Vinberg pairs and Higgs bundles

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- θ defines an automorphism of $\mathfrak{g}$ (denoted also by θ), determining a $\mathbb{Z}/m$-**grading** of $\mathfrak{g}$ (we write $\mathbb{Z}/m$ instead of $\mathbb{Z}/m\mathbb{Z}$) :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i \text{ with } \mathfrak{g}_i = \{x \in \mathfrak{g} \text{ such that } \theta(x) = \zeta^i x\},$$

where ζ is a primitive m-th root of unity. One has

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Vinberg θ-pairs

- Let $\mu_m = \{z \in \mathbb{C}^* \text{ such that } z^m = 1\}$. Having $\mathbb{Z}/m$-grading on $\mathfrak{g}$ is equivalent to having a homomorphism

$$\tilde{\theta} : \mu_m \to \mathrm{Aut}(\mathfrak{g}).$$

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- Since $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$, all the subspaces $\mathfrak{g}_i$ are stable under the adjoint action of G^θ.
- The pairs $(G^\theta, \mathfrak{g}_i)$ are called **Vinberg θ-pairs** (also **θ-groups** or **Vinberg θ-representations**).

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- Since $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$, all the subspaces $\mathfrak{g}_i$ are stable under the adjoint action of G^θ.
- The pairs $(G^\theta, \mathfrak{g}_i)$ are called **Vinberg θ-pairs** (also **θ-groups** or **Vinberg θ-representations**).
- Sometimes it is convenient in the Vinberg pair to replace G^θ by any subroup between the connected component of the identity of G^θ and its normalizer in G.

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Cartan: Given a holomorphic involution θ of G there is a **compact conjugation** τ of G, so that $\sigma := \tau\theta = \theta\tau$ is a conjugation of G. This gives a bijection

$$\mathrm{Aut}_2(G)/\sim \longleftrightarrow \mathrm{Conj}(G)/\sim,$$

where equivalence is conjugation by an inner automorphism of G.

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Let $G = \mathrm{SL}(V)$. Define on $\mathfrak{g} = \mathfrak{sl}(V)$ the $\mathbb{Z}/m$-**grading** given by

$$\mathfrak{g}_i = \{A \in \mathfrak{sl}(V) \text{ such that } A(V_j) \subset V_{j+i} \text{ for every } j \in \mathbb{Z}/m\}$$

In this situation

$$G^\theta = \mathrm{S}(\prod_{i \in \mathbb{Z}/m} \mathrm{GL}(V_i)),$$

and

$$\mathfrak{g}_1 = \bigoplus_{i \in \mathbb{Z}/m} \mathrm{Hom}(V_i, V_{i+1}).$$

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  This can be represented by the diagramme

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-2}} V_{m-1}, \qquad V_{m-1} \xrightarrow{f_{m-1}} V_0.$$

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- For other classical groups the action of G^θ on $\mathfrak{g}_1$ can be interpreted in terms of a cyclic quiver with some extra structure.

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- Recall first that if $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra, and $W(\mathfrak{t})$ is the Weyl group, the **Chevalley restriction theorem** establishes an isomorphism

$$\mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W(\mathfrak{t}).$$

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- Similarly, if θ is an **involution** of G and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the **Cartan decomposition** defined by θ, and $W(\mathfrak{a})$ is the **little Weyl group** defined by a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_1$, there is also a **Chevalley restriction theorem** studied by **Kostant–Rallis** (1971):

$$\mathfrak{g}_1 /\!\!/ G^\theta \cong \mathfrak{a}/W(\mathfrak{a}).$$

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- Key concept is that of **Cartan subspace**: linear subspace $\mathfrak{a} \subset \mathfrak{g}_1$ which is abelian as a Lie algebra, consisting of semisimple elements, and maximal with these two properties.

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- Key concept is that of **Cartan subspace**: linear subspace $\mathfrak{a} \subset \mathfrak{g}_1$ which is abelian as a Lie algebra, consisting of semisimple elements, and maximal with these two properties.
- The **little Weyl group**

$$W(\mathfrak{a}) = N_{G_\theta}(\mathfrak{a})/C_{G_\theta}(\mathfrak{a})$$

is a finite linear group generated by semisimple transformations of $\mathfrak{a}$ fixing a hyperplane. Hence $\mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$ is a **polynomial ring**, and the restriction of polynomial functions $\mathbb{C}[\mathfrak{g}_1] \to \mathbb{C}[\mathfrak{a}]$ induces an **isomorphism of invariant polynomial rings** $\mathbb{C}[\mathfrak{g}_1]^{G^\theta} \to \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$, or equivalently,

