Séminaire de géométrie spectrale
Nalini Anantharaman
Collège de France
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Spectral projectors on hyperbolic surfaces

Jean-Philippe Anker (Université d'Orléans)
<anker@univ-orleans.fr>

Ongoing collaboration with

Pierre Germain (Imperial College)
Tristan Léger (NYU & Princeton University)

Content

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The problem

$$X \begin{cases} \text{Riemannian manifold (complete)} \\ \text{dimension } n \end{cases} \qquad \text{e.g. } X = \mathbb{R}^n, \mathbb{T}^n, \mathbb{S}^n, \mathbb{H}^n$$

Δ Laplacian, $D = \sqrt{-\Delta}$

$P_{\lambda,\eta} = \mathbf{1}_{[\lambda-\eta,\lambda+\eta]}(D)$ projector in a spectral window

Problem

$$\text{Estimate } \| P_{\lambda,\eta} \|_{L^2 \to L^p} \text{ for } \begin{cases} p > 2 \\ \text{large frequency } \lambda \geq 0 \\ \text{small width } \eta > 0 \end{cases}$$

The problem (continued)

Remark 1 (TT^* trick)

$$\| P_{\lambda,\eta} \|_{L^2 \to L^p} = \| \overbrace{P^*_{\lambda,\eta}}^{P_{\lambda,\eta}} \|_{L^{p'} \to L^2} = \| \overbrace{P_{\lambda,\eta} P^*_{\lambda,\eta}}^{P_{\lambda,\eta}} \|^{1/2}_{L^{p'} \to L^p}$$

As usual $2 < p \leq \infty$ and $1 \leq p' < 2$ are dual indices : $\frac{1}{p} + \frac{1}{p'} = 1$

Remark 2 (smooth version)

We can replace $\mathbf{1}_{[\lambda-\eta,\lambda+\eta]}(D)$ by $\psi(\frac{D-\lambda}{\eta})$
where ψ is a smooth bump function

Related problem

Estimate $\| dP_\lambda \|_{L^{p'} \to L^p}$ where $dP_\lambda = \delta_\lambda(D) = \lim_{\eta \to 0} \frac{1}{2\eta} P_{\lambda,\eta}$

Comment.

$dP_\lambda \quad \leftrightsquigarrow \quad$ eigenfunctions

$P_{\lambda,\eta} \quad \leftrightsquigarrow \quad$ quasimodes

Stein-Tomas restriction theorem

- The Fourier transform
$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x,\xi\rangle}\, dx$$
of a function $f \in L^1(\mathbb{R}^n)$ is a continuous function (vanishing at infinity) and thus it makes sense to restrict it to the unit sphere $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n \,|\, \|\xi\| = 1\}$
- The Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{R}^n)$ runs through $L^2(\mathbb{R}^n)$ and thus it makes no sense to restrict it to $\mathbb{S}^{n-1}$

Nevertheless

Stein-Tomas restriction theorem

Let $p \geq p_{\mathrm{ST}} = 2\,\frac{n+1}{n-1}$. Then
$$\big\| \widehat{f}\,|_{\mathbb{S}^{n-1}} \big\|_{L^2} \lesssim \|f\|_{L^{p'}} \qquad \forall\, f \in \mathcal{S}(\mathbb{R}^n)$$

Stein-Tomas restriction theorem (continued)

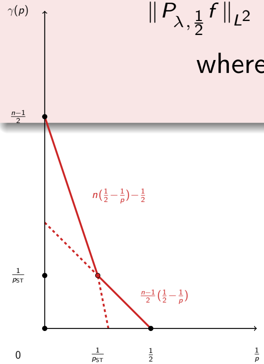
By rescaling and interpolation, one gets the following sharp result

Corollary (restriction to an annulus of width 1)

Let $p>2$. Then there exists $C>0$ such that

$$\underbrace{\| \mathbf{1}_{\lambda-\frac{1}{2}\leq\|\xi\|\leq\lambda+\frac{1}{2}} \widehat{f} \|_{L^2}}_{\| P_{\lambda,\frac{1}{2}} f \|_{L^2}} \leq C\,\lambda^{\gamma(p)} \|f\|_{L^{p'}} \qquad \forall\, f\in\mathcal{S}(\mathbb{R}^n)$$

where
$$\gamma(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) & \text{if } 2<p\leq p_{\mathrm{ST}} \\ n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2} & \text{if } p\geq p_{\mathrm{ST}} \end{cases}$$



