

**Cours 2023-2024:**

**La perception des objets mathématiques élémentaires:**

**Formes géométriques, motifs et graphiques**

***Perception of elementary mathematical objects:***

***Geometric shapes, patterns, and graphics***

Stanislas Dehaene

Chaire de Psychologie Cognitive Expérimentale

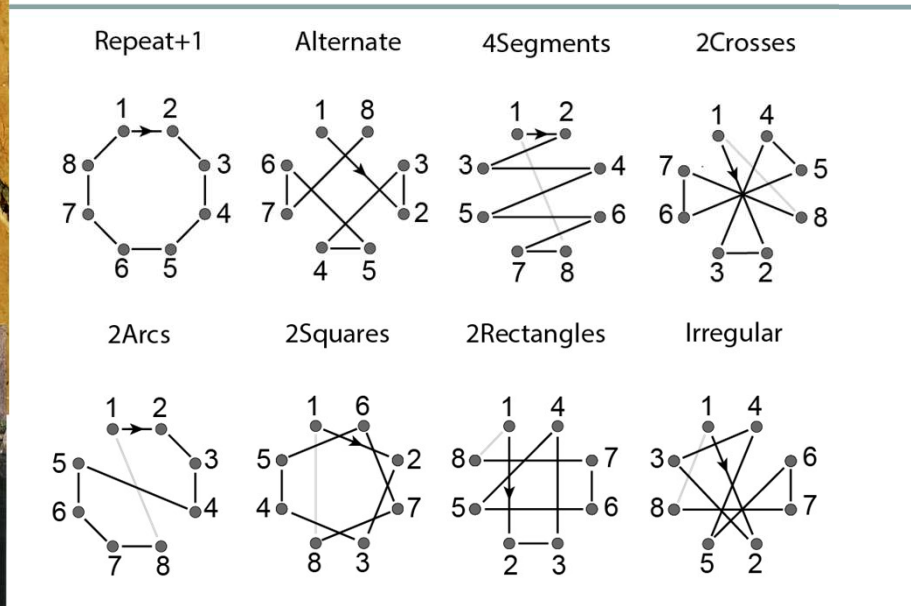
Cours n°4

**Perception des quadrilatères et singularité de l'espèce humaine en géométrie**

***Perception of quadrilaterals : a human singularity for geometry***



## Last course: the perception of geometric and musical sequences implies a “language of thought” encoding recursive repetitions with variations



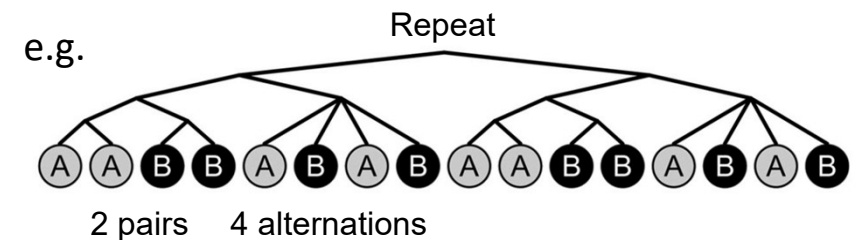
### A language for geometrical sequences

1. A small list of numerical and geometrical **primitives**
2. A single recursive rule : **Repetition** (possibly with variations)

Zig-Zag = repeat 4 times (repeat 2 times (symmetry) )  
 changing the start point by +1

We propose that auditory sequences are internally encoded using a **compression algorithm** that

- Detects recurrent sequences
- Compresses them as “loops” in a **language of thought** similar to a computer language
- Operates in a recursive manner



The key operation is **repetition with variation**.

In fact, the **very same language** that accounts for visuo-spatial sequences, **unchanged**, predicts the subjective and objective complexity of a binary **auditory** sequence.

