Verification of Functional Data Structures

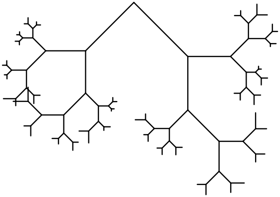
Correctness and Complexity

Tobias Nipkow

Technical University of Munich

Tobias Nipkow, Jasmin Blanchette,
Manuel Eberl, Alejandro Gómez-Londoño,
Peter Lammich, Christian Sternagel,
Simon Wimmer, Bohua Zhan

Functional Algorithms, Verified!



datatype $'a\ tree = Leaf \mid Node\ ('a\ tree)\ 'a\ ('a\ tree)$

$$T_{insert}\ a\ t + \Phi(insert\ a\ t) - \Phi(t) \le 3 \cdot \lg(|t| + 3) + 2$$

Quod Erat Demostrandum
QED
Isabelle/HOL

A compendium of functional data structures and algorithms

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Formally verified (in Isabelle/HOL)

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functional data structures and algorithms

Formally verified (in Isabelle/HOL)

Both functional correctness and (amortized) runnig time

Inspired by ...

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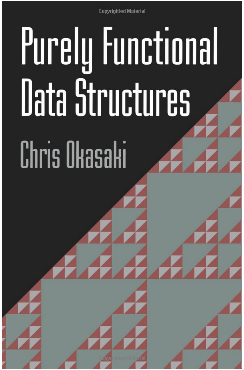
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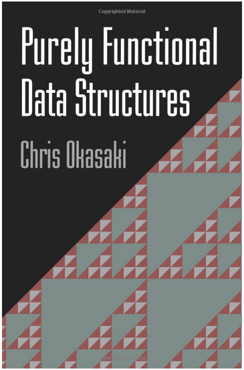
Purely Functional Data Structures

Chris Okasaki

Inspired by ...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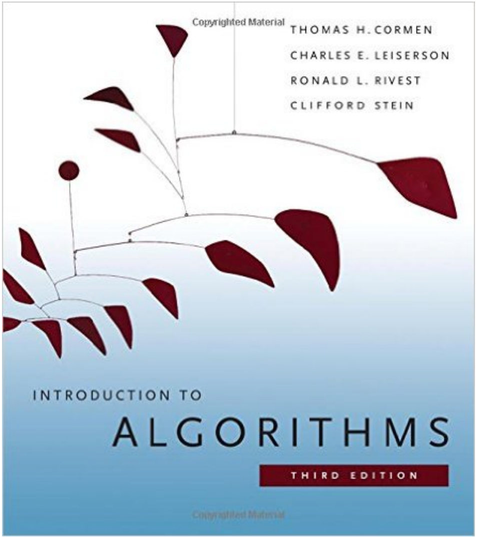
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… but with both textual
and (online) machine checked proofs

Topics

Sorting, Selection, Binary Trees, Binary Search Trees, Abstract Data Types, 2-3 Trees, Red-Black Trees, AVL Trees, Just Join, Braun Trees, Tries, Huffman's Algorithm, Priority Queues, Leftist Heaps, Leftist Heaps, Dynamic Programming, Amortized Analysis, Queues, Splay Trees, Skew Heaps, Pairing Heaps

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Graph Algorithms,
$\alpha\beta$-Search, Quadtrees,
Burrows-Wheeler Transformation
...

❶ Time

❷ Real Time Queue

❸ Real Time Double-Ended Queue

❹ Skew Heap

1. Time
2. Real Time Queue
3. Real Time Double-Ended Queue
4. Skew Heap

The idea

Running time complexity $\approx$ number of function calls

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For each $f :: \tau \to \tau'$

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Running time complexity $\approx$ number of function calls

For each $f :: \tau \to \tau'$
there is a $T_f :: \tau \to nat$

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Proofs about both f and T_f follow the same principles:
induction, case analyses, equational reasoning, logic, ...

The idea

Running time complexity $\approx$ number of function calls

For each $f :: \tau \to \tau'$
there is a $T_f :: \tau \to nat$ that counts function calls

Proofs about both f and T_f follow the same principles:
induction, case analyses, equational reasoning, logic, ...

Where does T_f come from?