Further development :
Strichartz estimates

Sogge's result and its consequences

Sogge's theorem

Let X be a compact Riemannian manifold. Then there exists $\eta_0 > 0$ such that

$$\| P_{\lambda,\eta_0} \|_{L^2 \to L^p} \approx \lambda^{\gamma(p)}$$

for $p > 2$ and λ large

Remark

This result is local and holds true for X with *bounded geometry* :

- injectivity radius bounded from below
- uniform local geometry
  in all small balls $B(x, r_0)$ of fixed radius $r_0 > 0$

Corollary

Let X be a Riemannian manifold with bounded geometry. Then there exists $\eta_0 > 0$ such that

$$\max_{\lambda-\eta_0\leq\mu\leq\lambda+\eta_0} \left\| P_{\mu,\eta} \right\|_{L^2\to L^p} \gtrsim \lambda^{\gamma(p)}\,\eta^{\frac{1}{2}}$$

for $p > 2$, λ large and η small

Let say $\quad \left\| P_{\lambda,\eta} \right\|_{L^2\to L^p} \gtrsim \lambda^{\gamma(p)}\,\eta^{\frac{1}{2}}$

Back to problem

- Behavior in λ of $\left\| P_{\lambda,\eta} \right\|_{L^2\to L^p}$ should be always $\lambda^{\gamma(p)}$
- Behavior in η depends on the global geometry of the manifold

Examples

- Sharp result for $\mathbb{R}^n$:

$$\| P_{\lambda,\eta} \|_{L^2 \to L^p} \approx \lambda^{\gamma(p)} \times \begin{cases} \eta^{\frac{n+1}{2}(\frac{1}{2}-\frac{1}{p})} & \text{if } 2 < p \leq p_{\mathrm{ST}} \\ \eta^{\frac{1}{2}} & \text{if } p \geq p_{\mathrm{ST}} \end{cases}$$

- Conjecture for $\mathbb{T}^n$: under the assumption $\eta > \lambda^{-1}$,

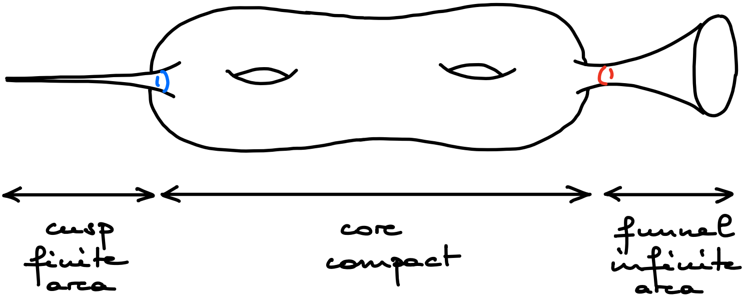
$$\| P_{\lambda,\eta} \|_{L^2 \to L^p} \approx \lambda^{\gamma(p)} \times \begin{cases} \eta^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})} & \text{if } 2 < p \leq p_{\mathrm{ST}} \\ \eta^{\frac{1}{2}} & \text{if } p \geq p_{\mathrm{ST}} \end{cases}$$

Partial results [Bourgain, Demeter, Germain, Myerson] ([1])

[1] See Germain's survey [arXiv:2306.16981]

Hyperbolic surfaces [2]

$$X \begin{cases} \text{Riemannian manifold (complete, connected)} \\ \text{dimension } \; n = 2 \\ \text{curvature } \; -1 \end{cases}$$



[2] Borthwick's book, 2016

Hyperbolic surfaces (continued)

Example (universal cover)

Hyperbolic plane $\mathbb{H} = \mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

Riemannian metric $ds^2 = \frac{d|z|^2}{(\operatorname{Im} z)^2}$

Isometry group $\operatorname{Isom}(\mathbb{H}) = \underbrace{\operatorname{Isom}^+(\mathbb{H})}_{G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{PSL}(2,\mathbb{R})/\{\pm \mathrm{Id}\}} \sqcup \operatorname{Isom}^-(\mathbb{H})$

