Poincaré Series and Convex Bodies on Flat Tori.

Yannick Bonthonneau, **Nguyen Viet Dang**, Matthieu Léautaud, Gabriel Rivière

Why interested in Poincaré series ?

Naive **misunderstanding** : Poincaré series could give crystalline measures.

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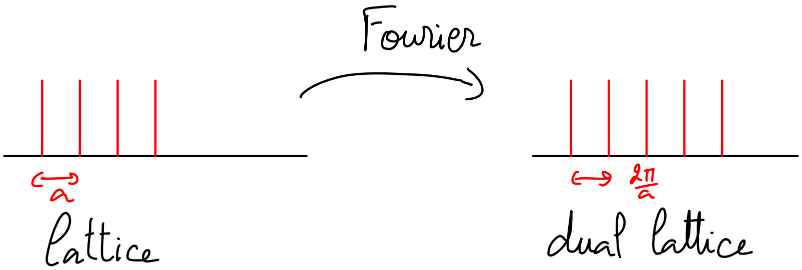
Naive **misunderstanding** : Poincaré series could give crystalline measures.
What are crystalline measures ?

The Dirac comb.

Sum of exponentials converges in $\mathcal{S}'(\mathbb{R})$:

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{iks} = \sum_{k\in\mathbb{Z}} \delta(s-2\pi k).$$



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In other words, Dirac comb is a periodic measure supported on lattice $\mathbb{Z}a$:

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Fourier transform $\widehat{\mu}$ supported by dual lattice by Poisson summation.

Meyer

2016

Guinand

1959

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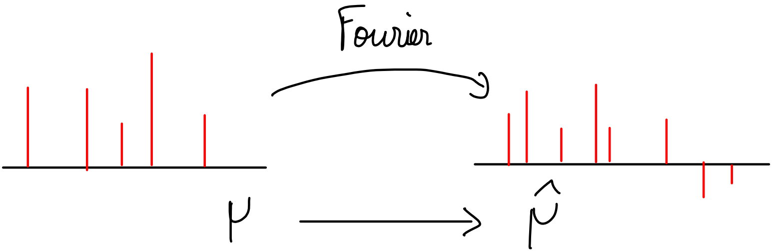
$\mu'(x) = \sum_{k\in\mathbb{Z}} \delta^{(1)}(x - ka), a > 0$ is a crystalline distribution.



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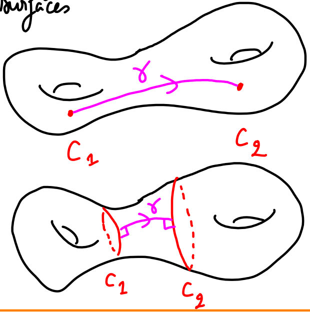
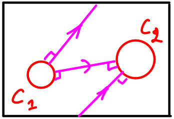
Challenge: find exotic crystalline measures and distributions.
Results by Kahane–Mandelbrojt, Guinand, Lev–Olevskii, Meyer, Sarnak–Kurasov, Viazovska–Radchenko. Look for crystalline measures carried by length spectras.

Poincaré series for amateurs in pictures.

Negatively curved surfaces

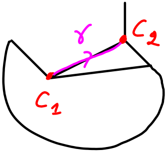


Convex bodies in tori



$\mathbb{T}^d$

Graphs

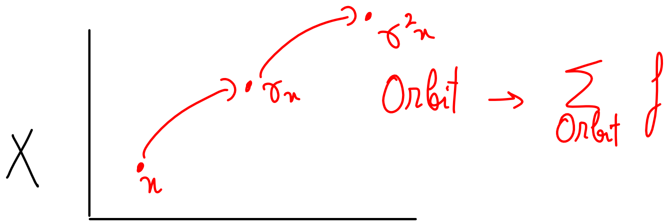


Dirichlet Series

$$\eta(s) = \sum_{\gamma : C_1 \to C_2} e^{-s\ell(\gamma)}$$

Poincaré series by and for serious people.

F. Paulin: Regards croisé sur les séries de Poincaré et leurs applications : Group Γ acts on X, subgroup $\Gamma_0 \subset \Gamma$, f Γ_0-invariant on X :



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$$\boxed{P(x) = \sum_{\gamma \in \Gamma/\Gamma_0} f(\gamma x).}$$

Spectral interpretation: trace formula

Trace formula principle: relate spectrum to integral geometry.