# Iconic and symbolic drawings in the Lascaux cave



Marie Amalric



Fosca Al Roumi



Samuel Planton



Mathias Sablé Meyer

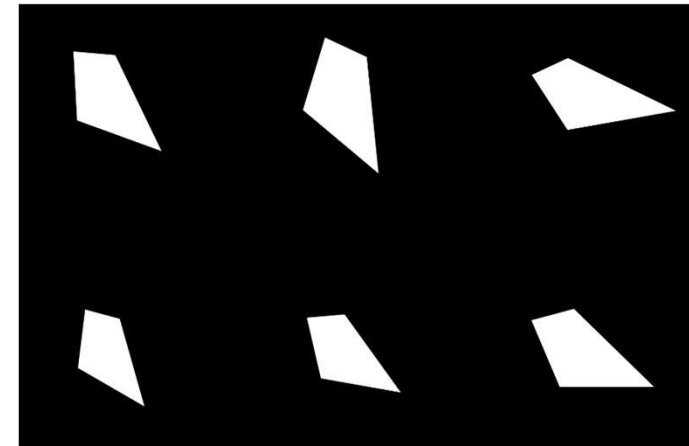
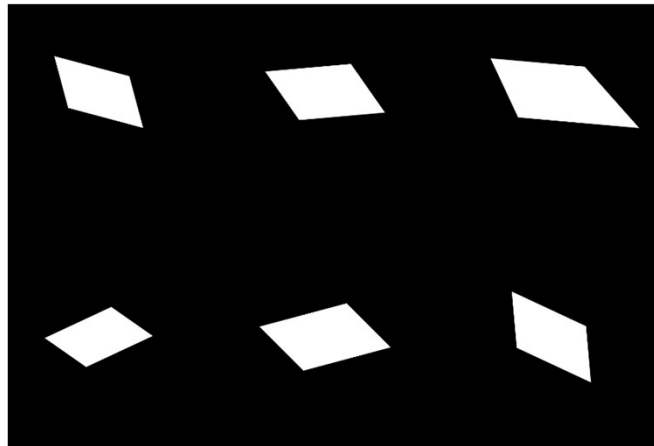
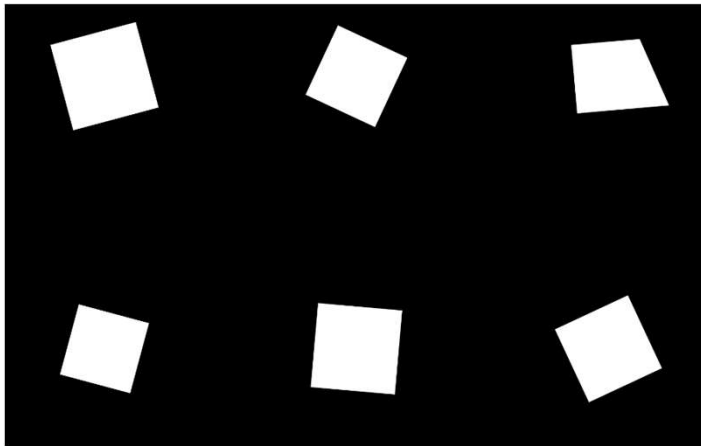




# How do human and non-human primates perceive quadrilaterals?

Sablé-Meyer, M., Fagot, J., Caparos, S., Kerkoerle, T. van, Amalric, M., & Dehaene, S. (2021). Sensitivity to geometric shape regularity in humans and baboons : A putative signature of human singularity. *PNAS*, *118*(16). <https://doi.org/10.1073/pnas.2023123118>

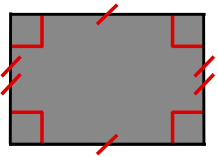
Detect the outlier shape !



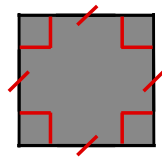
# Does shape regularity predict perceptual complexity?

We used 11 quadrilaterals ranging from highly regular (square) to fully irregular