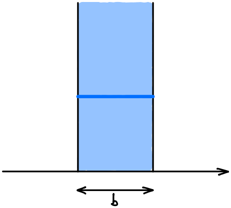
Other definition of hyperbolic surfaces

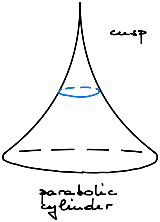
$X = \Gamma \backslash \mathbb{H}$, where Γ is a discrete torsion free subgroup of G

$+$ finiteness assumption

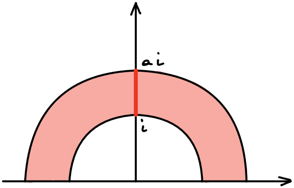
Further examples

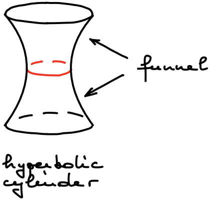
- Parabolic cylinder : $\Gamma = \{z \mapsto z + nb \mid n \in \mathbb{Z}\}$ with $b > 0$





- Hyperbolic cylinder : $\Gamma = \{z \mapsto a^n z \mid n \in \mathbb{Z}\}$ with $a > 1$





- Modular surface : $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ not torsion free

Critical exponent

Equivalent definitions of the critical exponent δ of Γ

- $\delta = \limsup_{R\to+\infty} \frac{1}{R} \log \overbrace{\left|\{\gamma\in\Gamma \mid d(x,\gamma.y)\leq R\}\right|}^{\text{counting function}}$
- $\delta = \inf\{s>0 \mid \underbrace{\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma.y)}}_{\text{Poincaré series}} < \infty\}$

Remarks

- Both definitions are independent of $x, y \in \mathbb{H}$
- $0 \leq \delta \leq 1$
- $\delta = \begin{cases} 0 & \text{for the hyperbolic cylinder} \\ \frac{1}{2} & \text{for the parabolic cylinder} \\ 1 & \text{for the modular surface} \end{cases}$

Results

Proposition [A-Germain-Léger 2023]

If X has cusps, then $\|P_{\lambda,\eta}\|_{L^2\to L^p} = \infty$

Theorem [A-Germain-Léger 2023]

Assume that X has funnels (infinite area) and no cusps

- Optimal upper bound when $0 \leq \delta < \frac{1}{2}$:

$$\|P_{\lambda,\eta}\|_{L^2\to L^p} \lesssim \lambda^{\gamma(p)}\,\eta^{\frac{1}{2}}$$

- Upper bound when $\frac{1}{2} \leq \delta < 1$: for every $\varepsilon > 0$ and $N > 0$,

$$\|P_{\lambda,\eta}\|_{L^2\to L^p} \lesssim \lambda^{\gamma(p)+\varepsilon}\,\eta^{\frac{1}{2}-\varepsilon}$$

under the condition $\eta > \lambda^{-N}$

Remarks

- In dimension $n=2$, $p_{\mathrm{ST}}=6$ and
$$\gamma(p)=\begin{cases}\frac{1}{4}-\frac{1}{2p} & \text{if } 2<p\leq 6\\ \frac{1}{2}-\frac{2}{p} & \text{if } p\geq 6\end{cases}$$
- Replace $D=\sqrt{-\Delta}$ by $D=\sqrt{-\Delta-\frac{1}{4}}$
- The first part of the theorem holds true more generally

  for locally symmetric spaces $\begin{cases}\text{rank } 1\\ \text{convex cocompact}\end{cases}$

  and for $0\leq\delta<\rho$ ($\Rightarrow$ infinite volume).