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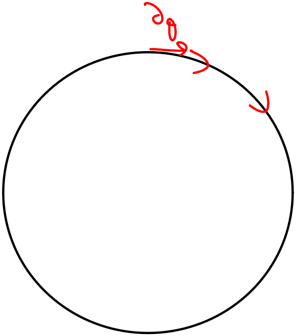
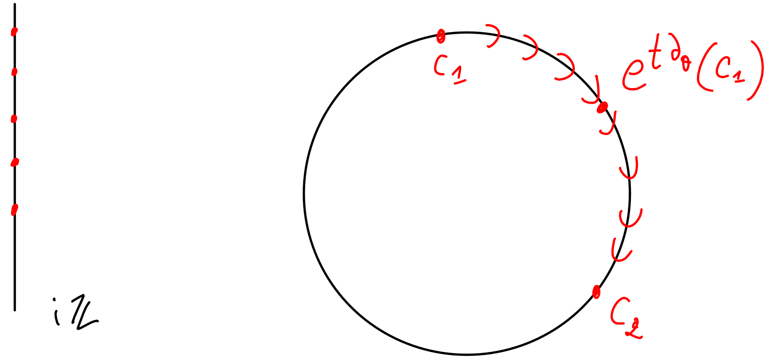


Spectral interpretation: trace formula

Trace formula principle: relate spectrum to integral geometry. Vector field ∂_θ generates flow $e^{-t\partial_\theta}$ on $\mathbb{S}^1$. Spectrum $\sigma(\partial_\theta) = i\mathbb{Z}$ with eigenfunctions $(e^{ik\theta})_{k\in\mathbb{Z}}$.



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$$\int_{\mathbb{S}^1} \left(e^{t\partial_\theta *}\delta_{c_1}\right) \wedge \delta_{c_2} d\theta = \int_{\mathbb{S}^1} \delta(t+\theta-c_1)\delta(\theta-c_2)d\theta$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \delta(t+\theta-c_1+2\pi n)\delta(\theta-c_2)d\theta$$

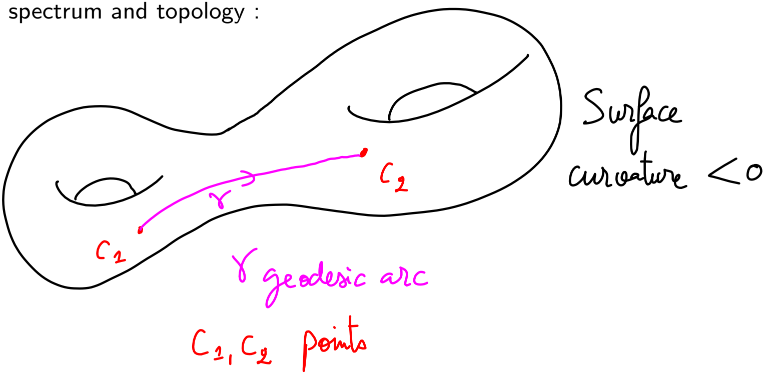
$$= \underbrace{\sum_{n\in\mathbb{Z}} \delta(t+2\pi n+c_2-c_1)}_{\text{integral geometric side}} = \underbrace{\sum_{\mathbb{Z}} e^{itn} \left(\left\langle \delta_{c_1}, e^{in.}\right\rangle \left\langle \delta_{c_2}, e^{-in.}\right\rangle\right)}_{\text{spectral side}}.$$

First result.

G. Rivière

From spectra to geometry.

Relates length spectrum and topology :



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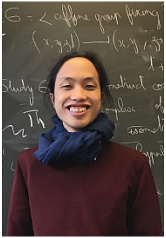
Theorem (D.–Rivière 2020)

ζ holomorphic when $Re(s) >> 1$ has meromorphic extension to $\mathbb{C}$.

$$\boxed{\zeta_{c_1,c_2}(0) = \frac{1}{\chi(M)} - \delta_{c_1,c_2}.} \tag{1}$$

About graphs.