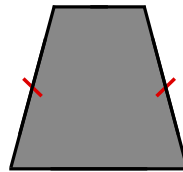
Rectangle



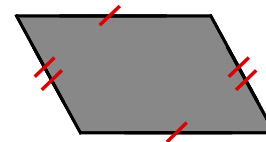
Square



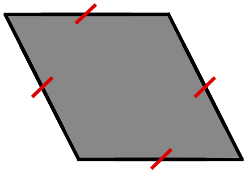
Iso-Trapezoid



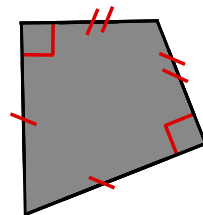
Parallelogram



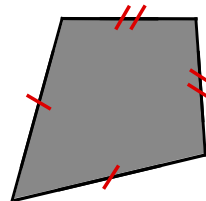
Rhombus



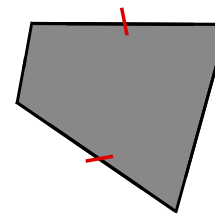
Right Kite



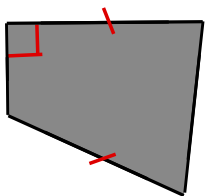
Kite



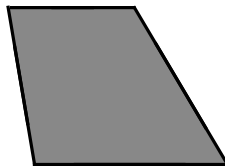
Hinge



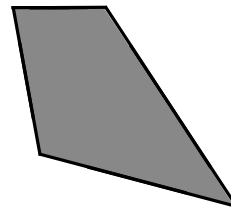
Right Hinge



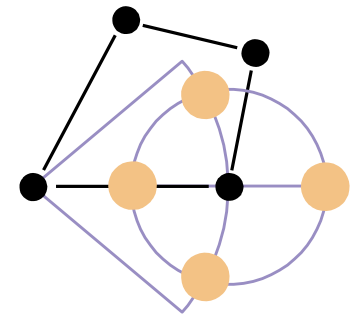
Trapezoid



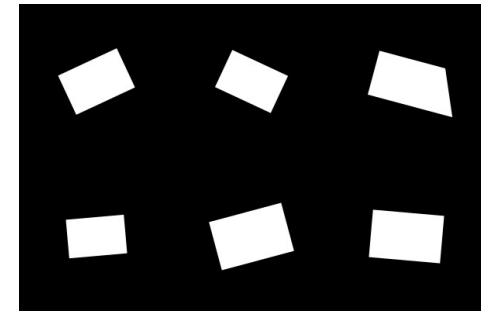
Irregular



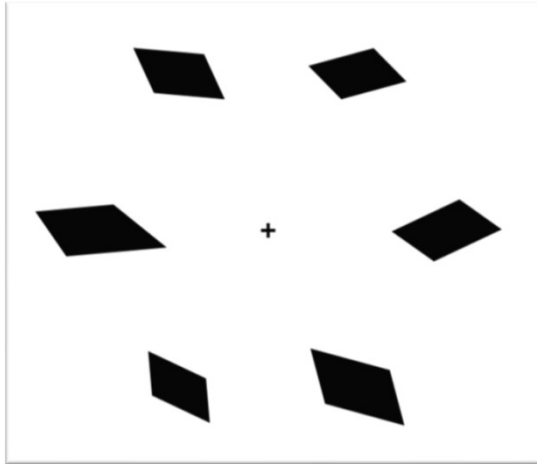
Deviants involve a displacement of the bottom right vertex.



Example display :