  Moreover, in this case, $\|dP_\lambda\|_{L^{p'}\to L^p}\lesssim\lambda^{2\gamma(p)}$

Basic tool

Spherical Fourier transform on $\mathbb{H}$

There is a Fourier transform on $\mathbb{H}$ and an inverse transform, which reduce to

$$\mathcal{F}f(\xi) = \int_{\mathbb{H}} f(x)\,\varphi_\xi(x)\,dx = 2\pi \int_0^\infty f(r)\,\varphi_\xi(r)\,(\sinh r)\,dr$$

and

$$f(r) = \tfrac{1}{2\pi} \int_0^\infty \mathcal{F}f(\xi)\,\varphi_\xi(r)\,(\tanh \pi\xi)\,\xi\,d\xi$$
$$= -\tfrac{1}{2^{3/2}\pi^2} \int_r^\infty \tfrac{\partial}{\partial s}\,\widehat{\mathcal{F}f}(s)\,\tfrac{ds}{\sqrt{\cosh s - \cosh r}}$$

for radial functions $f(x) = f(r)$, where $r = d(x,i)$. These formulae involve the spherical functions $\varphi_\xi(x) = \varphi_\xi(r)$, which can be expressed in terms of special functions (Legendre or hypergeometric)

Remark. Analogy with the Fourier transform of radial functions on $\mathbb{R}^n$ (Hankel transform), which involves modified Bessel functions

Another helpful tool

Kunze-Stein phenomenon on G [Kunze-Stein 1964]

$$L^2(G) * L^{2-\varepsilon}(G) \subset L^2(G) \qquad \forall\, 0 < \varepsilon \leq 1$$

The right convolution by a radial kernel $\mathcal{K}$ on $\mathbb{H}$ satisfies actually

Kunze-Stein phenomenon on $\mathbb{H}$ [Herz/Stein 1970]

$$\| f * \mathcal{K} \|_{L^2} \lesssim \| f \|_{L^2} \int_0^\infty |\mathcal{K}(r)|\, e^{\frac{r}{2}}\, r\, dr$$

The same operator satisfies

Kunze-Stein phenomenon on X [Fotiadis-Mandouvalos-Marias 2018/Zhang 2019]

Assume that $0 \leq \delta < \frac{1}{2}$ and let $0 < \varepsilon < \frac{1}{2} - \delta$. Then, for every $p > 2$,

$$\| f * \mathcal{K} \|_{L^p} \lesssim \| f \|_{L^{p'}} \left[\int_0^\infty |\mathcal{K}(r)\, e^{(\delta+\varepsilon) r}|^{\frac{p}{2}}\, e^{(\frac{1}{2}-\delta-\varepsilon) r}\, r\, dr \right]^{\frac{2}{p}}$$

Idea of proof when $0 \le \delta < \frac{1}{2}$

- Use the inverse spherical Fourier transform to express and estimate the kernel on $\mathbb{H}$

$$p_{\lambda,\eta}(x,y) = C \int_0^\infty \left[\psi\left(\tfrac{\xi-\lambda}{\eta}\right) + \psi\left(\tfrac{\xi+\lambda}{\eta}\right)\right] \varphi_\xi(r)\,(\tanh \pi\xi)\,\xi\,d\xi$$
$$= C\eta \int_r^\infty \tfrac{\partial}{\partial s}\left[\cos(\lambda s)\,\widehat{\psi}(\eta s)\right] \frac{ds}{\sqrt{\cosh s - \cosh r}}$$

where $r = d(x,y)$ and ψ is an even Schwartz function whose Fourier transform has compact support

$$\implies \quad |p_{\lambda,\eta}(x,y)| \lesssim \begin{cases} \lambda\,\eta & \text{for small } r = d(x,y) \\ \lambda^{\frac{1}{2}}\,\eta\, e^{-\frac{r}{2}} & \text{for large } r = d(x,y) \end{cases}$$

- Estimate the kernel on $X = \Gamma \backslash \mathbb{H}$

$$p^\Gamma_{\lambda,\eta}(x,y) = \sum_{\gamma\in\Gamma} p_{\lambda,\eta}(\gamma.x, y) = \sum_{\gamma\in\Gamma} p_{\lambda,\eta}(x, \gamma.y)$$

- Estimate related kernels
- Use interpolation and/or the Kunze-Stein phenomenon on X

