}{(\lambda_i(t)-\lambda_j(t))^2}\,\mathrm{d}t. \end{aligned} \tag{3}$$

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GOAL: Let $\mathbf{u}_i$ be the eigenvectors of a Wigner matrix W, then

$$\mathbf{E}\left[\sqrt{\frac{N}{2\langle \mathring{A}^2\rangle}}\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_i\rangle\right]^n \to (n-1)!!\mathbf{1}(n \text{ even}), \qquad \mathring{A} = A - \langle A\rangle .$$

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$$\mathrm{d}W_t = \frac{\mathrm{d}\widehat{B}_t}{\sqrt{N}}, \qquad W_0 = W. \tag{2}$$

The flow (2) adds a Gaussian component of size $\sqrt{t}$ to W_0.

Need only $t \sim N^{-1+\epsilon}$. This Gaussian component can later be removed by simple perturbation theory known as Green function comparison theorem (GFT).

The flow (2) induces the Dyson Brownian Motion (DBM) for eigenvalues and eigenvectors:

$$\begin{aligned} \mathrm{d}\lambda_i(t) &= \frac{dB_{ii}(t)}{\sqrt{N}} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\,\mathrm{d}t \\ \mathrm{d}\mathbf{u}_i(t) &= \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{\mathrm{d}B_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}\mathbf{u}_j(t) - \frac{1}{2N}\sum_{j\neq i}\frac{\mathbf{u}_i(t)}{(\lambda_i(t)-\lambda_j(t))^2}\,\mathrm{d}t. \end{aligned} \tag{3}$$

Remark: The BMs $B_{ij}(t)$, for $i \neq j$, and $B_{ii}(t)$ are independent!

Example for $n = 2$

By Ito's formula (from now on $\langle A \rangle = 0$, i.e. $A = \mathring{A}$):

$$\mathrm{d}\,\mathbf{E}\left[|\langle \mathbf{u}_i, A\mathbf{u}_i\rangle|^2 \middle| \boldsymbol{\lambda}\right] = \sum_{k \neq i} \frac{\mathbf{E}\left[|\langle \mathbf{u}_k, A\mathbf{u}_i\rangle|^2 \middle| \boldsymbol{\lambda}\right] - \mathbf{E}\left[|\langle \mathbf{u}_i, A\mathbf{u}_i\rangle|^2 \middle| \boldsymbol{\lambda}\right]}{N(\lambda_k - \lambda_i)^2} + \dots$$

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$$f_t(i,j) = \mathbf{E}\left[2|\langle \mathbf{u}_i, A\mathbf{u}_j\rangle|^2 + \langle \mathbf{u}_i, A\mathbf{u}_i\rangle \langle \mathbf{u}_j, A\mathbf{u}_j\rangle \middle| \boldsymbol{\lambda}\right]$$

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For general n: Closed equation for a certain lin. combination of overlaps f_t [Bourgade-Yau-Yin (2018)]

$$\partial_t f_t = \mathcal{L}(t) f_t, \tag{4}$$

for a certain operator $\mathcal{L}(t)$.

We explain the basic construction from [Bourgade-Yau-Yin (2018)]

Particle representation 1

Representation: We can think of $f_t = f_t(i_1, \dots, i_n)$ as a function of "n-particle configurations" on **N**.

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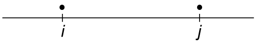

Figure 3: Particle configuration for $n = 2, f_t = f_t(i, j)$.

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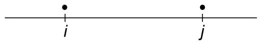

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$f_t(i,j) = 2|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2 + \langle \mathbf{u}_i, A\mathbf{u}_i \rangle \langle \mathbf{u}_j, A\mathbf{u}_j \rangle$ takes value in the "doubled space":

Figure 4: $|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2 + |\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2$

Figure 5: $\langle \mathbf{u}_i, A\mathbf{u}_i \rangle \langle \mathbf{u}_j, A\mathbf{u}_j \rangle$

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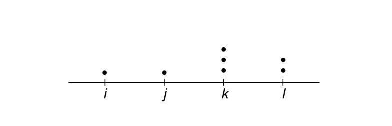

