Semiclassical techniques in infinite dimension

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In memory of Steve Zelditch

Outline

Some problems

Bosonic mean-field asymptotics=semiclassical asymptotics

Non mean-field, non semiclassical problems

Example 1

$$i\partial_t\psi = -\sum_{j=1}^{N}\Delta_{x_j}\psi + \frac{1}{N}\sum_{1\leq j\leq j'\leq N} V(x_{j'}-x_j)\psi$$

$V(x)=V(-x) \quad \psi(x_{\sigma(1)},\ldots,x_{\sigma(N)})=\psi(x_1,\ldots,x_N)$ (bosons).

$\frac{1}{N}=\varepsilon$, bosonic $\odot^N\mathfrak{h}\subset Fock_b(\mathfrak{h})$:

$$\begin{aligned} i\varepsilon\partial_t\psi &= \left[\varepsilon d\Gamma(-\Delta)^{Wick}+\varepsilon^2\frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)a^*(x)a^*(y)a(x)a(y)\ dxdy\right]\psi \\ &= \mathcal{E}^{Wick,\varepsilon}\psi \end{aligned}$$

$$\mathcal{E}(z,\overline{z}) = \int_{\mathbb{R}^d}|\nabla z|^2+\frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)|z(x)|^2|z(y)|^2\ dxdy\,.$$

Wick,ε-quantization:
Replace $z(x)$ by $a_\varepsilon(x)=\sqrt{\varepsilon}a(x)$ and $\overline{z}(x)$ by $a_\varepsilon^*(x)=\sqrt{\varepsilon}a^*(x)$.

Wick: $a()$ always on the right-hand side.

ε-quantization: $[a_\varepsilon(x),a_\varepsilon^*(y)]=\varepsilon\delta(x-y)$
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$$a_\varepsilon(g) = \sum_{j=1}^{d} \overline{g_j}\sqrt{\varepsilon}\frac{(\partial_{x_j} + x_j)}{\sqrt{2}} \qquad a_\varepsilon^*(f) = \sum_{j=1}^{d} f_j\sqrt{\varepsilon}\frac{(-\partial_{x_j} + x_j)}{\sqrt{2}}$$

$\varepsilon = 2h$ $a_\varepsilon(e_j)$ and $a_\varepsilon^*(e_j)$ are unitarily equivalent to $(h\partial_{x_j} + x_j)$ and $(-h\partial_{x_j} + x_j)$. Mean field $\sim$ semiclassical.

Finite dimension: $b(x, hD_x) = b^{Wick}(x, hD_x) + O(h)$ and $b(\sqrt{h}x, \sqrt{h}D_x) = b^{Wick}(\sqrt{h}x, \sqrt{h}D_x) + O(h)$.

Studying $(\mathscr{E}^{Wick,\varepsilon} - E)u^\varepsilon = (\mathscr{E}(\sqrt{h}x, \sqrt{h}D_x) - E)u^{2h} = o(h^0)$ starts with the study of the characteristic set $\mathscr{E}(x,\xi) - E = 0$ or $\mathscr{E}(z,\overline{z}) - E = 0$ in the phase space.

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Bosonic mean field asymptotics

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Example 3: Spin-photon and Maxwell-Bloch [1]

The one particle space is $\mathfrak{h}_{\mathbb{C}} = \{z \in L^2(\mathbb{R}^3;\mathbb{C}^3), k.z(k) = 0\} = \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{h}_{\mathbb{R}}$. The total system space is

$$Fock_b(\mathfrak{h}_{\mathbb{C}}) \otimes (\mathbb{C}^2)^{\otimes M} \quad , \quad H^{\varepsilon}_{free} = \varepsilon d\Gamma(|k|) \otimes \mathrm{Id}_{(\mathbb{C}^2)^{\otimes M}}$$

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Magnetic field: $\vec{B}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int_{\mathbb{R}^3} i\chi(k)|k|^{-1/2} \left[e^{ik.x} k \wedge a_\varepsilon(k) - e^{-ik.x} k \wedge a^*_\varepsilon(k)\right] \, dk$

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Interaction term: $H_{int} = \sum_{\lambda=1}^{M} \sum_{j=1}^{3} (\beta_j + B_j(x_\lambda)) \otimes \sigma_j(x_\lambda) = \sum_{\lambda=1}^{M} (\vec{\beta} + \vec{B}(x_\lambda)).\vec{\sigma}(x_\lambda)$

where x_λ is the position of spin λ and $\sigma_j(x_\lambda) = \mathrm{Id}_{\mathbb{C}^2} \otimes \cdot \otimes \sigma_j \otimes \cdots \otimes \mathrm{Id}_{\mathbb{C}^2}$.