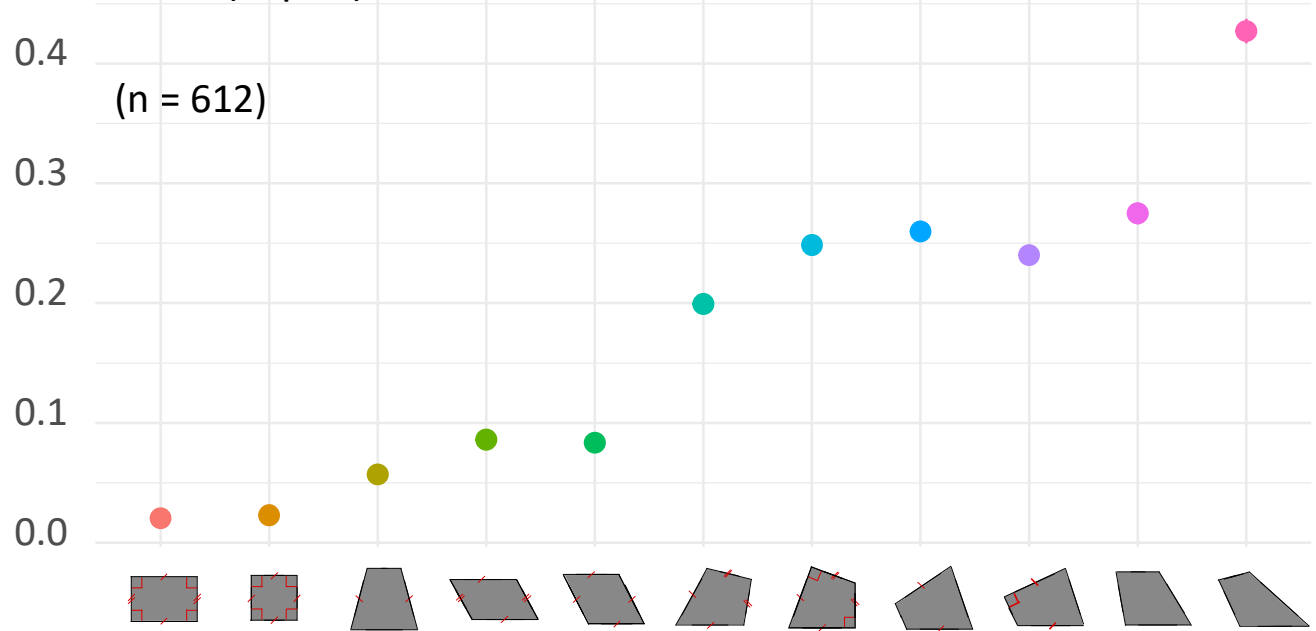


# Human adults: a major effect of shape regularity



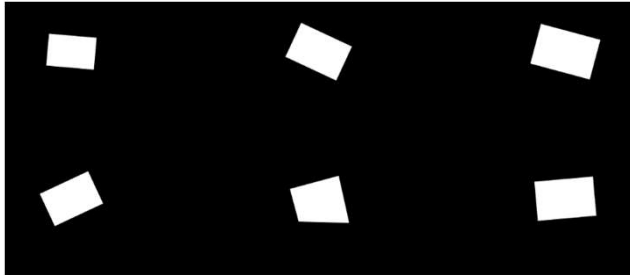
- Performance is above chance for all shapes, but varies from 7% to 42% errors.
- Response time follows the same pattern.
- The position, rotation and size of the outlier have either no significant effect or significant effects with almost no explained variance

Error Rate (exp. 1)

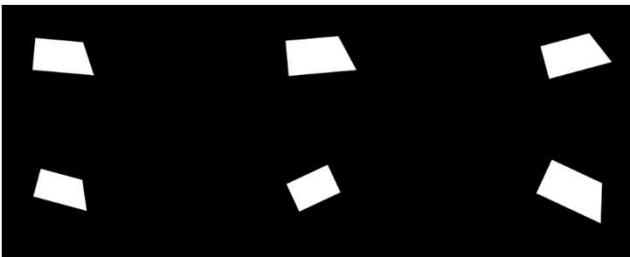


# The effect of shape regularity is highly replicable in human adults

Normal layout

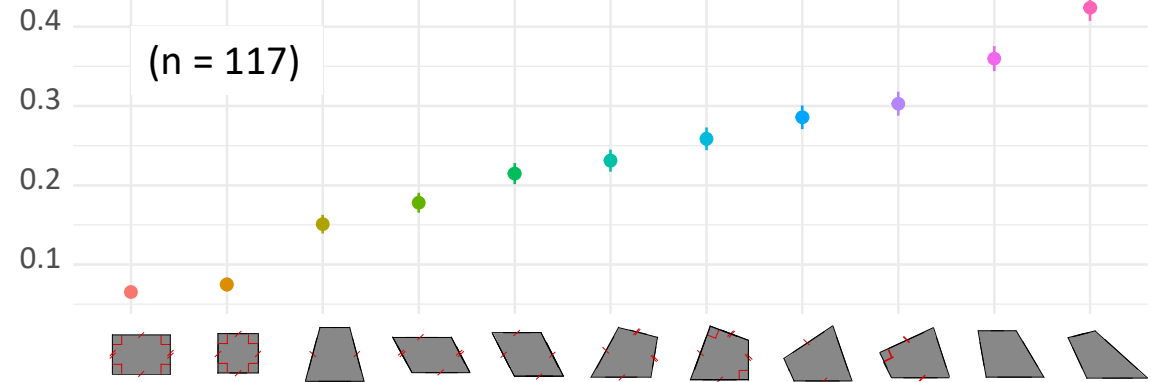


Swapped layout

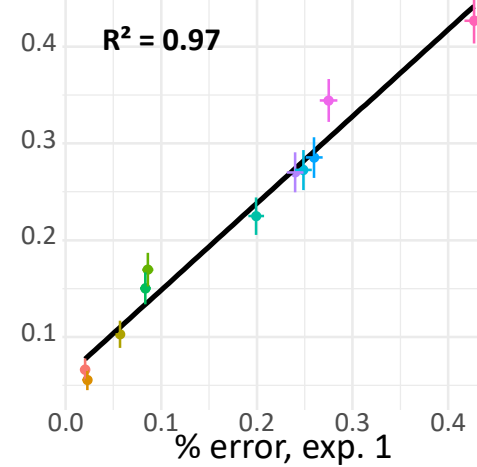


- Replication with a new group of 117 subjects and a different presentation mode of the stimuli.