$$\mathfrak{g}_1 /\!\!/ G^\theta \cong \mathfrak{a}/W(\mathfrak{a}).$$

Vinberg θ-pairs

- The fact that $W(\mathfrak{a})$ is a finite linear group generated by complex reflections implies that $\mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})} = \mathbb{C}[f_1, \cdots, f_r]$ is a polynomial algebra generated by r **algebraically independent polynomials** $f_1, \cdots, f_r$ whose degrees $d_1, \cdots, d_r$ are determined by the grading. Here r is the dimension of $\mathfrak{a}$, an invariant called the **rank of** $(G^\theta, \mathfrak{g}_1)$.

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- This was extended by Kostant–Rallis (1971) to obtain the **Kostant–Rallis section** in the symmetric pair case for the quotient map $\mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G^\theta$.

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- The existence of a similar section for Vinberg's θ-pairs for θ of higher order was conjectured by Popov (1976), but only proved more recently in full generality by Reeder–Levy–Yu–Gross (2012). In this context, such a section is referred as a **Kostant–Weierstrass section**.

Vinberg θ-pairs

- There are many **interesting applications** of Vinberg theory: GIT and representation theory, Classification of trivectors of 9-dimensional space $\mathbb{C}^9$ (Elashvili–Vinberg), arithmetic theory of elliptic curves and Jacobians (Bhargava–Shankar, Bhargava–Gross,...), moduli space of genus 2 curves (Rains–Sam), description of the moduli space of vector bundles on curves of small genus, del Pezzo surfaces and mysterious duality (Iqbal–Neitzke–Vafa), etc.

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- The **goal of this talk** is to discuss the role of Vinberg θ-pairs in **Higgs bundle theory**.
- First, I will review some old results put in the larger context of Vinberg's theory.
- Then I will introduce some new problems and work in progress in the study of the geometry of moduli spaces of **cyclic Higgs bundles**.

Higgs pairs

- X compact Riemann surface of genus $g \geq 2$ with canonical line bundle K
- G reductive complex Lie group with Lie algebra $\mathfrak{g}$
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- $\rho : G \to \mathrm{GL}(V)$ a representation of G in a complex vector space V
- A **(G, V)-Higgs pair on** X is a pair (E, φ) consisting of a holomorphic principal G-bundle $E \to X$ and
  $\varphi \in H^0(X, E(V) \otimes K)$,
  where $E(V) = E \times_G V$ is the vector bundle associated to the representation ρ.

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- There are suitable notions of (semi,poly) stability. Consider the **moduli space of polystable (G, V)-Higgs pairs**:

$$\mathcal{M}(G, V)$$

Higgs bundles

- When ρ is the **adjoint representation** $G \to \mathrm{GL}(\mathfrak{g})$ $(G, \mathfrak{g})$-Higgs pairs are the G-**Higgs bundles** introduced by **Hitchin** (1987).

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  $\mathcal{M}(G)$ is a quasi-projective variety and has a **hyperkähler structure** on the smooth locus.
- An important feature of G-Higgs bundles is their relation to representations of the fundamental group of X:
  The G-**character variety** of the fundamental group of X is defined as

$$\mathcal{R}(G) = \mathrm{Hom}(\pi_1(X), G) /\!\!/ G.$$

  The **non-abelian Hodge correspondence** (**Hitchin 1987, Donaldson 1987, Simpson 1988, Corlette 1988**) for G semisimple establishes a homeomorphism

$$\mathcal{M}(G) \cong \mathcal{R}(G).$$

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and the Vinberg θ-pairs $(G^\theta, \mathfrak{g}_i)$.

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- The moduli spaces associated to Vinberg pairs do appear naturally inside the moduli space $\mathcal{M}(G)$ of G-Higgs bundles as fixed point subvarieties for a certain action of a cyclic group.
  This is studied in joint paper with **S. Ramanan** (2019).