Example

$f\ xs\ ys = \textsf{case}\ xs\ \textsf{of}\ [\,] \Rightarrow ys \mid x\#xs \Rightarrow x\ \#\ f\ xs\ ys$

Example

$$f\ xs\ ys = \textsf{case}\ xs\ \textsf{of}\ [] \Rightarrow ys \mid x\#xs \Rightarrow x\ \#\ f\ xs\ ys$$

$$T_f\ xs\ ys = \textsf{case}\ xs\ \textsf{of}\ [] \Rightarrow 1 \mid x\#xs \Rightarrow 1 + 1 + T_f\ xs\ ys$$

Example

$f\ xs\ ys = \textsf{case}\ xs\ \textsf{of}\ [] \Rightarrow ys \mid x\#xs \Rightarrow x\ \#\ f\ xs\ ys$

$T_f\ xs\ ys = \textsf{case}\ xs\ \textsf{of}\ [] \Rightarrow 1 \mid x\#xs \Rightarrow 1 + 1 + T_f\ xs\ ys$

Principle: T_f is abstract interpretation of f

Example

$f\ xs\ ys =$ case xs of $[] \Rightarrow ys \mid x\#xs \Rightarrow x\ \#\ f\ xs\ ys$

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Can be automated

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Principle: T_f is abstract interpretation of f

Can be automated (easily for call-by-value)

Additive constants can be reduced to 1

Alternative: Monadic approach

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- Purpose: define f and T_f simultaneously

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- Implementation: define function on $(value, time)$ pairs

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Notation: $\{x \leftarrow e_1; e_2\} = bind\ e_1\ (\lambda x.\ e_2)$

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How to define your algorithms:

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How to define your algorithms:

Define monadic $fm :: \cdots \rightarrow (\tau, nat)$

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How to define your algorithms:

Define monadic $fm :: \cdots \to (\tau, nat)$
Then define $f = value \circ fm$ and $T_f = time \circ fm$

Example

$$fm\ []\ ys = return\ []$$

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Example

$fm\ []\ ys = return\ []$

$fm\ (x\#xs)\ ys = \{xys \leftarrow fm\ xs\ ys; return(x\ \#\ xys)$

$f\ xs\ ys = val(fm\ xs\ ys) \qquad T_f\ xs\ ys = time(fm\ xs\ ys)$

For proving properties of f and T_f:

Derive original recursive definitions of f and T_f
by automatic inductive proof

The rest of the presentation, mostly

- Focus on persistence and constant time access

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- Focus on persistence and constant time access
- No need to analyze time
  because all functions non-recursive

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2. Real Time Queue
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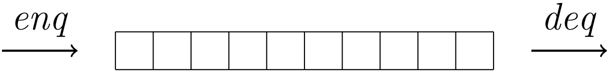
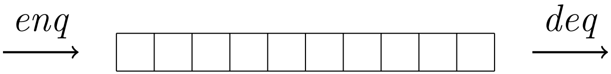
Queue

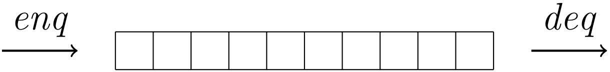


Queue



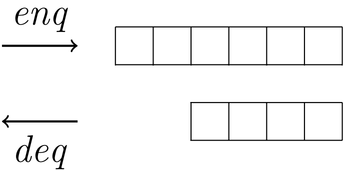
How to implement a functional queue efficiently?

Queue

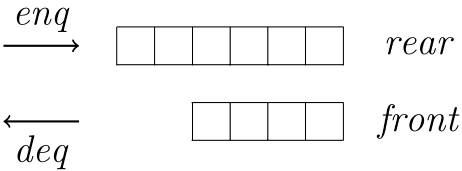


How to implement a functional queue efficiently?
As a list: either enq or deq take linear time

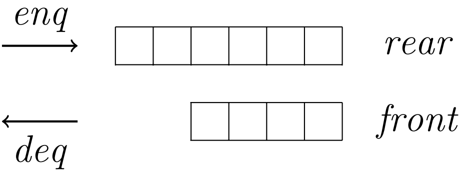
Two stacks



Two stacks

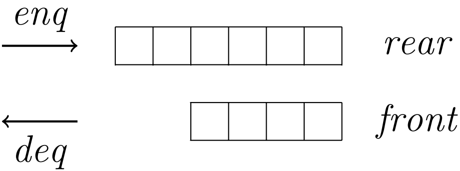


Two stacks



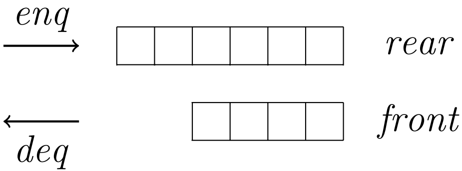
Problem: what if $front$ becomes empty?

Two stacks



Problem: what if $front$ becomes empty?
Need to reverse $rear$ — linear time!

Two stacks



enq $\longrightarrow$ $rear$

$\longleftarrow$ deq $front$

Problem: what if $front$ becomes empty?
Need to reverse $rear$ — linear time!