Dang



Mehmeti

Theorem (Dang-Mehmeti)

Schottky group Γ *acting on Berkovich line* $\mathbb{P}^{1,an}_k$, *surface* $\mathbb{P}^{1,an}_k \setminus$ *limit set*$/\Gamma$, g *number of generators of* Γ *genus*,

$$\zeta_{c,c}(s) = \sum_{\gamma \in \Gamma} e^{-s\ell(c,\gamma c)}, \quad \zeta_{c,c}(0) = \frac{1}{1-g} - 1.$$

Similar results on graphs by Anantharaman.

Simpler problem: two convex subsets on torus.

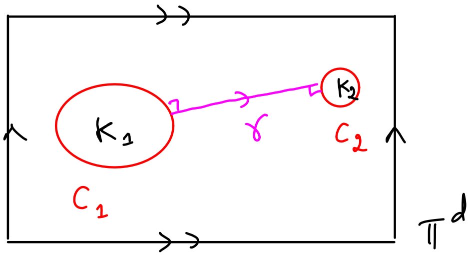


Bonthonneau Léautaud Rivière

Convex subsets in torii.

$K_1, K_2 \subset \mathbb{R}^d$ convex and $\partial K_1, \partial K_2$, analytic, strictly convex hypersurfaces of $\mathbb{R}^d$.
$\mathfrak{p} : \mathbb{R}^d \mapsto \mathbb{T}^d$ projects on $\mathbb{T}^d$,

$$c_1 = \mathfrak{p}\left(\partial K_1\right), c_2 = \mathfrak{p}\left(\partial K_2\right).$$



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Consider

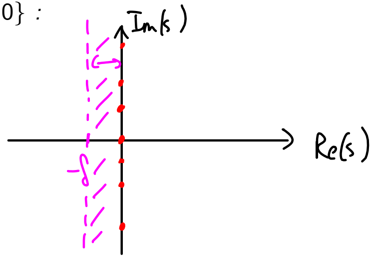
$$\zeta(c_1, c_2, s) = \sum_{\gamma} e^{-s\ell(\gamma)}$$

holomorphic on $Re(s) > 0$.

Work in progress

Theorem (Bonthonneau-D-Léautaud-Rivière, in progress)

Define $\Lambda = \{\pm i|\xi|; \xi \in \mathbb{Z}^d\}$, *near* $i|\xi| \in \Lambda \setminus \{0\}$ *:*



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$$\zeta(s) = \frac{1}{(s - i|\xi|)^{\frac{d+1}{2}}} (a_0 + (s - i|\xi|)a_1 + \dots) \quad + \quad \underbrace{\log(s - i|\xi|)\,(b_0 + (s - i|\xi|)b_1 + \dots)}_{\text{if } n \text{ odd}}$$

$$\dots \quad + \quad \underbrace{(b_0 + (s - i|\xi|)b_1 + \dots)}_{\text{if } n \text{ even}}$$

where $(a_i, b_i)_{i=0}^{\infty}$ *depend on* K_1, K_2, ξ.

Near $s=0$:

$$\zeta(s) = \sum_{\ell=1}^{d} \frac{a_\ell(K_1, K_2)}{s^\ell} + O(1),$$

a_ℓ related to mixed volumes

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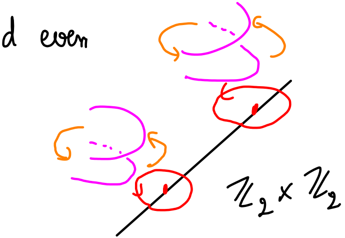
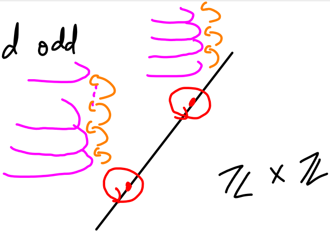
where $(a_i, b_i)_{i=0}^{\infty}$ *depend on* K_1, K_2, ξ.
$\zeta_{c_1,c_2}(s)$ *has multivalued analytic continuation on strip in picture.*





Nilsson class

ARKIV FÖR MATEMATIK Band 5 nr 32

1.64 04 3 Communicated 26 February 1964 by Å. Pleijel and L. Gårding

Some growth and ramification properties of certain integrals on algebraic manifolds

By Nils Nilsson

1. Introduction and statement of results

We are going to work in complex n-space C^n with elements $x=(x_1,\dots,x_n)$, etc., and shall consider a certain class $A(C^n)$ of analytic functions on C^n with special growth and ramification properties. To prepare the definition of $A(C^n)$ we make the following definition.