Interacting Hamiltonian: $H^\varepsilon = \varepsilon d\Gamma(|k|) + H^\varepsilon_{int}$.

Quantum evolved observable $A^\varepsilon(t) = e^{\frac{it}{\varepsilon} H^\varepsilon} A e^{-i\frac{t}{\varepsilon} H^\varepsilon}$.

The operators $\vec{B}^\varepsilon(t)$, $\vec{E}^\varepsilon(t)$, $\vec{S}^\varepsilon(t)$ with $\vec{S}^\varepsilon(0) = \mathrm{Id}_{Fock_b(\mathfrak{h})} \otimes \vec{\sigma}$, have the principal part $\vec{B}^{Wick}_0 \otimes \mathrm{Id}$, $\vec{E}^{Wick}_0 \otimes \mathrm{Id}$ and $\mathrm{Id}_{Fock_b(\mathfrak{h}_{\mathbb{C}})} \otimes \vec{S}_0(t)$ where

- $\vec{B}_0, \vec{E}_0$ solve the vaccuum Maxwell equations
- $\frac{d}{dt} S_0(t, x_\lambda) = 2(\vec{\beta} + \vec{B}_0(x_\lambda, t)) \wedge \vec{S}_0(t, x_\lambda)$ (Bloch equation)

Correction terms can be computed.

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$$Fock_b(\mathfrak{h}_{\mathbb{C}}) \otimes (\mathbb{C}^2)^{\otimes M} \quad , \quad H^{\varepsilon}_{free} = \varepsilon d\Gamma(|k|) \otimes \mathrm{Id}_{(\mathbb{C}^2)^{\otimes M}}$$

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Correction terms can be computed.

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Comparison between Example 1 and Example 3:

In Example 1 the Hamiltonian is quartic and the underlying dynamics is nonlinear.

$$\text{Hartree} \quad i\partial_t z = -\Delta z + (V * |z|^2) z \,.$$

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Consequences: In Example 3 observables can be propagated; in Example 1 observables cannot be propagated in any reasonnable pseudodifferential class.

Example 1: Random homogeneization of wave propagation [2]

$$i\varepsilon\partial_t u^{\varepsilon}_{\omega} = -\Delta_x u^{\varepsilon}_{\omega} + \sqrt{2\varepsilon}\mathcal{V}(x,\omega)u^{\varepsilon}_{\omega}$$

where $\mathcal{V}(x,\omega)$ is a centered real gaussian field sucht that $\mathbb{E}(\mathcal{V}(x,\omega)\mathcal{V}(y,\omega)) = F^{-1}\left(|F(V)|^2\right)(x-y)$.

After using the invariance translation and interpreting gaussian random fields in the bosonic Fock space it becomes

$$i\varepsilon\partial_t f_{\xi} = (\xi + d\Gamma(D_y))^2 f_{\xi} + \sqrt{2\varepsilon}\phi(V)f_{\xi}$$

with $(f_{\xi})_{\xi\in\mathbb{R}^d} \in L^2(\mathbb{R}^d_{\xi}) \otimes Fock_b(L^2(\mathbb{R}^d, dy;\mathbb{C}))$.

The term $\sqrt{2\varepsilon}\phi(V) = \sqrt{\varepsilon}a(V) + \sqrt{\varepsilon}a^*(V)$ is semiclassical.
The main term $(\xi + d\Gamma(D_y))^2$ is quartic and not semiclassical (mean-field) in the field variable.
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Example 2: Bose-Einstein non interacting gas [3]