Vinberg θ-pairs and cyclic Higgs bundles

- $\operatorname{Aut}(G)$ acts on $\mathcal{M}(G)$: For $\alpha \in \operatorname{Aut}(G)$ and a G-Higgs bundle (E, φ) we defined

$$\alpha \cdot (E, \varphi) := (\alpha(E), \alpha(\varphi)) \text{ where } \alpha(E) = E \times_\alpha G.$$

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  Consider the homomorphism $\mu_m \to \mathrm{Aut}(G) \times \mathbb{C}^*$ defined by $\zeta \mapsto (\theta, \zeta)$. Let Γ be the image.
- Γ is isomorphic to μ_m and acts on $\mathcal{M}(G)$ by the rule

$$(E, \varphi) \mapsto (\theta(E), \zeta\theta(\varphi)).$$

- **Extension of structure group** defines a finite map

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G).$$

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- Since the action of θ depends only on the class of θ in $\mathrm{Out}(G)$, there are other subvarieties in $\mathcal{M}(G)^\Gamma$.
- Let $\mathrm{Aut}_m(G) \subset \mathrm{Aut}(G)$ the be set of elements of order m. There is a map

$$\mathcal{cl} : \mathrm{Aut}_m(G)/\sim \to \mathrm{Out}_m(G)$$

called the clique map. For an element $a \in \mathrm{Out}_m(G)$ we refer to the set $\mathcal{cl}_m^{-1}(a)$ as the **clique** defined by a.

- One has that $\mathcal{cl}_m^{-1}(a) = H^1(\mathbb{Z}/m, \mathrm{Ad}(G))$, the first **Galois cohomology** set.

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- Let $\mathcal{M}_*(G) \subset \mathcal{M}(G)$ be the **smooth locus** (stable and simple G-Higgs bundles), then one has

$$\mathcal{M}_*(G)^\Gamma \subset \bigcup_{[\theta'] \in H^1(\mathbb{Z}/m, \mathrm{Ad}(G))} \widetilde{\mathcal{M}}(G^{\theta'}, \mathfrak{g}_1').$$

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- The elements in $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$ are called **cyclic G-Higgs bundles** and have been studied in relation to various geometric aspects of moduli spaces of Higgs bundles.

- One has that $\mathcal{cl}_m^{-1}(a) = H^1(\mathbb{Z}/m, \mathrm{Ad}(G))$, the first **Galois cohomology** set.
- Let $\mathcal{M}_*(G) \subset \mathcal{M}(G)$ be the **smooth locus** (stable and simple G-Higgs bundles), then one has

$$\mathcal{M}_*(G)^\Gamma \subset \bigcup_{[\theta'] \in H^1(\mathbb{Z}/m, \mathrm{Ad}(G))} \widetilde{\mathcal{M}}(G^{\theta'}, \mathfrak{g}_1').$$

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- The moduli spaces $\mathcal{M}(G^\theta, \mathfrak{g}_i)$ for general $\mathfrak{g}_i$ in the $\mathbb{Z}/m$-grading of $\mathfrak{g}$ do also show up as fixed point in $\mathcal{M}(G)$. But now the homomorphism $\mu_m \to \mathrm{Aut}(G) \times \mathbb{C}^*$ is defined by $\zeta \mapsto (\theta, \zeta^i)$ and consider the action of the image Γ.

$\mathbb{Z}/2$-gradings and non-abelian Hodge correspondence

- If $m = 2$, θ defines the $\mathbb{Z}/2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

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- Let G^σ be the real form of G defined by the conjugation $\sigma = \theta\tau$, with τ a compact conjugation commuting with θ. Consider the G^σ**-character variety** of the fundamental group of X defined by

$$\mathcal{R}(G^\sigma) = \mathrm{Hom}(\pi_1(X), G^\sigma) /\!\!/ G^\sigma,$$

In this case one also has a **non-abelian Hodge correspondence** establishing a homeomorphism

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \cong \mathcal{R}(G^\sigma),$$

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  In this case one also has a **non-abelian Hodge correspondence** establishing a homeomorphism

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- The subvarieties $\widetilde{\mathcal{M}}(G^\theta, \mathfrak{g}_1)$ are in the fixed point locus for the action of $\mathbb{Z}/2$ on $\mathcal{M}(G)$ sending $(E, \varphi) \mapsto (\theta(E), -\theta(\varphi))$ and define **Lagrangian subvarieties** of $\mathcal{M}(G)$.

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- Let $G_0 < G$ be the centralizer of ζ; G_0 acts on each $\mathfrak{g}_i$. **Important result** due to **Vinberg** (1975): For $i \neq 0$, $\mathfrak{g}_i$ is a **prehomogeneous vector space** for G_0. This means that $\mathfrak{g}_i$ (for $i \neq 0$) has a unique open dense G_0-orbit.