Definition. A path γ: $x=x(t)=(x_1(t),\dots,x_n(t))$, $t_1\leqslant t\leqslant t_2$, in C^n is said to be of class $\mathcal{A}$ if it consists of a finite number of pieces where the components $x_i(t)$ are regular algebraic functions of t. If the number of pieces is not greater than n_1 and the algebraic functions all have degrees $\leqslant n_2$ (i.e. each of them may be defined by a polynomial of degree $\leqslant n_2$), then we shall say that γ has the rank (n_1,n_2). (It then also has the rank (n_1',n_2'), if $n_1\leqslant n_1'$ and $n_2\leqslant n_2'$).
We now define the class $A(C^n)$.

Definition. $A(C^n)$ consists of all functions f such that there exists an algebraic manifold V_f in C^n of the form $p(x)=0$, where $p(x)$ is a complex polynomial not identically zero, such that

(*a*) f is a regular analytic and in general many-valued function on the whole of (C^n-V_f)
(*b*) all the determinations of f in the neighbourhood of any point in (C^n-V_f) span a linear space over C of finite dimension
(*c*) there is a point $x^0\in(C^n-V_f)$, a real number a, a complex polynomial $R(x)$, $R(x)\neq 0$ when $x\in(C^n-V_f)$, and for every determination f_0 of f at x^0 and every (n_1,n_2) a real number C such that

$$|f_{0\gamma}(x)|\leqslant C(|x|+1)^a|R(x)|^{-1}\quad(\forall x\in\gamma)$$

for all paths γ in C^n of rank (n_1,n_2) starting at x^0 and not meeting V_f. Here $f_{0\gamma}$ is the function on γ obtained by analytic continuation along γ out from f_0, and $|x|=(\sum_{i=1}^n|x_i|^2)^{\frac{1}{2}}$.

Remark. It may be proved that the condition (*c*) is not changed if we only permit pairs (n_1,n_2) of the form $(n_1,1)$.
Instead of $f\in A(C^n)$ we shall also say that $f(x)$ is of class A in x. If a function f is defined and regular analytic in a neighbourhood of a point $x^0\in C^n$ (or on any

1964

Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

163

Pierre Deligne
Institut des Hautes Etudes Scientifiques
Bures-sur-Yvette/France

Equations Différentielles à Points Singuliers Réguliers

1970

Nilsson class

Definition

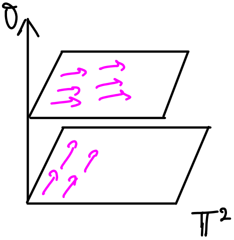
f Nilsson near $z = 0$ if moderate growth + finite determinations:

$$f = \sum_{finite} a_i(z) \log(z)^{v_i} z^{\mu_i},$$

$v_i \in \mathbb{N}$, $\mu_i \in \mathbb{C}$, a_i *holomorphic.*

Multivariate version has nice functorial properties, applications in PDE, oscillatory integrals, Feynman integrals by Boutet de Monvel, Leray, Malgrange, Pham . . .

Transport by the geodesic flow.



$M = S^*\mathbb{T}^d$ **contact** manifold. Coordinates $(x;\theta) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$.

$$e^{tV}(x;\theta) = (x + t\theta;\theta) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$$

Geodesic flow generator $V = \theta.\partial_x$: $\boxed{(e^{tV} f)(x;\theta) = f(x + t\theta;\theta)}$:
Given a distribution $\omega \in \mathcal{D}'(S\mathbb{T}^d)$:

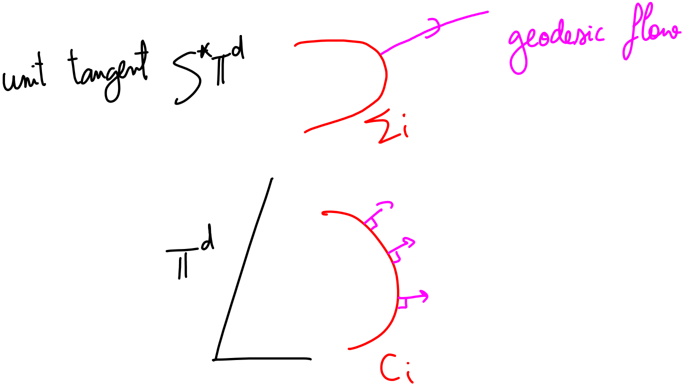
$$e^{-tV}\omega(x;\theta) = \omega(x - t\theta;\theta).$$

$f = e^{-tV}\omega$ solves **transport equation**:

$$\partial_t f + \theta.\partial_x f = 0, f(0,.) = \omega.$$

Transporting currents.