Consider a Gibbs state $\varrho_\varepsilon = Z_\varepsilon^{-1}\Gamma_\varepsilon(e^{-\beta(H-\mu(\varepsilon))} = Z_\varepsilon^{-1}e^{-d\Gamma_\varepsilon(\beta(H-\mu(\varepsilon)))}$ where $H = \frac{h}{2}(-\Delta_x + x^2 - d)$ and $Z_\varepsilon = \mathrm{Tr}[\Gamma_\varepsilon(e^{-\beta(H-\mu(\varepsilon))})]$.
The mean-field limit as $\varepsilon \to 0$ and $h > 0$ fixed, for $d \geq 2$ and $\beta\mu(\varepsilon) = -\frac{\varepsilon}{\nu_C}$ says that the p-th reduced density matrix converges weakly to

$$\gamma_0^{(p)} = p!\nu_C^p |\psi_0^{\otimes p}\rangle\langle\psi_0^{\otimes p}|$$

where $\psi_0(x) = \pi^{-d/4}e^{-|x|^2/2}$.
When $p = 1$, $\tilde{b} = \mathrm{Id}_{L^2(\mathbb{R}^d)}$, $b^{Wick} = d\Gamma_\varepsilon(\mathrm{Id}) = N_\varepsilon = \varepsilon N_1$ one gets

$$\mathrm{Tr}\left[\gamma_0^1\right] = \nu_C < \liminf_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon N_\varepsilon\right].$$

Missing mass: By taking a thermodynamic limit with $\varepsilon = \varepsilon(h) = h^d$ and $h \to 0$, physics tells us that there are two phases a condensate phase at the quantum scale and a classical Gibbs gas.

Is there a mathematical translation of this ? Are there mathematical objects catching those two scales ?

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The above weak convergence says that for all compact operator $\tilde{b} : \odot^p L^2(\mathbb{R}^d) \to \odot^p L^2(\mathbb{R}^d)$ and $b(z) = \langle z^{\otimes p}, \tilde{b}z^{\otimes p}\rangle$,

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Example 3: Bose-Hubbard models and ETH [4][5][6]

Hamiltonian for N-bosons on k-sites

$$\tilde{H}_{k,N} = -\frac{1}{2}\sum_{|i-j|=1} J_{ij} a_i^* a_j + \frac{\Lambda}{N}\sum_{j=1}^{k} n_j(n_j-1) \quad , \quad n_j = a_j^* a_j\,.$$

Eigenstate Thermalization Hypothesis means that eigenvectors $|E_\alpha\rangle$ with energy $E_\alpha \in [\overline{E}-\Delta E, \overline{E}+\Delta E]$ satisfy

$$\langle E_\alpha\,, \mathscr{O} E_\beta\rangle \sim \delta_{\alpha,\beta} f(\overline{E}) + \underbrace{e^{-S(\overline{E})/2}}_{small\ \ factor} f_2(\overline{E}, E_\alpha - E_\beta) R_{\alpha,\beta}$$

where $S(\overline{E})$ is interpreted as or related to an entropy and $(R_{\alpha,\beta})_{\alpha,\beta}$ behave like a random (gaussian) matrices.

The scaling of ΔE , therefore the observable $\mathscr{O}$ but also of $k = k(N)$ must be specified. $k(N) = Cte$ finite dimensional semiclassical problem, $k = k_0 N$ thermodynamic limit, or $N \to \infty$ and then $k \to \infty$. . .
All can be put in $Fock_b(\ell^2(\mathbb{Z}))$ with some $k(N)$-dependent mean-field Hamiltonian. The scaling of the observables and $k = k(N)$ lead to non exactly mean-field or semiclassical asymptotics.

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For a fixed $k \in \mathbb{N}$,

$$H_{k,\varepsilon} = \frac{1}{N}\tilde{H}_N = -\frac{\varepsilon}{2}\sum_{|i-j|=1} J_{ij}a_i^* a_j + \Lambda\varepsilon^2 \sum_{j=1}^{k} n_j(n_j-1)$$

$$= -\sum_{|i-j|=1} J_{ij}a_{i,\varepsilon}^* a_{j,\varepsilon} + \Lambda\sum_{j=1}^{k} a_{j,\varepsilon}^* a_{j,\varepsilon}^* a_{j,\varepsilon} a_{j,\varepsilon} ,$$

with $\varepsilon = 1/N$ is a pure mean-field (semiclassical) problem in finite dimension.

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Wick quantization

Polynomial symbols in $z,\overline{z}$ can always be ε-Wick quantized.