Figure 6: Argument of $f(i, j, k, k, k, l, l)$

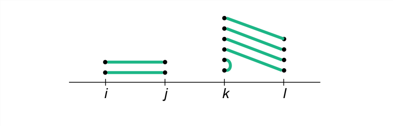

Figure 7: $\mathbf{E}\left[|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2 \, \langle \mathbf{u}_k, A\mathbf{u}_k \rangle \, |\langle \mathbf{u}_k, A\mathbf{u}_l \rangle|^4 \middle| \boldsymbol{\lambda} \right]$

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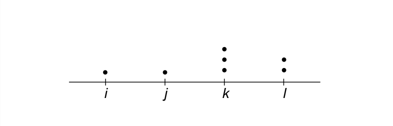

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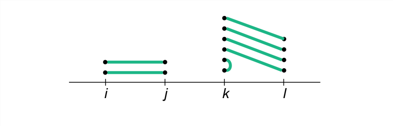

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Action of $\mathcal{L}(t)$:

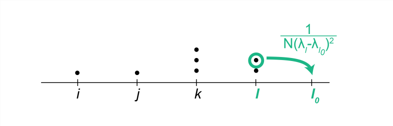

Figure 8: Argument of $\mathcal{L}f_t$:
$f(i, j, k, k, k, l, l) \to f(i, j, k, k, k, l, l_0)$

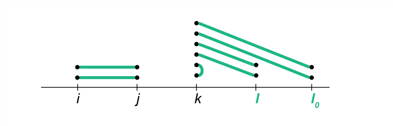

Figure 9:
$\mathbf{E}\left[|\langle \mathbf{u}_i, A\mathbf{u}_j \rangle|^2 \langle \mathbf{u}_k, A\mathbf{u}_k \rangle \, |\langle \mathbf{u}_k, A\mathbf{u}_l \rangle|^2 |\langle \mathbf{u}_k, A\mathbf{u}_{l_0} \rangle|^2 \middle| \boldsymbol{\lambda} \right]$

Sketch of the proof II

Recall: We have $\partial_t f_t = \mathcal{L}(t) f_t$. For $n = 2$:

$$\partial_t f_t(i,j) = \sum_{k \neq i} \frac{f_t(k,j) - f(i,j)}{N(\lambda_k - \lambda_i)^2} + \sum_{k \neq i} \frac{f_t(i,k) - f(i,j)}{N(\lambda_k - \lambda_j)^2}.$$

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Note that this is the discretisation of the $\sqrt{-\Delta} = |p|$ operator in $1d$
$\implies \partial_t f_t = \mathcal{L}(t) f_t$ is a (discrete) heat equation with fractional Laplacian

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- For $\partial_t f_t = \mathcal{L}(t) f_t$, want: heat kernel "$e^{-\mathcal{L}(t)}$" averages in all directions.
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Gain: $\left\langle \Im G\, A\, \Im G'\, A\right\rangle$ much easier to understand than $\langle \mathbf{u}_j, A\Im G(\lambda_i) A\mathbf{u}_j\rangle$!

Sketch of the proof IV

Recall: We consider $\partial_t f_t = \mathcal{L}(t) f_t$ (discrete heat equation with fraction Laplacian on **N**).

Want heat kernel "$e^{-\mathcal{L}(t)}$" averages in all directions.

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- Heuristically:

$$\mathcal{L}(t) = \sum_{r=1}^{n} \mathcal{L}_r(t), \qquad \mathcal{L}_r(t) \approx |p_r| := \sqrt{-\Delta_r},$$

i.e. $\mathcal{L}(t)$ (=infinitesimally the heat kernel) averages only in one coordinate direction. One direction is not enough, local laws require averaging in ALL directions.

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- To get more averaging: Replace $\mathcal{L}(t) = \sum_r |p_r|$ by the regularised product

$$\mathcal{A}(t) := \frac{1}{\eta} \prod_{r=1}^{n} \left(1 - e^{-\eta |p_r|}\right) \qquad \left(\sim \eta^{n-1} \prod_{r=1}^{n} |p_r| \quad \text{morally}\right)$$

  with $\eta \sim N^{-1} \Longrightarrow$ Average in any direction.
  The replacement is possible on the level of Dirichlet form, $D(f) := \langle f, \mathcal{L} f \rangle$.