$\Sigma_i = \{(x;\theta), x \in c_i, \theta \perp T_x c_i\}$ **unit normal to** c_i,



Transporting currents.

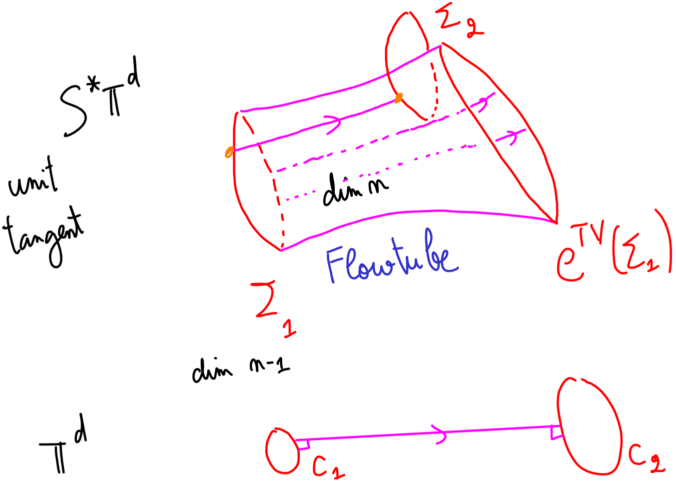
Margulis 1969 phd

$\Sigma_i = \{(x;\theta), x \in c_i, \theta \perp T_x c_i\}$ **unit normal to** c_i, de Rham current of integration $[\Sigma_i]$.

Transporting currents.

$\Sigma_i = \{(x;\theta), x \in c_i, \theta \perp T_x c_i\}$ **unit normal to** c_i, de Rham current of integration $[\Sigma_i]$.
Geometrically, submanifold $e^{[0,T]V}(\Sigma_1)$ emanating from Σ_1 and counting:

$$\text{Number of arcs length} \leqslant T = |e^{[0,T]V}(\Sigma_1) \cap \Sigma_2|.$$



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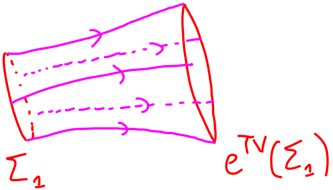
Proposition (Representation of $e^{[0,T]V}(\Sigma_1)$ as current)

Currents $[\Sigma_i]$ in $\mathbb{ST}^d$, V generates geodesic flow.

$$\underbrace{[e^{[0,T]V}(\Sigma_1)] = -\int_0^T \iota_V e^{-tV}[\Sigma_1]dt}_{\textit{Integration current}}$$



Intersection formula.

Proof.

$\partial \int_0^T -\iota_V e^{-tV}[\Sigma_1] dt$

Intersection formula.

Lie-Cartan $\mathcal{L}_V = d i_V + i_V d$

Proof.

$$\partial \int_0^T -\iota_V e^{-tV}[\Sigma_1]dt = -\int_0^T d\iota_V e^{-tV}[\Sigma_1]dt - \underbrace{\int_0^T \iota_V de^{-tV}[\Sigma_1]dt}_{=0}$$

Intersection formula.

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$$= e^{-TV}[\Sigma_1] - [\Sigma_1].$$

□

Intersection formula.

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□

Assume all intersections **transverse**, weight intersections:

$$\zeta(s) = \sum_\gamma e^{-s\ell(\gamma)} = -\int_0^\infty \left\langle [\Sigma_2], \iota_V e^{-tV}[\Sigma_1] \right\rangle e^{-ts} dt \quad (2)$$

Intersection formula.

Proof.

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Key observation, the **resolvent appears**:

$$\boxed{\zeta_{c_1,c_2}(s) = -\left\langle [\Sigma_2], \iota_V (V+s)^{-1}[\Sigma_1] \right\rangle} \quad (3)$$

Singularities in s « Ruelle resonances »

Degond
1986

Hyperbolicity $\overset{\text{trade}}{\longleftrightarrow}$ Analyticity

Anosov

Idea from kinetic theory, Degond (1985).