However: *amortized* running time of each operation (averaged over a sequnce of operations) is constant

Challenge: *Real Time Queue*

All operations have *worst-case* constant running time

One solution: *laziness*

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Implementation with eager/call-by-value evaluation?

Real Time Queue

with call-by-value

- Do not wait for $front = []$

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- Do not wait for $front = []$
- Compute new front $front\ @\ rev\ rear$
  early and incrementally

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- Incremental reversal by pair of stacks:
  $([a, b, c], [])$

Real Time Queue

with call-by-value

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- Compute new front $front\ @\ rev\ rear$ early and incrementally
- Incremental reversal by pair of stacks: $([a, b, c], []) \to ([b, c], [a])$

Real Time Queue

with call-by-value

- Do not wait for $\mathit{front} = []$
- Compute new front $\mathit{front} \mathbin{@} \mathit{rev}\ \mathit{rear}$
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- Incremental reversal by pair of stacks:
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Real Time Queue

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- Compute new front $front \mathbin{@} rev\ rear$
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  $([a, b, c], []) \to ([b, c], [a]) \to ([c], [b, a]) \to ([], [c, b, a])$
- Using a ‘copy’ of $front$ and $rear$

Real Time Queue

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- Using a ‘copy’ of $front$ and $rear$
  “shadow queue”

Real Time Queue

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- Do not wait for $front = []$
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- Incremental reversal by pair of stacks: $([a, b, c], []) \rightarrow ([b, c], [a]) \rightarrow ([c], [b, a]) \rightarrow ([], [c, b, a])$
- Using a ‘copy’ of $front$ and $rear$ “shadow queue”
- In parallel with enq and deq calls

Reversal strategy

Aim: $(r, f) \rightarrow^* ([], f \ @ \ rev \ r)$

Reversal strategy

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In two phases:

Reversal strategy

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In two phases:

1. Reverse $\quad r = [b_1, \ldots, b_m] \to^m [b_m, \ldots, b_1] =: r'$

Reversal strategy

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In two phases:

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   and $f = [a_1, \ldots, a_n] \rightarrow^{n} [a_n, \ldots, a_1] =: f'$

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2. Reverse f' onto r':
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When to start?

Reversal strategy

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When to start? When $m = n + 1$!

Reversal strategy

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In two phases:

1. Reverse $r = [b_1, \ldots, b_m] \to^{m} [b_m, \ldots, b_1] =: r'$
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When to start? When $m = n + 1$!

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- Need to finish before original front becomes empty

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When to start? When $m = n + 1$!

- Requires $n + 1 + n$ steps
- Need to finish before original front becomes empty
- Need to perform 2 steps per enq/deq
- $+1$ initial step

Complication

Complication

deq from the original $front$

Complication

deq from the original $front$

Cannot easily remove them from the shadow queue

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Solution:

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enq into new (initially empty) $rear$.

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deq from the original $front$

Cannot easily remove them from the shadow queue

Solution:

- Remember how many elements have been removed
- Better: how many elements are still valid

enq into new (initially empty) $rear$.
Reversal fast enough to ensure $|new\ rear| \leq |new\ front|$ at the end

Implementation

The shadow queue

datatype $'a\ status =$

$\quad Idle\ |$

$\quad Rev\ (nat)\ ('a\ list)\ ('a\ list)\ ('a\ list)\ ('a\ list)\ |$

$\quad App\ (nat)\ ('a\ list)\ ('a\ list)\ |$

$\quad Done\ ('a\ list)$

Shadow step

$$exec :: {}'a\ status \Rightarrow {}'a\ status$$

$$exec\ Idle = Idle$$

$$\begin{aligned}&exec\ (Rev\ ok\ (x \# f)\ f'\ (y \# r)\ r')\\ &= Rev\ (ok + 1)\ f\ (x \# f')\ r\ (y \# r')\end{aligned}$$

$$exec\ (Rev\ ok\ []\ f'\ [y]\ r') = App\ ok\ f'\ (y \# r')$$

$$exec\ (App\ (ok + 1)\ (x \# f')\ r') = App\ ok\ f'\ (x \# r')$$

$$exec\ (App\ 0\ f'\ r') = Done\ r'$$

$$exec\ (Done\ v) = Done\ v$$

Dequeue from shadow queue

$$invalidate :: {}'a\ status \Rightarrow {}'a\ status$$

$$invalidate\ Idle = Idle$$

$$invalidate\ (Rev\ ok\ f\ f'\ r\ r') = Rev\ (ok - 1)\ f\ f'\ r\ r'$$

$$invalidate\ (App\ (ok + 1)\ f'\ r') = App\ ok\ f'\ r'$$

$$invalidate\ (App\ 0\ f'\ (x \# r')) = Done\ r'$$

$$invalidate\ (Done\ v) = Done\ v$$

The whole queue

$$\textbf{record}\ 'a\ queue = \begin{array}{ll} front & :: 'a\ list \\ lenf & :: nat \\ rear & :: 'a\ list \\ lenr & :: nat \\ status & :: 'a\ status \end{array}$$