Sketch of the proof V

Replace $\mathcal{L}(t) \approx \sum_r |p_r|$ (average in one direction) by the regularised product

$$\mathcal{A}(t) := \frac{1}{\eta} \prod_{r=1}^{n} \left(1 - e^{-\eta |p_r|}\right) \qquad \left(\sim \eta^{n-1} \prod_{r=1}^{n} |p_r| \quad \text{morally}\right)$$

with $\eta \sim N^{-1} \implies$ Average in any direction.

- Indeed heuristically:

$$\mathcal{L}(t) f_t = \sum_{r=1}^{n} \sum_{k \neq i_r} \frac{f(i_r \to k) - f_t(i_1, \dots, i_n)}{N(\lambda_k - \lambda_{i_r})^2},$$

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The replacement is possible on the level of Dirichlet form, $D(f) := \langle f, \mathcal{L} f \rangle$.

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- Main technical steps:
  (i) the energy method for DBM [Marcinek-Yau (2020)] analysing

$$\partial_t\|f_t\|_2^2 = -2D_t(f_t) \le 0.$$

  (ii) local laws for $\langle GAGA\dots\rangle$ with $\langle A\rangle = 0$ and $\langle A^2\rangle$ errors [Cipolloni, E, Schröder (2022)].

Multi-resolvents local laws with $\langle A^2 \rangle$ errors

Theorem (Cipolloni, E, Schröder 2022)

For any deterministic observable A, with $\langle A \rangle = 0$, it holds:

$$\left| \langle G(z_1)AG(z_2)A \rangle - m(z_1)m(z_2) \langle A^2 \rangle \right| \le \frac{\langle A^2 \rangle}{\sqrt{N\eta}} \ll \langle A^2 \rangle \qquad (5)$$

for any $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ such that $\eta := \Im z_1 \sim \Im z_2 \ge N^{-1+\epsilon}$.

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Remark: (6) is a special case of a general multi-resolvents local laws for $G_1A_1 \dots G_kA_k$ with optimal dependence on A – we call it rank uniformity.

Rank uniform local law

Previous local laws for the resolvent $G(z)$ of Wigner matrices

$$\langle (G(z) - m(z))A \rangle \lesssim \frac{\|A\|}{N\eta}, \qquad \eta := \Im z \qquad \text{(averaged)}$$

$$\langle \mathbf{x}, (G(z) - m(z))\mathbf{y} \rangle \lesssim \sqrt{\frac{\rho}{N\eta}} \|\mathbf{x}\| \|\mathbf{y}\|, \qquad \rho := \Im m \qquad \text{(isotropic)}$$

Note: $A = |\mathbf{y}\rangle\langle\mathbf{x}|$ for the averaged law gives an isotropic estimate off by a huge factor $\sqrt{N/\rho\eta}$ because $\|A\|$ is far from optimal for lower rank observables.

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Theorem [Rank-uniform local law (single resolvent)] (Cipolloni, E, Schröder 2022)

For any deterministic observable A and for any $N\eta\rho \gg 1$.

$$\left| \langle (G(z) - m(z))A \rangle \right| \lesssim \frac{|\langle A \rangle|}{N\eta} + \frac{\sqrt{\rho}\, \langle |\mathring{A}|^2 \rangle^{1/2}}{N\sqrt{\eta}}, \qquad \mathring{A} = A - \langle A \rangle . \tag{6}$$

Unifies and extends the isotropic and averaged local laws. Multi-resolvent versions also hold.

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Method: A nonlinear hierarchy of *master inequalities* for the quantities

$$\psi_k := \max \frac{N^{(3-k)/2}\sqrt{\eta}}{\sqrt{\rho}} \frac{1}{\prod_i \langle|A_i|^2\rangle^{1/2}} \left|\langle G(z_1)A_1G(z_2)\dots A_k - \prod_i m(z_i)A_i\rangle\right|,$$

where max runs over all z_i, $\Im z_i = \eta$ and deterministic A_i's with $\langle A_i\rangle = 0$, AND a *reduction inequality* stating roughly $\psi_{2k} \ll (\psi_k)^2$ to close the hierarchy.

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We proved:

- **Eigenstate Thermalisation Hypothesis for Wigner matrices:** eigenvector overlaps with deterministic A are $\lesssim N^{-1/2}$.
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Main technical steps:

- Energy estimates for multi indexed DBM.
- Dramatically improved local law for traceless observables.
- New hierarchy of master inequalities and its closure for ψ_k.

THANK YOU VERY MUCH FOR YOUR ATTENTION!