$\mathbb{Z}$-gradings and Hodge bundles

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- For a $\mathbb{Z}$-grading we consider $(G_0, \mathfrak{g}_i)$-**Higgs pairs** over X. Let (E, φ) be a $(G_0, \mathfrak{g}_i)$-Higgs pair. Extending the structure group defines a G-Higgs bundle (E_G, φ), where $E_G = E \times_{G_0} G$, and we use $E(\mathfrak{g}_i) \subset E_G(\mathfrak{g})$.

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- A result of **Simpson** (1992) states that the $\mathbb{C}^*$-**fixed points** in the moduli space of G-Higgs bundles (under the action of rescaling the Higgs field) are Hodge bundles for some $\mathbb{Z}$-grading.
- Via de non-abelian Hodge correspondence, Hodge bundles correspond to holonomies of **complex variations of Hodge structure**.

Vinberg $\mathbb{C}^*$-pairs and the Toledo character

- Without loss of generality, we can consider the Vinberg $\mathbb{C}^*$-pair $(G_0, \mathfrak{g}_1)$. Let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit.

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- Let $e \in \mathfrak{g}_1$ and (h, e, f) be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. We define the **Toledo rank** of e by

$$\mathrm{rk}_T(e) = \frac{1}{2}\chi_T(h),$$

and the **Toledo rank** of $(G_0, \mathfrak{g}_1)$ by

$$\mathrm{rk}_T(G_0, \mathfrak{g}_1) = \mathrm{rk}_T(e) \quad \text{for } e \in \Omega.$$

Hodge bundles and the Toledo invariant

- Let (E, φ) be a $(G_0, \mathfrak{g}_1)$-Higgs pair and $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ be the **Toledo character** associated to $(G_0, \mathfrak{g}_1)$.
  For a **rational number** q sufficiently large $q\chi_T$ lifts to a character $\widetilde{\chi}_T : G_0 \to \mathbb{C}^*$.

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- **Theorem: Arakelov–Milnor inequality (Biquard–Collier–G–Toledo 2021)**. If (E,φ) is semistable, then

$$\tau(E,\varphi)\geq -\operatorname{rk}_T(\varphi)(2g-2),$$

  where $\operatorname{rk}_T(\varphi)=\operatorname{rk}_T(\varphi(x))$ for a generic $x\in X$.
  In particular,

$$\tau(E,\varphi)\geq -\operatorname{rk}_T(G_0,\mathfrak{g}_1)(2g-2).$$

Hodge bundles and cyclic Higgs bundles

- Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-**grading** with grading element $\zeta \in \mathfrak{g}_0$, and let $G_0 < G$ be the centralizer of ζ.

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- Let $m \geq 2$ be the smallest integer for which $\mathfrak{g}_j = 0$ for every $|j| \geq m$. Then one has a $\mathbb{Z}/m$-**grading** defined by

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m} \overline{\mathfrak{g}}_i,$$

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- Assume that the automorphism of $\mathfrak{g}$ of order m defining this $\mathbb{Z}/m$-grading lifts to an automorphism θ of G, and that $G^\theta = G_0$.
- Let X be a compact Riemann surface. We want to study $(G_0, \overline{\mathfrak{g}}_1)$-**Higgs pairs** over X. These correspond, as explained above, to θ-cyclic G-Higgs bundles.

Hodge bundles and cyclic Higgs bundles

- Let (E, φ) be a $(G_0, \overline{\mathfrak{g}}_1)$-Higgs pair over X. Notice that

$$E(\overline{\mathfrak{g}}_1) = E(\mathfrak{g}_1) \oplus E(\mathfrak{g}_{1-m}),$$

Hence we can decompose $\varphi = \varphi^+ + \varphi^-$ with

$$\varphi^+ \in H^0(X, E(\mathfrak{g}_1) \otimes K) \text{ and } \varphi^- \in H^0(X, E(\mathfrak{g}_{1-m}) \otimes K).$$

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- Consider the Toledo invariant $\tau^+ = \tau(E, \varphi^+)$ using the Toledo character χ_T^+ of $(G_0, \mathfrak{g}_1)$ determined by the grading element ζ. If $\varphi^- = 0$ one has that the semistability of (E, φ) implies the Arakelov–Milnor inequality, but actually one has the following stronger result:
  Theorem (G-González 2023). If $(E, \varphi^+ + \varphi^-)$ is semistable, then

$$\tau^+ \geq -\operatorname{rk}_T(\varphi^+)(2g-2).$$

Hodge bundles and cyclic Higgs bundles

- We can also consider the Toledo invariant $\tau^- = \tau(E, \varphi^-)$ using the Toledo character χ_T^- of $(G_0, \mathfrak{g}_{1-m})$. The grading element defining the Toledo character χ_T^- is $\zeta^- = \frac{\zeta}{1-m}$. Now, if $\varphi^+ = 0$ one has

$$\tau^- \geq -\operatorname{rk}_T(\varphi^-)(2g-2).$$

In general we can not guarantee this inequality if $\varphi^+ \neq 0 \neq \varphi^-$.