Idea of proof when $\frac{1}{2} \leq \delta < 1$

- **Decompositions.** Given bump functions $\psi \in C_c^\infty(\mathbb{R})$ and $\theta \in \mathcal{S}(\mathbb{R})$ such that $\theta > 0$ and supp $\widehat{\theta}$ is compact, write first

$$\psi\left(\tfrac{D-\lambda}{\eta}\right) = \theta(D-\lambda)^2 \int_{-\infty}^{+\infty} Z(t)\, e^{itD^2}\, dt$$

in terms of the Schrödinger group e^{itD^2}, where $Z = Z_{\lambda,\eta}$ denotes the Fourier transform of

$$\tau \longmapsto \begin{cases} \frac{1}{2\pi} \frac{\psi\left(\frac{\sqrt{\tau}-\lambda}{\eta}\right)}{\theta(\sqrt{\tau}-\lambda)^2} & \text{if } \tau > 0 \\ 0 & \text{otherwise} \end{cases}$$

Given a smooth partition of unity $1 = \sum_{j=0}^m \chi_j$ corresponding to the decomposition $X = \underbrace{X_0}_{\text{core}} \cup \underbrace{\left(\bigcup_{j=1}^m X_j\right)}_{\text{funnels}}$, split up next

$$\psi\left(\tfrac{D-\lambda}{\eta}\right) = \theta(D-\lambda) \sum_j \chi_j \int_{-\infty}^{+\infty} Z(t)\, \theta(D-\lambda)\, e^{itD^2}\, dt$$

Idea of proof when $\frac{1}{2} \leq \delta < 1$ (continued)

- **Estimates in the core.** Main tools
  - Sogge's theorem
  - Resolvent estimates [Bourgain-Dyatlov 2018] **in dim n=2**
    $$\|\chi_0(D^2-\lambda^2 \pm i0)^{-1}\chi_0\|_{L^2\to L^2} \lesssim \lambda^{-1+2\varepsilon} \quad \text{for } \lambda \text{ large}$$
  - Kato's local L^2 smoothing theorem yields
    $$\|\chi_0\,\theta(D-\lambda)\,e^{itD^2}f\|_{L^2_t L^2_x} \lesssim \lambda^{-\frac{1}{2}+\varepsilon}\|f\|_{L^2}$$
- **Estimates in the funnels.** Tools from the case $0 \leq \delta < \frac{1}{2}$
  - improved Strichartz estimates for $\theta(D-\lambda)\,e^{itD^2}$:
    $$\|\theta(D-\lambda)\,e^{itD^2}f\|_{L^q_t(L^p_x)} \lesssim \lambda^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}\|f\|_{L^2} \quad (p>2,\ q\geq 2)$$
  - global L^p smoothing estimate for e^{itD^2}:
    $$\|D^{\frac{1}{2}-\gamma(p)}\,e^{itD^2}f\|_{L^p_x(L^2_t)} \lesssim \|f\|_{L^2} \quad (p>2)$$
  - commutator estimates
- Piece results together (method goes back to Staffilani-Tataru)

Low frequency estimate

Theorem [A-Germain-Léger 2023]

Assume that

- X has funnels (infinite area) and no cusps
- $0 \leq \delta < \frac{1}{2}$

Then

$$\|P_{\lambda,\eta}\|_{L^2 \to L^p} \lesssim (\lambda+\eta)\,\eta^{\frac{1}{2}}$$

for $p > 2$, $0 \leq \lambda < 1$ and $0 < \eta < 1$

Remark

Again this result holds true more generally

for locally symmetric spaces $\begin{cases} \text{rank } 1 \\ \text{convex cocompact} \end{cases}$

and for $0 \leq \delta < \rho$ ($\Rightarrow$ infinite volume).

Moreover, in this case, $\|dP_\lambda\|_{L^{p'} \to L^p} \lesssim \lambda^2$

Some references

- J.-Ph. Anker, P. Germain & T. Léger : *Spectral projectors on hyperbolic surfaces*, preprint [arXiv:2306.12827]
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- P. Germain : *L^2 to L^p bounds for spectral projectors on thin intervals in Riemannian manifolds*, preprint [arXiv:2306.16981]
- P. Germain & T. Léger : *Spectral projectors, resolvent, and Fourier restriction on the hyperbolic space*, J. Funct. Anal. 285 (2023), no. 2, Paper No. 109918