enq and deq

$$enq\ x\ q =$$
$$check\ (q(\!|rear := x\ \#\ rear\ q,\ lenr := lenr\ q + 1|\!))$$

$$deq\ q =$$
$$check$$
$$(q(\!|lenf := lenf\ q - 1,\ front := tl\ (front\ q),$$
$$status := invalidate\ (status\ q)|\!))$$

$$\begin{array}{l}
check\ q = \\
(\mathsf{if}\ lenr\ q \le lenf\ q\ \mathsf{then}\ exec2\ q \\
\quad \mathsf{else\ let}\ newstate = \\
\qquad\qquad\qquad Rev\ 0\ (front\ q)\ []\ (rear\ q)\ [] \\
\qquad\quad \mathsf{in}\ exec2 \\
\qquad\qquad\quad (q(\!|lenf := lenf\ q + lenr\ q, \\
\qquad\qquad\qquad\quad status := newstate, \\
\qquad\qquad\qquad\quad rear := [],\ lenr := 0|\!)))
\end{array}$$

$$\begin{array}{l}
exec2\ q = \ (case\ exec\ (exec\ q)\ of \\
\qquad\qquad\qquad\quad Done\ fr \Rightarrow q(\!|status = Idle,\ front = fr|\!)\ | \\
\qquad\qquad\qquad\quad newstatus \Rightarrow q(\!|status = newstatus|\!))
\end{array}$$

Model oriented specification

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Model data structure by existing mathematical types

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Example: $queue$ by $list$

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Correctness proof of an implementation:
define α and prove Spec

Queue specification

interface $empty :: {'a}\ queue$

$enq :: {'a} \Rightarrow {'a}\ queue \Rightarrow {'a}\ queue$

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specification

$$invar\ q \Longrightarrow list\ (enq\ x\ q) = list\ q\ @\ [x]$$

Queue specification

interface $\mathit{empty} :: {}'a\ \mathit{queue}$

$\mathit{enq} :: {}'a \Rightarrow {}'a\ \mathit{queue} \Rightarrow {}'a\ \mathit{queue}$

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invariant $\mathit{invar} :: {}'a\ \mathit{queue} \Rightarrow \mathit{bool}$

specification

$\mathit{invar}\ q \Longrightarrow \mathit{list}\ (\mathit{enq}\ x\ q) = \mathit{list}\ q\ @\ [x]$

$\mathit{invar}\ q \Longrightarrow \mathit{list}\ (\mathit{deq}\ q) = \mathit{tail}\ (\mathit{list}\ q)$

$\mathit{invar}\ q \wedge \mathit{list}\ q \neq [] \Longrightarrow \mathit{first}\ q = \mathit{head}\ (\mathit{list}\ q)$

$\vdots$

Correctness

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- tricky because of invariant and abstraction function
  700 lines of Isabelle (by Alejandro Gómez-Londoño)

$status$ invariant

$$\begin{aligned}
&inv\_st\ (Rev\ ok\ f\ f'\ r\ r') = \\
&(|f| + 1 = |r| \wedge |f'| = |r'| \wedge ok \leq |f'|) \\
&inv\_st\ (App\ ok\ f'\ r') = (ok \leq |f'| \wedge |f'| < |r'|) \\
&inv\_st\ Idle = True \\
&inv\_st\ (Done\ \_) = True
\end{aligned}$$