$b(z,\overline{z}) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p}\rangle$, $\tilde{b} \in \mathscr{L}(\mathfrak{h}^{\odot p};\mathfrak{h}^{\odot q})$,

$$b^{Wick,\varepsilon}|_{\mathfrak{h}^{\odot n+p}} = \varepsilon^{(p+q)/2} \frac{\sqrt{(n+p)!(n+q)!}}{n!} \mathbf{S}_{n+q}(\tilde{b} \otimes \mathrm{Id}_{\mathfrak{h}^{\odot n}})\mathbf{S}_{n+p}$$

When $\mathfrak{h} = L^2(\mathbb{R}^d, dx_1)$ and $\tilde{b}$ has the Schwartz kernel $\tilde{b}(x,y) \in \mathscr{S}'_{sym}(\mathbb{R}^{d(q+p)})$,

$$b^{Wick,\varepsilon} = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(x,y) a_\varepsilon^*(x_1)\cdots a_\varepsilon^*(x_q) a_\varepsilon(y_1)\cdots a_\varepsilon(y_p)\, dx dy$$

makes sense as a weakly defined operator on $\oplus_n^{alg} \mathscr{S}_{sym}(\mathbb{R}^{dn})$.

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$$b^{Wick,\varepsilon} = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(x,y) a_\varepsilon^*(x_1)\cdots a_\varepsilon^*(x_q) a_\varepsilon(y_1)\cdots a_\varepsilon(y_p)\,dx\,dy$$

makes sense as a weakly defined operator on $\oplus_n^{alg}\mathscr{S}_{sym}(\mathbb{R}^{dn})$.

In finite dimension they are both defined by using integrals with the Lebesgue measure.

Weyl:

$$b^{Weyl,\varepsilon}(x,y)=[b(\sqrt{\varepsilon/2}.,\sqrt{\varepsilon/2}.)]^{Weyl}(x,y)=\int_{\mathbb{R}^d} e^{i\xi.(x-y)}a(\sqrt{\varepsilon/2}\frac{x+y}{2},\sqrt{\varepsilon/2}\xi)~\frac{d\xi}{(2\pi)^d}$$

With $z=x+i\xi$, $\mathrm{Re}\,\langle z_1,z_2\rangle=x_1.x_2+\xi_1.\xi_2$ and $(\mathrm{Re}\,\langle z,\zeta\rangle)^{Wick,\varepsilon}=a_\varepsilon(\zeta)+a_\varepsilon^*(\zeta)=\sqrt{2}\phi_\varepsilon(\zeta)$.

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By writing $b(z)=\int_{\mathbb{C}^d} e^{2i\pi\,\mathrm{Re}\,\langle z,\zeta\rangle}(Fb)(\zeta)~L(d\zeta)$ with $(Fb)(\zeta)=\int_{\mathbb{C}^d} e^{-2i\pi\,\mathrm{Re}\,\langle\zeta,z\rangle}b(z)~L(dz)$ we get

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$$b^{A-Wick,\varepsilon}=(G_{\varepsilon/2}*b)^{Weyl}\quad G_h=h^{-d}G(./\sqrt{h})\quad G(z)=\frac{e^{-|z|^2}}{\pi^d}.$$

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Probabilities 1: Reims group (Amour, Jager, Nourrigat...)

One idea consists in replacing Lebesgue measures by gaussian measures [1][7].

$\pi^{-d/4}e^{-|x|^2/2}\times$ is unitary from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, \pi^{-d/2}e^{-|x|^2}dx)$
Weyl quantization and Anti-Wick quantization are expressed with the Gaussian measure $(\pi\varepsilon)^{-d}e^{-\frac{|z|^2}{\varepsilon}}L(dz)$.

We can then extend the definition of ε-Weyl quantization to the case of an infinite dimensional phase-space by using gaussian measures and Wiener spaces. This leads to some good semiclassical algebras with asymptotic expansions of extended Moyal products.

Drawback: Two gaussian measures are quasi-equivalent when their covariance matrix differ by a Hilbert-Schmidt operator.
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Probabilities 2: Construction of nonlinear Gibbs measures

For most nonlinear PDEs the dynamics cannot enter in a single gaussian measured space. An alternative consists in constructing an invariant nonlinear Gibbs measure for a given nonlinear PDE.

This problem has received a strong attention by the mathematical community in the last decades. Among the many contributors: Bourgain, Burq, Tzvetkov...(see [8] for a survey).

