Determinants of the Laplacian and random surfaces

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Determinants and path integrals

If A is a positivive self-adjoint operator on $\mathbb{R}^N$, then we have the classical gaussian integral formula

$$\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Ax,x\rangle}dx = (\det A)^{-1/2}.$$

If (M,g) is a compact connected Riemannian manifold, then one could define by analogy the functional integral by

$$Z(m) := \int \exp\left(-\frac{1}{2}\int_M (|\nabla_g\phi|^2 + m^2\phi^2)d\mathrm{Vol}_g\right)\mathcal{D}\phi$$

$$= (\det(\Delta_g + m^2))^{-1/2},$$

where Δ_g is the (positive) Laplacian on M and provided that one can give a rigourous meaning to $\det(\Delta_g + m^2)$.

Set $m = 0$ for simplicity and let

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_j \leq \ldots.$$

denote the discrete spectrum of Δ_g. For all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough, we know by Weyl's law that the spectral zeta function

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}$$

is well defined and holomorphic. The regularized determinant is then usually defined by

$$\log \det(\Delta_g) := -\zeta'(0),$$

provided one can prove an analytic extension to $s = 0$ of ζ. Practically, one performs a meromorphic continuation by noticing that for large $\mathrm{Re}(s)$ we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}(\mathrm{Tr}(e^{-t\Delta_g}) - 1)dt,$$

and use the short times asymptotics of the heat kernel: uniformly in $x \in M$, one has as $t \to 0$

$$e^{-t\Delta_g}(x,x) = t^{-d/2}\left(\sum_{j=0}^{p} a_j(x)t^j\right) + O(t^{p-d/2}),$$

where a_j are smooth functions on M and $d = \dim(M)$. From a statistical physics point of view, $Z(m)$ is a partition function and

$$\frac{\log(Z(m))}{\mathrm{Vol}(M)}$$

is an intensive physical quantity. A relevant problem is to understand the thermodynamical limit when $\mathrm{Vol}(M) \to +\infty$, in particular we want to discuss the behaviour of

$$\frac{\log \det(\Delta_g)}{\mathrm{Vol}(M)}$$

for sequences of manifolds with $\mathrm{Vol}(M) \to +\infty$.

An example on the torus

Let Δ_L be the flat Laplacian on the torus $\mathbb{T}_L = \mathbb{R}^2/(\mathbb{Z} \oplus iL\mathbb{Z})$, then we have a classical identity [1]

$$\det(\Delta_L) = L^2 |\eta(iL)|^4,$$

where $\eta(\tau)$ is the Dedekind Eta modular form, defined for all $\mathrm{Im}(\tau) > 0$ by

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2i\pi n\tau}\right).$$

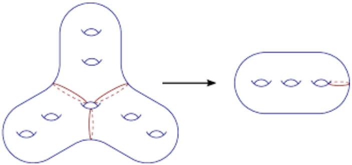
The proof follows directly from Poisson summation formula and Kronecker limit formula for Eisenstein series. As a consequence,

$$\lim_{L \to +\infty} \frac{\log \det(\Delta_L)}{\mathrm{Vol}(\mathbb{T}_L)} = \frac{-\pi}{12}.$$

[1] See for example Kierlanczyk's PhD Thesis, MIT 1986.

Large cyclic covers

Let $M = \widetilde{M}/\Gamma$ be with $H^1(M, \mathbb{Z})$ infinite. Consider surjective homomorphisms $\pi_N : H^1(M, \mathbb{Z}) \to \mathbb{Z}_N$. Then $\Gamma_N := \mathrm{Ker}(\pi_N)$ defines an N-cyclic cover of M via $M_N := \widetilde{M}/\Gamma_N$, and $\mathrm{Vol}(M_N) = N\mathrm{Vol}(M)$.