Queue invariant

$$
\begin{array}{l}
\mathit{invar}\ q = \\
(\mathit{lenf}\ q = |\mathit{front\_list}\ q| \wedge \\
\ \mathit{lenr}\ q = |\mathit{rev}\ (\mathit{rear}\ q)| \wedge \\
\ \mathit{lenr}\ q \leq \mathit{lenf}\ q \wedge \\
\ (\textsf{case}\ \mathit{status}\ q\ \textsf{of} \\
\qquad \mathit{Rev}\ \mathit{ok}\ f\ f'\ r\ r' \Rightarrow \\
\qquad\quad 2 * \mathit{lenr}\ q \leq |f'| \wedge \\
\qquad\quad \mathit{ok} \neq 0 \wedge 2 * |f| + \mathit{ok} + 2 \leq 2 * |\mathit{front}\ q| \\
\quad |\ \mathit{App}\ \mathit{ok}\ f\ r \Rightarrow \\
\qquad\quad 2 * \mathit{lenr}\ q \leq |r| \wedge \mathit{ok} + 1 \leq 2 * |\mathit{front}\ q| \\
\quad |\ \_ \Rightarrow \mathit{True}) \wedge \\
\ (\exists\, \mathit{rest}.\ \mathit{front\_list}\ q = \mathit{front}\ q\ @\ \mathit{rest}) \wedge \\
\ (\nexists\, \mathit{fr}.\ \mathit{status}\ q = \mathit{Done}\ \mathit{fr}) \wedge \mathit{inv\_st}\ (\mathit{status}\ q))
\end{array}
$$

Abstraction function

$$
\begin{array}{l}
\mathit{list}\ q = \mathit{front\_list}\ q\ @\ \mathit{rear\_list}\ q \\
\mathit{front\_list}\ q = \\
(\textsf{case}\ \mathit{status}\ q\ \textsf{of} \\
\quad \mathit{Idle} \Rightarrow \mathit{front}\ q \\
\ \mid \mathit{Rev}\ \mathit{ok}\ f\ f'\ r\ r' \Rightarrow \mathit{rev}\ (\mathit{take}\ \mathit{ok}\ f')\ @\ f\ @\ \mathit{rev}\ r\ @\ r' \\
\ \mid \mathit{App}\ \mathit{ok}\ f'\ x \Rightarrow \mathit{rev}\ (\mathit{take}\ \mathit{ok}\ f')\ @\ x \\
\ \mid \mathit{Done}\ f \Rightarrow f)
\end{array}
$$

The inventors

Robert Hood and Robert Melville.
Real-Time Queue Operation in Pure LISP.
Information Processing Letters, 1981.

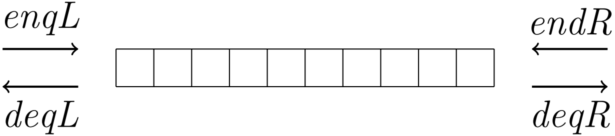
1. Time

2. Real Time Queue

3. Real Time Double-Ended Queue

4. Skew Heap

Double-Ended Queue ("Deque")



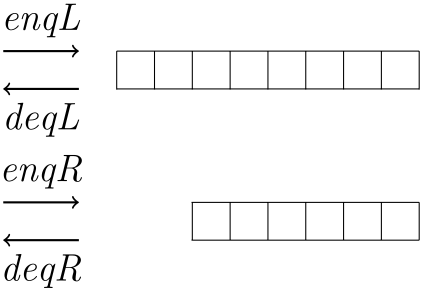
$enqL$

$deqL$

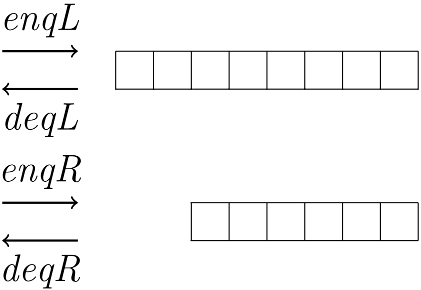
$endR$

$deqR$

Two stacks



Two stacks



$\xrightarrow{enqL}$

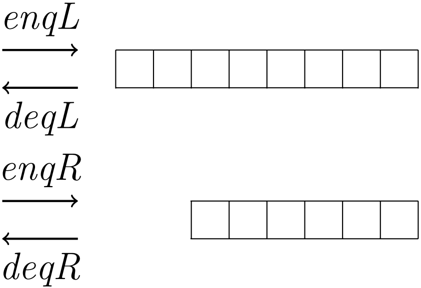
$\underset{deqL}{\xleftarrow{\hspace{2em}}}$

$\xrightarrow{enqR}$

$\underset{deqR}{\xleftarrow{\hspace{2em}}}$

Amortized constant time enq/deq:

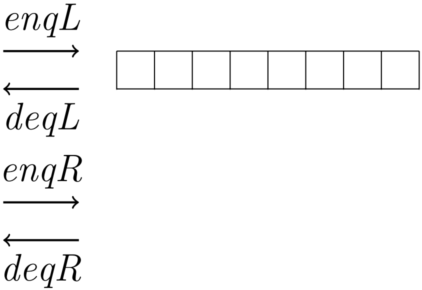
Two stacks



Amortized constant time enq/deq:

If one stack becomes empty,

Two stacks



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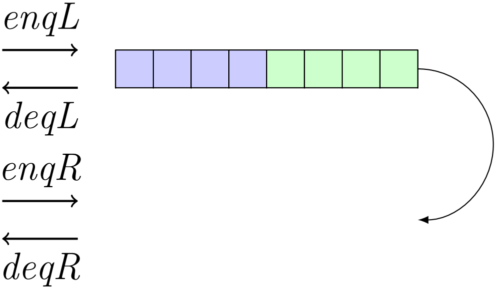

Amortized constant time enq/deq:

If one stack becomes empty,
reverse *the botttom half* of the other one

Real Time Deque

One solution: *laziness*

Real Time Deque

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Implementation with eager/call-by-value evaluation?

Real Time Deque

Call-by-value

- Do not wait for $[\,]$

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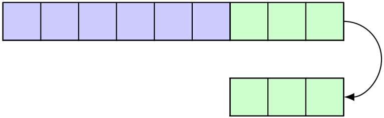

Main invariant

S is smaller stack, B bigger stack, $m = |S|$, $n = |B|$.

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When $3m \approx n$

Main invariant

S is smaller stack, B bigger stack, $m = |S|$, $n = |B|$.

$$3m \geq n$$

When is $3m \geq n$ destroyed by enq or deq?
When $3m \approx n$ ($\approx$ means we ignore the fine details)

Rebalancing strategy

Start: $B = B_{12} @ B_3$ where $|B_{12}| = 2m$ and $|B_3| = m$.

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$B_{12} @ B_3$

S

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$$B_{12} @ B_3 \quad \rightarrow^{2m} \quad \overleftarrow{B}_{12}$$

$$B_3$$

$$S$$

Rebalancing strategy

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$$\begin{array}{lll}
B_{12} @ B_3 & \rightarrow^{2m} & \overleftarrow{B_{12}} \\
 & & B_3 \\
S & \rightarrow^{m} & \overleftarrow{S}
\end{array}$$

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$$\begin{array}{llll}
B_{12} @ B_3 & \rightarrow^{2m} & \overleftarrow{B}_{12} & \rightarrow^{2m} & B_{12} \\
 & & B_3 & & \\
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$$\begin{tikzcd}
B_{12} @ B_3 \arrow[r, "2m"] & \overleftarrow{B_{12}} \arrow[rr, "2m"] & & B_{12} \\
& B_3 \arrow[r, "m"] & \overleftarrow{B_3} & \\
S \arrow[r, "m"] & \overleftarrow{S} & &
\end{tikzcd}$$

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$$
\begin{array}{lllllll}
B_{12} @ B_3 & \rightarrow^{2m} & \overleftarrow{B_{12}} & & & \rightarrow^{2m} & B_{12} \\
 & & B_3 & \rightarrow^{m} & \overleftarrow{B_3} & \rightarrow^{m} & S @ \overleftarrow{B_3} \\
S & \rightarrow^{m} & \overleftarrow{S} & & & &
\end{array}
$$

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$$\begin{array}{lllllll}
B_{12} @ B_3 & \rightarrow^{2m} & \overleftarrow{B_{12}} & & \rightarrow^{2m} & & B_{12} \\
 & & B_3 & \rightarrow^{m} & \overleftarrow{B_3} & \rightarrow^{m} & S @ \overleftarrow{B_3} \\
S & \rightarrow^{m} & \overleftarrow{S} & & & &
\end{array}$$

Requires $4m$ micro-steps, 4 per enq/deq step

Two deques

Rebalancing happens on shadow deque

Two deques

Rebalancing	happens on	shadow deque
enq/deq	happens on	current deque

Another complication

At the end of rebalancing:

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Need to combine results of rebalancing
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(Why not a problem with real time queue?)

Another detail

Deques of size ≤ 3 are represented as normal lists

No problems

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Rebalancing needs m steps
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Two extremes during rebalancing:

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- $m \times deq$ of S: S has m elements ✓
  in the end the stacks have size m and $2m$ ✓

The full story

500 lines of code

3900 lines of invariants, abstraction functions and proofs

(by Balazs Toth)

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Real-time deques, multihead turing machines, and purely functional programming. In *FPCA* 1993.

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Real-time deques, multihead turing machines, and purely functional programming. In *FPCA* 1993.

Already sketched in Hood's PhD thesis 1982

1. Time

2. Real Time Queue

3. Real Time Double-Ended Queue

4. **Skew Heap**

A *skew heap* is a self-adjusting heap (priority queue)

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Functions $insert$, $merge$ and del_min
have amortized logarithmic complexity.

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Functions $insert$, $merge$ and del_min
have amortized logarithmic complexity.