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In finite dimension Wigner measures associated with a family of states $(\varrho_\varepsilon)_{\varepsilon>0}$, $\varrho_\varepsilon \in \mathscr{L}^1(L^2(\mathbb{R}^d, dx))$, $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$, with the additional uniform condition $\mathrm{Tr}\,[\varrho_\varepsilon(1+N_\varepsilon)^{\delta/2}] \leq C_\delta$ for some $\delta > 0$, are probability measures [9] characterized after a subsequence extraction

$$\forall \zeta \in \mathbb{C}^d\,,\ \lim_{k\to\infty} \mathrm{Tr}\left[\varrho_{\varepsilon_k} W_{\varepsilon_k}(\sqrt{2}\pi\zeta)\right] = \int_{\mathbb{C}^d} e^{2i\pi \mathrm{Re}\,\langle \zeta, z\rangle}\, d\mu(z)\,.$$

Notation: $\mu \in \mathscr{M}(\varrho_\varepsilon, \varepsilon \in (0, \varepsilon_0))$ or $\{\mu\} = \mathscr{M}(\varrho_{\varepsilon_k}, \varepsilon \in \{\varepsilon_k, k \in \mathbb{N}\})$.

The same definition after a diagonal extraction works when $\mathbb{C}^d$ is replaced by an infinite dimensional separable Hilbert space owing to

$$Fock_b(F \oplus F^\perp) = Fock_b(F) \otimes Fock_b(F^\perp), \quad \dim F < +\infty$$
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Assumption: $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\left[\varrho_\varepsilon\right] = 1$, $\mathrm{Tr}\left(\varrho_\varepsilon(1+N_\varepsilon)^{\delta/2}\right) \leq C_\delta$

$\mu \in \mathscr{M}(\varrho_\varepsilon, \varepsilon \in \mathscr{E})$, $0 \in \overline{\mathscr{E}}, \mathscr{E} \subset]0, \varepsilon_0[$ can be reduced to $\mathscr{M}(\varrho_\varepsilon, \varepsilon \in \mathscr{E}) = \{\mu\}$ after a sequence extraction:

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Semiclassical techniques in infinite dimension
Francis Nier, LAGA, Univ. Paris XIII
Some problems
Quantization and probabilities
Semiclassical propagation in an infinite dimensional phase-space

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Wigner measure, Wick observables and reduced density matrices

In finite dimension $\mathfrak{h} = \mathbb{C}^d_z \sim \mathbb{R}^{2d}_X$, $1 + \varepsilon d/2 + N_\varepsilon = (1 + |X|^2)^{Weyl,\varepsilon}$. For any polynomial $b(z,\overline{z})$ of total degree p

$$\|(1+N_\varepsilon)^{-p/2}[b^{Wick,\varepsilon} - b^{Weyl,\varepsilon}](1+N_\varepsilon)^{-p/2}\| = \mathscr{O}(\varepsilon).$$

When ϱ_ε satisfies the assumptions with $\delta \geq 2p$, then

$$\lim_{\varepsilon\in\mathscr{E},\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon b^{Wick,\varepsilon}\right] = \int_{\mathbb{R}^d} \langle z^{\otimes p}, \tilde{b} z^{\otimes p}\rangle \, d\mu(z)$$

In infinite countable dimension: When $\tilde{b} \in \mathscr{L}^\infty(\mathfrak{h}^{\odot p})$ is compact and ϱ_ε satisfy the assumptions with $\delta \geq 2p$, then

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Counter-example: $\varepsilon = \frac{1}{n}$, $\varrho_\varepsilon = |W_{1/n}(ne_n)\Omega\rangle\langle W_{1/n}(ne_n)\Omega|$ where Ω is the vacuum state and $(e_n)_{n\in\mathbb{N}\setminus\{0\}}$ is a Hilbert basis of $\mathfrak{h}$. Then

$$\mathscr{M}(\varrho_\varepsilon, \varepsilon \in 1/(\mathbb{N}\setminus\{0\})) = \{\mu = \delta_0\} \quad \text{while} \quad \mathrm{Tr}\left[\varrho_\varepsilon N_\varepsilon\right] = |e_n|^2 = 1 \overset{n\to\infty}{\to} 1 > 0 = \int_{\mathfrak{h}} |z|^2 d\mu(z)$$

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$$\|(1 + N_\varepsilon)^{-p/2}[b^{Wick,\varepsilon} - b^{Weyl,\varepsilon}](1 + N_\varepsilon)^{-p/2}\| = \mathcal{O}(\varepsilon).$$