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- This happens, in particular, when $m = 2$. In this case the $\mathbb{Z}$-grading is of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the associated $\mathbb{Z}/2$-grading $\mathfrak{g} = \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$, with $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$ and $\bar{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, defines a **real form of Hermitian type**.

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- In this case, $\tau := \tau^+ = -\tau^-$ and we recover the following. **Theorem: Milnor–Wood inequality (Biquard–G–Rubio 2017)**. If $(E, \varphi^+ + \varphi^-)$ is semistable

$$-\operatorname{rk}_T(\varphi^+)(2g-2) \leq \tau \leq \operatorname{rk}_T(\varphi^-)(2g-2).$$

Hermitian real forms and cyclic Higgs bundles

- If $\mathfrak{g}$ is simple, then a real form of Hermitian type is one of the classical real Lie algebras $\mathfrak{su}(p,q)$, $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(2,n)$ or $\mathfrak{e}_6(-14)$, $\mathfrak{e}_7(-25)$ in the exceptional case.

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- Recall that $\mathcal{M}(G^\theta, \overline{\mathfrak{g}}_1)$ is homeomorphic to the character variety $\mathcal{R}(G^\sigma)$, where σ is an antiholomorphic involution of G corresponding to θ. In this case, the symmetric space defined by the quotient G^σ by its maximal compact subgroup is a **Hermitian symmetric space of the non-compact type**.

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- In this situation, the Milnor–Wood inequality given above leads to the **classical Milnor–Wood inequality**

$$|\tau| \leq 2r(g-1),$$

where r is the **rank of the symmetric space**. Here τ coincides with the original invariant defined by **Toledo** for a representation of $\pi_1(X)$ in G^σ for which the above bound is proved in general by **Burger–Iozzi–Wienhard**.

Hermitian real forms and cyclic Higgs bundles

- In this Hermitian situation the moduli space $\mathcal{M}(G^\theta, \bar{\mathfrak{g}}_1)$ for maximal Toledo invariant ($|\tau| = 2r(g-1)$) has very special **rigidity properties** (**Biquard–G–Rubio**).

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- In the maximal Toledo and tube situation, the corresponding character variety consists entirely of **discrete and faithful** representations (**Burger–Iozzi–Labourie–Wienhard** 2006): **higher Teichmüller spaces**.

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- In the maximal Toledo and tube situation, the corresponding character variety consists entirely of **discrete and faithful** representations (**Burger–Iozzi–Labourie–Wienhard** 2006): **higher Teichmüller spaces**.
- These rigidity properties generalize when the Toledo invariant is maximal to Hodge bundles (**Biquard–Collier–G–Toledo**) and to cyclic Higgs bundles obtained from Hodge bundles (**G–González**).

Vinberg θ-pairs and the Hitchin fibration

- Let $\theta \in \mathrm{Aut}(G)$ be of order m. Consider the $\mathbb{Z}/m$-grading defined by θ:
$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i,$$
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- Recall
$$\mathbb{C}[\mathfrak{g}_1]^{G^\theta} \to \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})} = \mathbb{C}[f_1, \cdots, f_r],$$
where $r = \dim \mathfrak{a} = \mathrm{rk}(G^\theta, \mathfrak{g}_1)$. Let $d_i = \deg f_i$

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- Evaluating the polynomials f_i on the Higgs field we have the **Hitchin map**:
$$h : \mathcal{M}(G^\theta, \mathfrak{g}_i) \to B(G^\theta, \mathfrak{g}_1) \cong \bigoplus_{i=1}^{r} H^0(X, K^{d_i}),$$

$m = 1$ **(Adjoint representation)**.

- h is the usual Hitchin map (Hitchin, 1987):

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- A symmetric pair version of the Fundamental Lemma in relation to this has been given by **Leslie** (2021).
- A section of the Hitchin map in this case was constructed by **G–Peón-Nieto–Ramanan** (2018).

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- There should be also a relative θ-version of the Fundamental Lemma.