Fourier transform in x, keep dependence in θ, for currents $U_1 = \iota_V[\Sigma_1], U_2 = [\Sigma_2]$:

$$\left\langle U_2, (V+s)^{-1} U_1 \right\rangle_{S^*\mathbb{T}^d} = \frac{1}{(2\pi)^d} \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \underbrace{(i\langle \xi, \theta \rangle + s)^{-1}} \widehat{U_1}(\xi,\theta)\widehat{U_2}(-\xi,\theta) d^{d-1}\theta.$$

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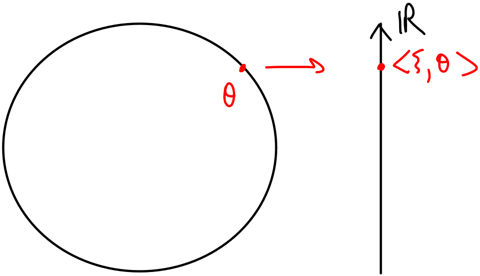
Each ξ, **multiplication** operator by **height function**:

$$\mathrm{m}_\xi : f \in L^2(\mathbb{S}^{d-1}) \longmapsto \langle \xi, . \rangle f \in L^2(\mathbb{S}^{d-1}).$$



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$$\mathrm{m}_\xi : f \in L^2(\mathbb{S}^{d-1}) \longmapsto \langle \xi, . \rangle f \in L^2(\mathbb{S}^{d-1}).$$

Sums over resolvent of m :

$$\frac{1}{(2\pi)^d} \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} (i\mathrm{m}_\xi + s)^{-1} \widehat{U_1}(\xi, \theta) \widehat{U_2}(-\xi, \theta) d^{d-1}\theta$$

Spectral theory of multiplication operator m_f.

f analytic Morse on $\mathbb{S}^{d-1}$, spectrum of $\mathrm{m}_f : L^2(\mathbb{S}^{d-1}) \mapsto L^2(\mathbb{S}^{d-1})$ is $[\inf(f), \sup(f)]$.

Ex: $m_{\sin(\theta)} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1) \quad \sigma = [-1, 1]$

Bargmann

$(m_{\sin(\theta)} - s)^{-1} : \mathcal{H}_\varepsilon = e^{-\varepsilon\Delta} L^2 \longrightarrow \mathcal{H}_\varepsilon^* , \quad \sigma = -1 \text{ --- } +1$

analytic functions   hyperfunctions

$$\int_0^{2\pi} (\sin(\theta) - s)^{-1} u_1(\theta) u_2(\theta)\, d\theta = \int_{[0,2\pi]} (\sin(z) - s)^{-1} u_1(z) u_2(z)\, dz$$

deform

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$$= \int_{e^{t\nabla Im(f)}\mathbb{S}^{d-1}} \frac{1}{(\tilde{f}-s)} \underbrace{\Omega}_{\text{hol.} \in \Lambda^{top}}$$

Spectral theory of multiplication operator m_f.

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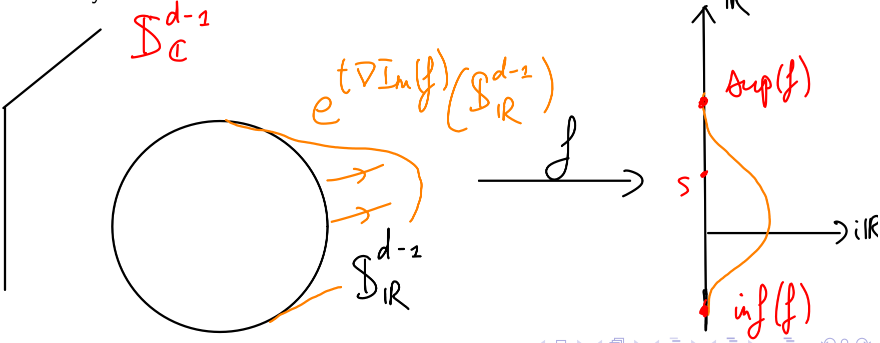
Stokes allows cycle deformation.



Thank you again for the invitation and for your attention.