When ϱ_ε satisfies the assumptions with $\delta \geq 2p$, then

$$\lim_{\varepsilon \in \mathcal{E}, \varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon b^{Wick,\varepsilon}\right] = \int_{\mathbb{R}^d} \langle z^{\otimes p}, \tilde{b} z^{\otimes p}\rangle \, d\mu(z)$$

In infinite countable dimension: When $\tilde{b} \in \mathcal{L}^\infty(\mathfrak{h}^{\odot p})$ is compact and ϱ_ε satisfy the assumptions with $\delta \geq 2p$, then

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Counter-example: $\varepsilon = \frac{1}{n}$, $\varrho_\varepsilon = |W_{1/n}(ne_n)\Omega\rangle\langle W_{1/n}(ne_n)\Omega|$ where Ω is the vacuum state and $(e_n)_{n \in \mathbb{N}\setminus\{0\}}$ is a Hilbert basis of $\mathfrak{h}$. Then

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Condition (P):

$$\forall p \in \mathbb{N}, \forall \tilde{b} \in \mathscr{L}(\mathfrak{h}^{\odot p}), \quad \lim_{\varepsilon \in \mathscr{E}, \varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon b^{Wick,\varepsilon}\right] = \int_{\mathfrak{h}} b(z,\bar{z})\; d\mu(z).$$

Condition (PI):

$$\forall p \in \mathbb{N}, \lim_{\varepsilon \in \mathscr{E}, \varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon N_\varepsilon^p\right] = \lim_{\varepsilon \in \mathscr{E}, \varepsilon \to 0} \mathrm{Tr}\left[\varrho_\varepsilon (|z|^{2p})^{Wick,\varepsilon}\right] = \int_{\mathfrak{h}} |z|^{2p}\; d\mu(z).$$

Norm convergence of $\gamma_\varepsilon^{(p)}$: The reduced density matrices $\gamma_\varepsilon^{(p)}$ converge in trace norm to

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Reconsider the Bose-Einstein non interacting gas, $\varrho_\varepsilon = Z_\varepsilon^{-1} e^{-\beta(H-\mu(\varepsilon))}$, $\mu(\varepsilon) = -\frac{\varepsilon}{\nu_C \beta}$, $H = \frac{h}{2}(-\Delta_x + x^2 - d)$.

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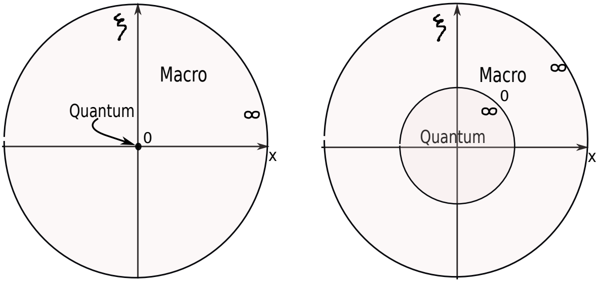
Second microlocalization picture



$$b(\sqrt{h}x, \sqrt{h}D_x, x, D_x)\,,\ b \in \mathscr{C}^\infty(\mathbb{R}^{2d} \times \mathbb{R}^{2d}; \mathbb{C})$$

$\operatorname{supp} b(., Y) \subset B(0, C_b) \subset \mathbb{R}^{2d}$ for all $Y \in \mathbb{R}^{2d}$

$\lim_{R\to\infty} b(X, R\omega) = a_\infty(X, \omega)$ in $\mathscr{C}^\infty(\mathbb{R}^{2d} \times \mathbb{S}^{2d-1})$.

Is there an Egorov theorem in infinite dimension ?

Measure transportation technique

Is there an Egorov type theorem in infinite dimension ?

Answer: No and Yes [4][9][11]

All the known and apparently relevant classes of semiclassical operators associated with $Fock_b(\mathfrak{h})$, $\mathfrak{h}$ separable Hilbert space, rely on a rigid linear structure:

- cylindrical symbols $b(z) = b(\pi_F(z))$ with the asymptotic equivalence of quantizations in finite dimension.
- Wick quantized polynomials.
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Already before considering a quantization, those classes of symbols are not preserved by a nonlinear transformation.

However it is possible to prove the propagation of Wigner measures.