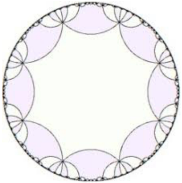
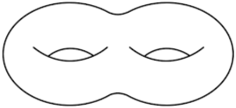
Thm (N.V. Dang 2023) For all $m > 0$, we have

$$\lim_{N \to +\infty} \frac{\log(\det(\Delta_N + m^2))}{\mathrm{Vol}(M_N)} = C_M,$$

where C_M depends on M. Moreover, the limit still exists if $m = 0$ and M is hyperbolic or a Torus.

Compact congruence covers of surfaces

- Let $\mathbb{H}^2$ be the usual real hyperbolic plane with curvature -1 and Γ a non-elementary discrete group of isometries.
- We assume that Γ is co-compact (no elliptic elements) so that $X = \Gamma\backslash\mathbb{H}^2$ is a compact hyperbolic surface.

- If $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is an arithmetic co-compact group, each γ has entries in the ring of integers $\mathcal{O}$ of a totally real number field and given a prime ideal $\mathcal{P} \subset \mathcal{O}$, one can define congruence subgroups $\Gamma(\mathcal{P}) = \{\gamma \in \Gamma \ \ \gamma \equiv I \mod \mathcal{P}\}$.

Thm (folklore) Let $\Delta_{\mathcal{P}}$ be the hyperbolic Laplacian on $X_{\mathcal{P}} := \Gamma(\mathcal{P})\backslash\mathbb{H}^2$, then there exists $E > 0$, universal, such that

$$\lim_{N(\mathcal{P})\to+\infty} \frac{\log\det(\Delta_{\mathcal{P}})}{\mathrm{Vol}(X_{\mathcal{P}})} = E.$$

- The proof follows from a uniform spectral gap result of Sarnak-Xue (1991) combined with the fact that the injectivity radius of $X_{\mathcal{P}}$ goes to infinity. General ideas from Bergeron-Venkatesh (2013) and the 7 Samurais (2017) [2] give the result.
- In higher dimension, similar results hold for the Laplacian on functions, however determinants of Laplacian on k-forms are much more subtle to analyze.

[2] See "On the growth of L^2-invariants for sequences of lattices in Lie groups" by ABBGNRS.

The Bergeron-Venkatesh conjecture (2013)

Let $M = \Gamma\backslash\mathbb{H}^3$ be a compact connected hyperbolic 3-manifold. Consider a decreasing sequence of finite index subgroups $\Gamma_n \subset \Gamma$ such that $\cap_n \Gamma_n = \{Id\}$. On each cover $M_n := \Gamma_n\backslash\mathbb{H}^3$, denote by $\Delta_n^{(k)}$ the Hodge-Laplacian acting on differential k-forms.

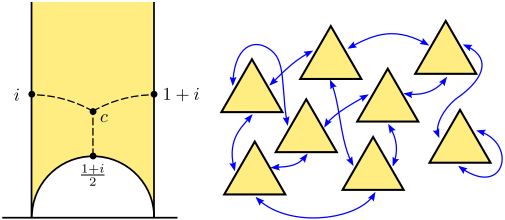
Conjecture:

$$\lim_{n\to\infty} \frac{1}{2}\sum_{k=0}^{3}(-1)^{k+1}k\frac{\log\det(\Delta_n^{(k)})}{\mathrm{Vol}(M_n)} = \frac{-1}{6\pi}.$$

- The *LHS* is called the analytic torsion and is related via Cheeger-Müller's Theorem to the growth of *torsion homology* in the covers M_n.
- The analog of the above result is known when the Laplacians are twisted by certain representations of Γ which guarantee a uniform spectral gap for all Laplacians.

Models of random hyperbolic surfaces

Brooks-Makover: (Random Belyi surfaces) Glue ideal hyperbolic triangles according to random 3-regular graphs (Bollobás) with $2n$ vertices. One obtains a random (non-compact) surface X_n with area $2\pi n$.



Random glueing of triangles

One can then compactify X_n by cutting cusps, filling conformally with discs and then getting a hyperbolic metric on the compactified surface X_n^C.