Functions $insert$ and del_min are defined via $merge$

Implementation type

Ordinary binary trees

Implementation type

Ordinary binary trees

Invariant: $heap$

$merge$

$$merge\ \langle\rangle\ t = t$$
$$merge\ h\ \langle\rangle = h$$

$merge$

$merge\ \langle\rangle\ t = t$
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Swap subtrees when descending:

$merge$

$merge\ \langle\rangle\ t = t$
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Swap subtrees when descending:

$merge\ (\langle l_1,\ a_1,\ r_1\rangle =: t_1)\ (\langle l_2,\ a_2,\ r_2\rangle =: t_2) =$
$(\textsf{if}\ a_1 \le a_2\ \textsf{then}\ \langle merge\ t_2\ r_1,\ a_1,\ l_1\rangle$
$\ \textsf{else}\ \langle merge\ t_1\ r_2,\ a_2,\ l_2\rangle)$

Functional correctness proofs

Straightforward

Logarithmic amortized complexity

Theorem

$$T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2$$
$$\leq 3 * \log_2\ (|t_1|_1 + |t_2|_1) + 1$$

Towards the proof

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Right heavy:

$rh\ l\ r = (\textsf{if}\ |l| < |r|\ \textsf{then}\ 1\ \textsf{else}\ 0)$

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Number of right heavy nodes on left spine:

$lrh\ \langle\rangle = 0$
$lrh\ \langle l,\ \_,\ r\rangle = rh\ l\ r + lrh\ l$

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Lemma

$2^{lrh\ t} \leq |t| + 1$

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Corollary

$lrh\ t \le \log_2\ |t|_1$

Towards the proof

Right heavy:

$rh\ l\ r = (\text{if } |l| < |r| \text{ then } 1 \text{ else } 0)$

Number of not right heavy nodes on right spine:

$rlh\ \langle\rangle = 0$

$rlh\ \langle l,\ \_,\ r\rangle = 1 - rh\ l\ r + rlh\ r$

Lemma

$2^{rlh\ t} \leq |t| + 1$

Corollary

$rlh\ t \leq \log_2\ |t|_1$

Potential

The potential is the number of right heavy nodes:

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$\Longrightarrow$ right heavy nodes are bad

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Lemma

$T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2$
$\leq lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1$

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The potential is the number of right heavy nodes:

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$\Phi\ \langle l,\ \_,\ r\rangle = \Phi\ l + \Phi\ r + rh\ l\ r$