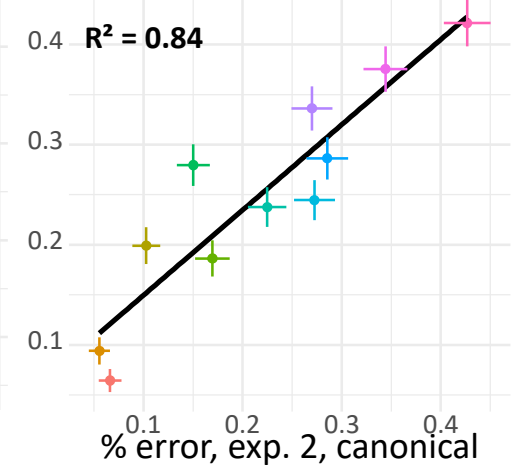
Error Rate (exp. 2)



% error, exp. 2, canonical

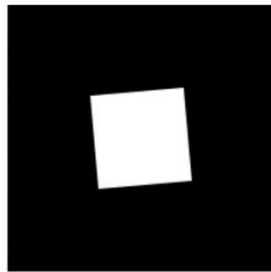


% error, exp. 2, swapped



# The effect of shape complexity is accessible to introspection

Subjects rated the subjective complexity or regularity of the 11 shapes.



Jugement de régularité:

84

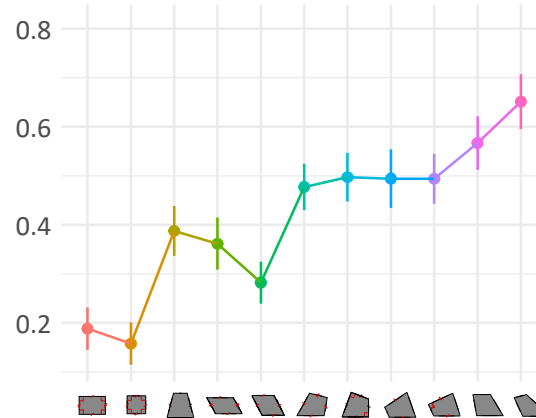


Jugement de régularité:

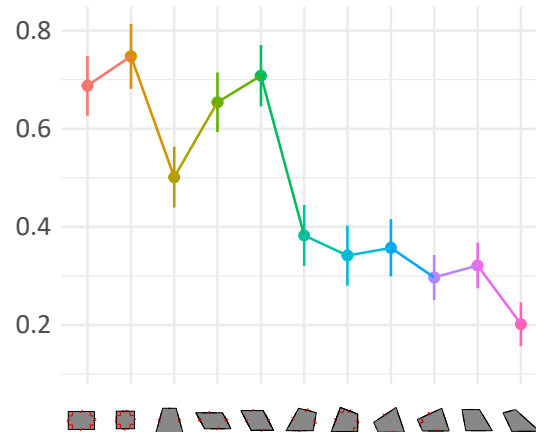
13



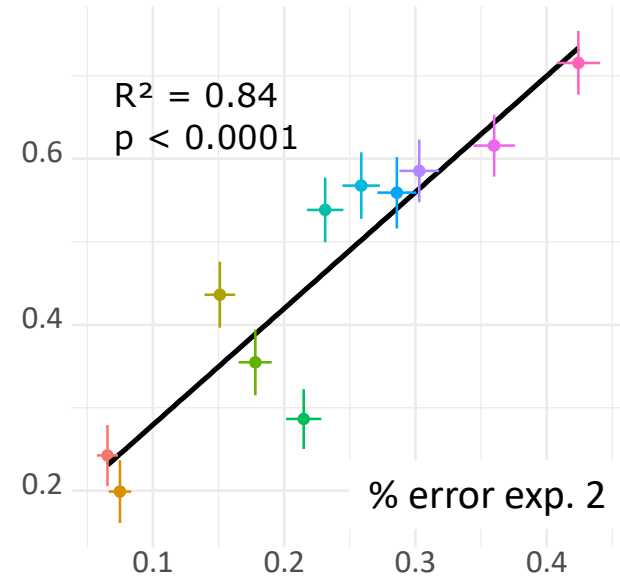
Judgment of Complexity



Judgment of Regularity



Error rate, subjective rating (n = 48)