Random covers: An n-sheeted connected riemannian cover

$$\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}^2 \to X = \Gamma\backslash\mathbb{H}^2$$

corresponds to a subgroup $\widetilde{\Gamma} \subset \Gamma$ of index n. On the other hand, we have a bijection

$$\left\{\widetilde{\Gamma} \subset \Gamma \ : \ [\Gamma : \widetilde{\Gamma}] = n\right\} \leftrightarrow \{\text{transitive hom. } \phi : \Gamma \to \mathcal{S}_n\},$$

where $\mathcal{S}_n$ is the symmetric group of permutations of $\{1, \dots, n\}$. We consider the uniform probability measure on the "discrete Teichmuller space" $\mathrm{Hom}(\Gamma, \mathcal{S}_n)$. It is known that

$$|\mathrm{Hom}(\Gamma, \mathcal{S}_n)| = (n!)^{2g-1} \sum_{\lambda \text{ irred}} \frac{1}{d_\lambda^{2g-2}} = (n!)^{2g-1}(2 + O(n^{-2})).$$

Γ acts on $\{1, \dots, n\}$ via $\phi_n \in \mathrm{Hom}(\Gamma, \mathcal{S}_n)$, let $S \subset \{1, \dots, n\}$ be a set of representative of the orbits and set

$$X_n = \bigsqcup_{i \in S} \Gamma_i\backslash\mathbb{H}^2,$$

where $\Gamma_i := \mathrm{Stab}_\Gamma(i) \subset \Gamma$ is a finite index subgroup of Γ.

- X_n is an n-sheeted cover of X, possibly non-connected, in general non-Galois.
- a.a.s. as $n \to +\infty$, X_n is connected.
- $\mathrm{Vol}(X_n) = n\mathrm{Vol}(X)$.

Smooth Weil-Petersson model:

Let $\mathcal{M}_g$ be the moduli space of compact hyperbolic surfaces with genus g, up to isometry, then $\mathcal{M}_g$ is endowed with a smooth volume form (Weil-Petersson volume). Moreover, the volume of the moduli space is finite, which by normalization gives a probability measure. Recall that by Gauss-Bonnet, if $X \in \mathcal{M}_g$, then we have $\mathrm{Vol}(X) = 4\pi(g-1)$.

Determinant of the Laplacian on typical surfaces

Thm 1 *(N. 2023)* For all previous models of random surfaces, there exists a universal $E > 0$ s.t. for all $\epsilon > 0$ we have as $\mathrm{Vol}(X) \to +\infty$,

$$\mathbb{P}\left(\frac{\log\det(\Delta_X)}{\mathrm{Vol}(X)} \in [E-\epsilon, E+\epsilon]\right) \to 1$$

Thm 2 *(N., Wu and He 2023)*

- In the random cover model, we have for all $\beta > 0$,

$$\lim_{n\to+\infty} \mathbb{E}_n\left(\frac{|\log\det\Delta_{X_n}|^\beta}{(\mathrm{Vol}(X_n))^\beta}\right) = E^\beta.$$

- In the Weil-Petersson model, we have for all $0 < \beta < 2$,

$$\lim_{g\to+\infty} \mathbb{E}_g\left(\frac{|\log\det\Delta_X|^\beta}{(\mathrm{Vol}(X))^\beta}\right) = E^\beta,$$

while for $\beta \geq 2$, we have $\mathbb{E}_g(|\log\det\Delta_X|^\beta) = +\infty$.

Ideas of proofs 1

If M_n is a sequence of compact hyperbolic manifolds with $M_n = \Gamma_n \backslash \mathbb{H}^d$, and $\mathrm{Vol}(M_n) \to +\infty$, we say that (M_n) converges in the sense of Benjamini-Schramm to $\mathbb{H}^d$ iff we have for all $R > 0$,

$$\lim_{n \to +\infty} \frac{\mathrm{Vol}((M_n)_{<R})}{\mathrm{Vol}(M_n)} = 0,$$

where $(M_n)_{<R}$ is the set of $x \in M_n$ such that $\mathrm{Inj}(x) < R$. From the 7 Samurais, one can extract the following fact: assume that (M_n) is such that

1. (M_n) converges in the sense of Benjamini-Schramm to $\mathbb{H}^d$.
2. (M_n) has a uniform spectral gap i.e. there exists $\beta > 0$ such that $\lambda_1(M_n) \geq \beta$ for all n.