Example: Consider $H^\varepsilon = \mathscr{E}^{Wick,\varepsilon}$ with

$$\mathscr{E}(z,\overline{z}) = \int_{\mathbb{R}^3} |\nabla z(x)|^2 \; dx + \frac{\lambda}{2}\int_{\mathbb{R}^6} \frac{|z(x)|^2|z(y)|^2}{|x-y|} \; dxdy$$

where the Hartree flow $\phi_{Hartree}$ associated with $i\partial_t z = -\Delta z + \lambda(\frac{1}{|x|} * |z|^2)z$ is well defined in $H^1(\mathbb{R}^3;\mathbb{C})$ which is a Borel subset of $\mathfrak{h} = L^2(\mathbb{R}^3;\mathbb{C})$.

Take $\varrho_\varepsilon \in \mathscr{L}^1(Fock_b(\mathfrak{h}))$ such that $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\left[\varrho_\varepsilon\right] = 1$ and $\mathrm{Tr}\left[\varrho_\varepsilon d\Gamma(1-\Delta)^{\delta/2}\right] \leq C_\delta$. Then

$$\left(\mathscr{M}(\varrho_\varepsilon, \varepsilon \in \mathscr{E}) = \{\mu_0\}\right) \Rightarrow \left(\mathscr{M}(e^{-\frac{it}{\varepsilon}H^\varepsilon}\varrho_\varepsilon e^{\frac{it}{\varepsilon}H_\varepsilon}), \varepsilon \in \mathscr{E}) = \{\phi_{Hartree}(t)_*\mu\}\right).$$

Additionally if $(\varrho_\varepsilon)_{\varepsilon\in\mathscr{E}}$ satisfies the condition (PI), it is satisfied for any given time $t \in \mathbb{R}$.

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Measure transportation technique, probabilities again [12]

One first checks after an Ascoli type argument that a sequence extraction allows to say

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Then one proves that $\mu(t)$ is a weak solution to $\partial_t \mu + \{\mathscr{E}, \mu\} = 0$, weak meaning after testing on cylindrical functions on $\mathbb{R}_t \times \mathfrak{h}$.

The projected measures $\pi_{F,*}\mu(t)$ solve a family of transport equations with non Lipschitz continuous vector fields, but with some uniform L^p-estimates. Using the theory of generalized flows (probalistic trajectory picture) and a stability with respect to $\dim F$ one can prove $\mu(t) = \phi_{Hartree}(t)_* \mu$.

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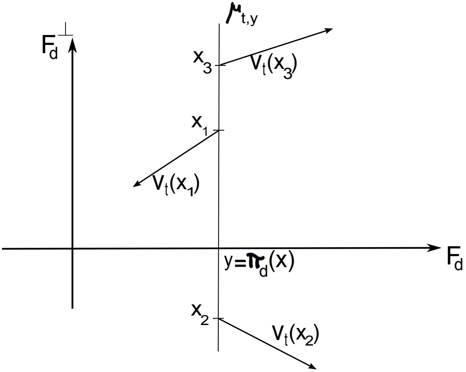
Measure transportation picture



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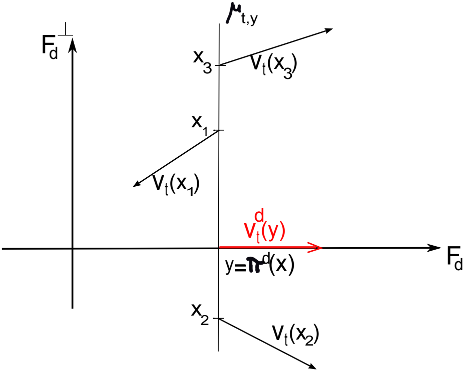

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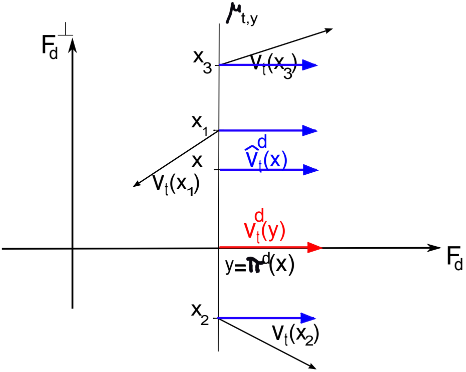

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THANK YOU!