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```
by(induction t1 t2 rule: merge.induct)(auto)
```

Node-Node case

Let $t_1 = \langle l_1,\ a_1,\ r_1\rangle$, $t_2 = \langle l_2,\ a_2,\ r_2\rangle$.

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Case $a_1 \leq a_2$. Let $m = merge\ t_2\ r_1$

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$$T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2$$

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$$
\begin{aligned}
&T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2\\
&= T\_merge\ t_2\ r_1 + 1 + \Phi\ m + \Phi\ l_1 + rh\ m\ l_1\\
&\quad - \Phi\ t_1 - \Phi\ t_2
\end{aligned}
$$

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$$\begin{aligned}
&T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2\\
= {}&T\_merge\ t_2\ r_1 + 1 + \Phi\ m + \Phi\ l_1 + rh\ m\ l_1\\
&\quad - \Phi\ t_1 - \Phi\ t_2\\
= {}&T\_merge\ t_2\ r_1 + 1 + \Phi\ m + rh\ m\ l_1\\
&\quad - \Phi\ r_1 - rh\ l_1\ r_1 - \Phi\ t_2
\end{aligned}$$

Node-Node case

Let $t_1 = \langle l_1,\ a_1,\ r_1 \rangle$, $t_2 = \langle l_2,\ a_2,\ r_2 \rangle$.
Case $a_1 \leq a_2$. Let $m = merge\ t_2\ r_1$

$$\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
= {} & T\_merge\ t_2\ r_1 + 1 + \Phi\ m + \Phi\ l_1 + rh\ m\ l_1 \\
& - \Phi\ t_1 - \Phi\ t_2 \\
= {} & T\_merge\ t_2\ r_1 + 1 + \Phi\ m + rh\ m\ l_1 \\
& - \Phi\ r_1 - rh\ l_1\ r_1 - \Phi\ t_2 \\
\leq {} & lrh\ m + rlh\ t_2 + rlh\ r_1 + rh\ m\ l_1 + 2 - rh\ l_1\ r_1
\end{aligned}$$

by IH

Node-Node case

Let $t_1 = \langle l_1,\ a_1,\ r_1\rangle$, $t_2 = \langle l_2,\ a_2,\ r_2\rangle$.
Case $a_1 \le a_2$. Let $m = merge\ t_2\ r_1$

$T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2$
$= T\_merge\ t_2\ r_1 + 1 + \Phi\ m + \Phi\ l_1 + rh\ m\ l_1$
$\quad - \Phi\ t_1 - \Phi\ t_2$
$= T\_merge\ t_2\ r_1 + 1 + \Phi\ m + rh\ m\ l_1$
$\quad - \Phi\ r_1 - rh\ l_1\ r_1 - \Phi\ t_2$
$\le lrh\ m + rlh\ t_2 + rlh\ r_1 + rh\ m\ l_1 + 2 - rh\ l_1\ r_1$
by IH
$= lrh\ m + rlh\ t_2 + rlh\ t_1 + rh\ m\ l_1 + 1$

Node-Node case

Let $t_1 = \langle l_1,\ a_1,\ r_1\rangle$, $t_2 = \langle l_2,\ a_2,\ r_2\rangle$.
Case $a_1 \le a_2$. Let $m = merge\ t_2\ r_1$

$$
\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
= {} & T\_merge\ t_2\ r_1 + 1 + \Phi\ m + \Phi\ l_1 + rh\ m\ l_1 \\
& - \Phi\ t_1 - \Phi\ t_2 \\
= {} & T\_merge\ t_2\ r_1 + 1 + \Phi\ m + rh\ m\ l_1 \\
& - \Phi\ r_1 - rh\ l_1\ r_1 - \Phi\ t_2 \\
\le {} & lrh\ m + rlh\ t_2 + rlh\ r_1 + rh\ m\ l_1 + 2 - rh\ l_1\ r_1 \\
& \text{by IH} \\
= {} & lrh\ m + rlh\ t_2 + rlh\ t_1 + rh\ m\ l_1 + 1 \\
= {} & lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1
\end{aligned}
$$

Main proof

$$T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2$$

Main proof

$$
\begin{aligned}
&T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
&\le lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1
\end{aligned}
$$

Main proof

$$
\begin{aligned}
&T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2\\
&\le lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1\\
&\le \log_2\ |merge\ t_1\ t_2|_1 + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1
\end{aligned}
$$

Main proof

$$
\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
& \leq lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1 \\
& \leq \log_2\ |merge\ t_1\ t_2|_1 + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& = \log_2\ (|t_1|_1 + |t_2|_1 - 1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1
\end{aligned}
$$

Main proof

$$
\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
& \leq lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1 \\
& \leq \log_2\ |merge\ t_1\ t_2|_1 + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& = \log_2\ (|t_1|_1 + |t_2|_1 - 1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& \leq \log_2\ (|t_1|_1 + |t_2|_1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1
\end{aligned}
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Main proof

$$
\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
& \le lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1 \\
& \le \log_2\ |merge\ t_1\ t_2|_1 + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& = \log_2\ (|t_1|_1 + |t_2|_1 - 1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& \le \log_2\ (|t_1|_1 + |t_2|_1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
& \le \log_2\ (|t_1|_1 + |t_2|_1) + 2 * \log_2\ (|t_1|_1 + |t_2|_1) + 1
\end{aligned}
$$

because $\log_2\ x + \log_2\ y \le 2 * \log_2\ (x + y)$ if $x,y > 0$

Main proof

$$
\begin{aligned}
& T\_merge\ t_1\ t_2 + \Phi\ (merge\ t_1\ t_2) - \Phi\ t_1 - \Phi\ t_2 \\
\le\ & lrh\ (merge\ t_1\ t_2) + rlh\ t_1 + rlh\ t_2 + 1 \\
\le\ & \log_2\ |merge\ t_1\ t_2|_1 + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
=\ & \log_2\ (|t_1|_1 + |t_2|_1 - 1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
\le\ & \log_2\ (|t_1|_1 + |t_2|_1) + \log_2\ |t_1|_1 + \log_2\ |t_2|_1 + 1 \\
\le\ & \log_2\ (|t_1|_1 + |t_2|_1) + 2 * \log_2\ (|t_1|_1 + |t_2|_1) + 1 \\
& \text{ because } \log_2\ x + \log_2\ y \le 2 * \log_2\ (x + y) \text{ if } x,y > 0 \\
=\ & 3 * \log_2\ (|t_1|_1 + |t_2|_1) + 1
\end{aligned}
$$

Sources

The inventors of skew heaps:

Daniel Sleator and Robert Tarjan.
Self-adjusting Heaps.
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Formalisation: TN

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Verification of lazy versions?