Then

$$\lim_{n \to \infty} \frac{\log \det \Delta_n}{\mathrm{Vol}(M_n)}$$

exists and is a universal constant.

Ideas of proofs 2

The proof of Theorem 1 uses similar ideas combined with the fact that for all 3 models, surfaces with small spectral gap and lots of closed geodesics with bounded length have small probability. More precisely, we have

- (1) There exists $C > 0$ such that as $\mathrm{Vol}(X) \to +\infty$,

$$\mathbb{P}\left(\lambda_1(X) \geq C\right) \to 1.$$

- (2) Set $N_X(L) : \#\{\gamma \in \mathcal{P}_X,\ k \in \mathbb{N}\ :\ k\ell(\gamma) \leq L\}$. For all $R > 0$, for all $1 > \alpha > 0$, as $\mathrm{Vol}(X) \to +\infty$

$$\mathbb{P}\left(N_X(R) \leq (\mathrm{Vol}(X))^{\alpha}\right) \to 1.$$

Here $\mathcal{P}_X$ denotes the set of primitive closed geodesics on the surface X.

Ideas of proofs 3

- Non effective uniform spectral gaps are due to Brooks-Makeover (2001) for their model. An effective bound is due to Petri in his Thesis (2017).
- Effective uniform spectral gaps are due to Magee-Naud-Puder (2022) in the random cover model.
- Effective Uniform spectral gaps in the Weil-Petersson model are due (with increasing quality) to Mirzakhani (2013), Wu-Xue/Wright-Lipnowski (2022), Anantharaman-Monk (2023).
- The bound (2) follows from Petri (2017) in the Brooks-Makeover case, Petri-Mirzakhani (2019) in the Weil-Petersson model, Magee-Puder (2023) in the random cover case.

Ideas of proofs 4

Proof of Theorem 2 requires more work and uses Theorem 1 plus some good enough a priori bounds of the determinant on moduli space.

$$|\log \det(\Delta_X)| \ll \mathrm{Vol}(X)\left(1 + |\log(\lambda_*(X))| + \frac{1}{\ell_0(X)}\right)$$

$$+\frac{\log^+ \ell_0^{-1}(X)}{\ell_0(X)} N_X(1),$$

where $\ell_0(X)$ is the shortest closed geodesic on X and $\lambda_*(X) = \min\{\lambda_1(X), 1/4\}$. This bound blows up when X has small closed geodesics and/or small spectral gap.

- A key observation by Mirzakhani is that

$$\int_{\mathcal{M}_g} \frac{1}{\ell_0(X)} dm_{WP}(X) \leq CV_g,$$

where $V_g = m_{WP}(\mathcal{M}_g)$ and $C > 0$ is uniform in g.

- Similarly, from Cheeger's inequality it follows that for all $0 < \alpha < 1$,

$$\int_{\mathcal{M}_g} \frac{1}{(\lambda_1(X))^\alpha} dm_{WP}(X) \leq CV_g.$$

Combining these estimates (and more work) yields Theorem 2 in the Weil-Petersson case. In the random cover case, $\ell_0(X)$ is uniformly bounded from below so only $\lambda_1(X)$ is a real obstacle. A combinatorial argument plus a min-max type estimate shows that for random covers of degree n, $\lambda_1(X_n) > C/n^{3/2}$, whenever X_n is connected.

Concluding remarks

- The analysis of moments of $\log \det(\Delta)$ require extra work in the Brooks-Makover model.
- In particular we need to show that for large n,

$$\mathbb{E}_n \left(\frac{1}{(\ell_0(X_n^C))^\alpha} \right)$$

is uniformly bounded for some $\alpha > 0$.

- This will require an "effective conformal compactification", see the paper of Mangoubi. [3]

[3] Dan Mangoubi. Conformal extension of metrics of negative curvature. J. Anal. Math., 91 (2003).