Mirror symmetry and big algebras

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- the Hecke transform of the Hitchin section $\mathcal{H}^\mu_c(\mathcal{O}_{W_0^+})$ is supported at a union of Lagrangian upward flows

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- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow W^+_{\mathcal{E}}$ closed $\Leftrightarrow h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathbb{A}$ proper
- e.g. $\mathcal{E} = (E, 0)$ very stable $\Leftrightarrow E$ very stable (Laumon 1988)

Lagrangian upward flows in $\mathbb{M}$

- $\mathbb{M} := \mathbb{M}_{\mathrm{PGL}_n} \ni (E,\Phi)$; $\Phi \in \mathrm{End}_0(E) \otimes K_C$
- $$\begin{array}{cccc} h: & \mathbb{M} & \to & \mathbb{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E,\Phi) & \mapsto & \det(x-\Phi) \end{array}$$ Hitchin map
- $\mathbb{C}^\times \curvearrowright \mathbb{M}$ by $(E,\Phi) \mapsto (E,\lambda\Phi)$; *semiprojective*:
  1. $\mathbb{M}^{\mathbb{C}^\times}$ projective
  2. $\lim_{\lambda\to 0} \lambda\mathcal{E}$ exists for every $\mathcal{E} \in \mathbb{M}$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times} \rightsquigarrow W_{\mathcal{E}}^+ := \{\mathcal{F} \in \mathbb{M} | \lim_{\lambda\to 0} \lambda\mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W_{\mathcal{E}}^+ \subset \mathbb{M}$ locally closed $\cong T_{\mathcal{E}}^+\mathbb{M}$
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- e.g. $\mathcal{E} = (E,0)$ very stable $\Leftrightarrow E$ very stable (Laumon 1988)
- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \to \mathbb{A}$$

becomes explicit!

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Hitchin map as spectrum of equivariant cohomology

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H^{2*}_{\mathrm{SL}_n} \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\llcorner"] & \mathbb{C}[\mathbb{A}] \arrow[u] \arrow[r, phantom, "\rightsquigarrow"] & & \mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) & \mathbb{A} \arrow[l, two heads] \arrow[ul, phantom, "\llcorner"]
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- construction by applying Kirillov M-operators to Φ_a

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Mirror of equivariant cohomology is Kirillov algebra

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- $\rightsquigarrow$ *classical limit of geometric Satake*

Construction of the big algebra

Construction of the big algebra

- $\{X_i\}$ basis for $\mathfrak{sl}_n$

Construction of the big algebra

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Corollary (Hausel 2022)

$H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^{\mu}) \cong \mathcal{M}^{\mu}$ *as* $H^*_{\mathrm{PGL}_n}$*-algebras*

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$$\begin{tikzcd}
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\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, twoheadrightarrow] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
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- $(\mathcal{B}^\mu)_\kappa := \mathcal{B}^\mu/(x - \kappa(x))_{x \in \mathcal{B}^\mu}$ *coinvariant algebra*
- $\Rightarrow \mathrm{tr}\left(\kappa : V^\mu_\lambda \to V^\mu_\lambda\right) = \dim \check{V}^\mu_\lambda$

Finer structure of big algebras

- (Yakimova 2022) $\rightsquigarrow$ construction of $\mathcal{B}^\mu$ generalises to other simple G, but explicit generators only in classical types and G_2
- $\mathrm{Spec}(\mathcal{M}^\mu) = \bigcup_{X_*^+ \ni \lambda \leq \mu} \mathrm{Spec}(\mathcal{M}^\mu_\lambda)$ irreducible components.
  $\mathcal{M}^\mu_\lambda \cong H^*_{\mathrm{G}}(\mathrm{G}/P_\lambda, \mathbb{C}) \cong H^*_{\mathrm{G}}(\mathrm{Gr}^\lambda_\mu, \mathbb{C})$; $\mathrm{Gr}^\mu = \coprod_{\mu \leq \lambda} \mathrm{Gr}^\mu_\lambda$ BB-dec.
- mutliplicity algebra of $\mathcal{M}^\mu_\lambda \subset \mathcal{B}^\mu_\lambda$ endows
  $IH^*(W^\mu_\lambda) \cong \mathcal{B}^\mu_\lambda / (\mathcal{M}^\mu_{\lambda +})$ a graded ring
  ringifying Kazhdan-Lusztig polynomials $IP_t(W^\mu_\lambda)$
  $IH^*(W^\mu_0) \cong \mathcal{B}^\mu / (\mathcal{M}^\mu_+) \rightsquigarrow$ (Brylinski 1989) limits of weight spaces
- G complex reductive, $\kappa \in \mathrm{Aut}(\mathrm{G})$ distinguished
  corresponds to a folding of the Dynkin diagram
- endoscopy group $\check{\mathrm{G}}_\kappa := \widecheck{\mathrm{G}^\kappa_0}$

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Finer structure of big algebras

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- for principal $\mathrm{H} \subset \mathrm{G}$ we expect a transfer $\mathcal{B}^\mu(\check{\mathrm{G}}) \twoheadrightarrow \mathcal{B}^\mu(\check{\mathrm{H}})$