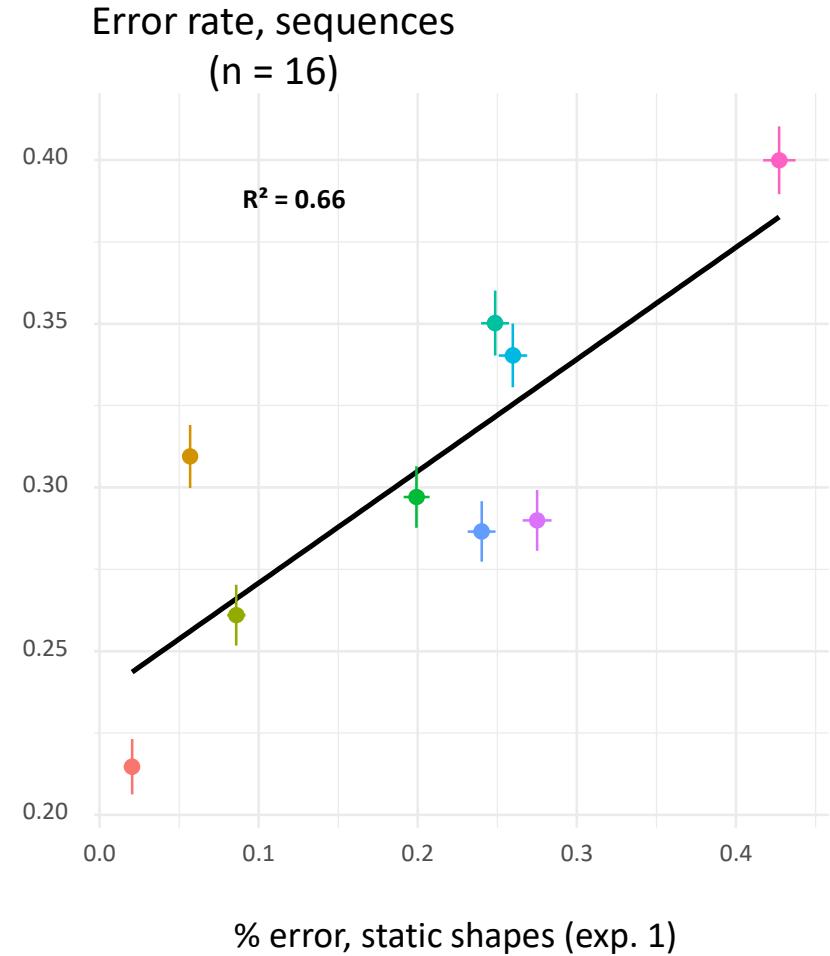
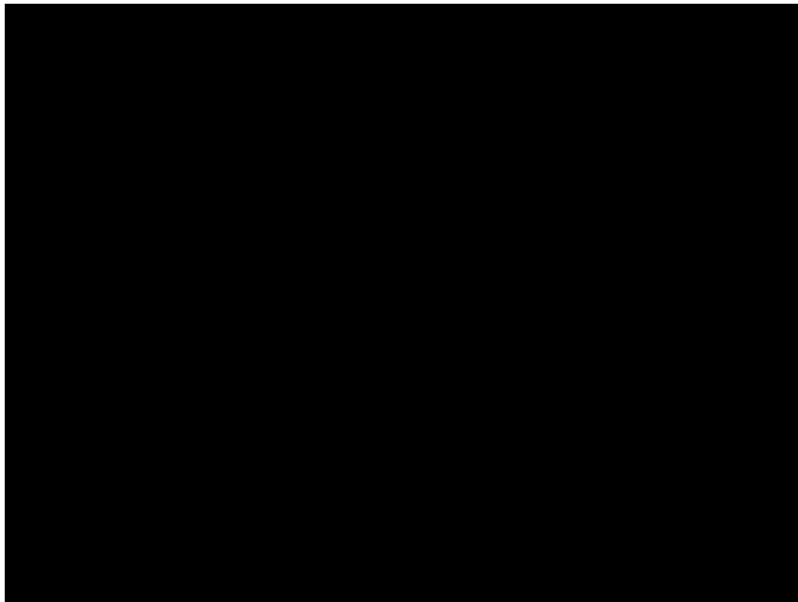




# The shape regularity arises from an abstract level: it persists even when the shapes are presented as sequences

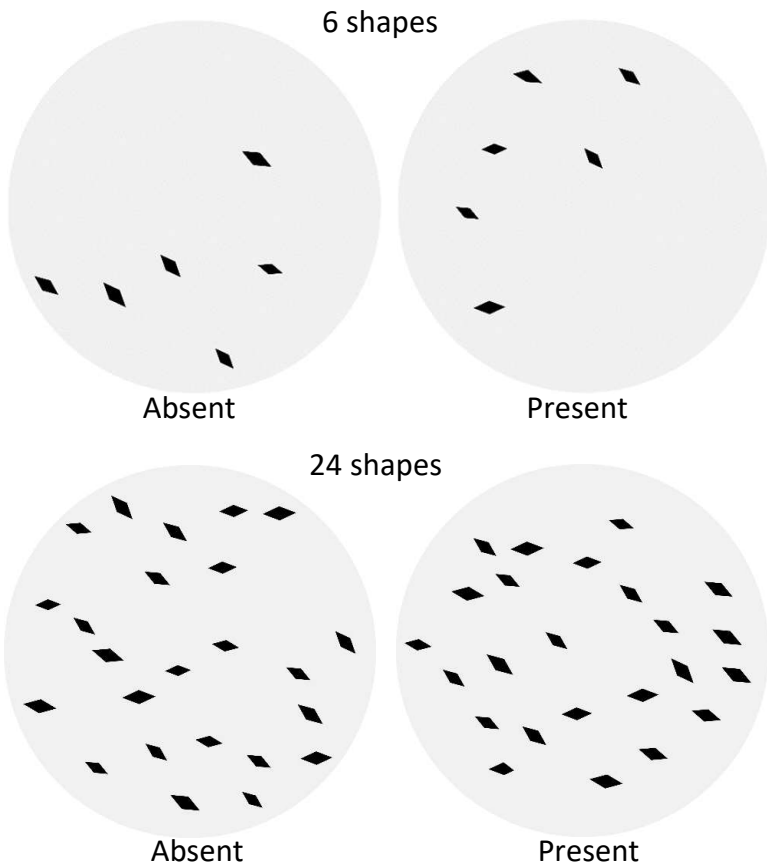
If the shape regularity effect arises from an abstract “language of thought”, then it should persist even when classical bottom-up shape recognition is prevented.

Once again, error rate varied strongly with shape regularity.

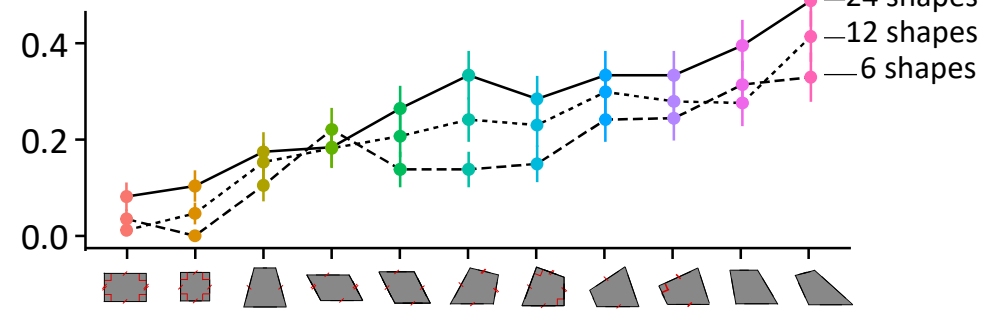


# Visual search: Evidence for an attention-dependent stage, not a pre-attentive pop-out

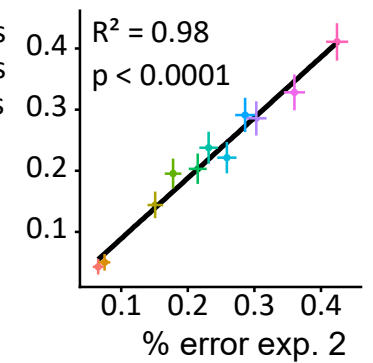
- Classical visual search paradigm: is a deviant present or absent?
- All shapes give rise to a **serial pattern** of visual search → no pop-out, but a search whose speeds varies with geometrical regularity.



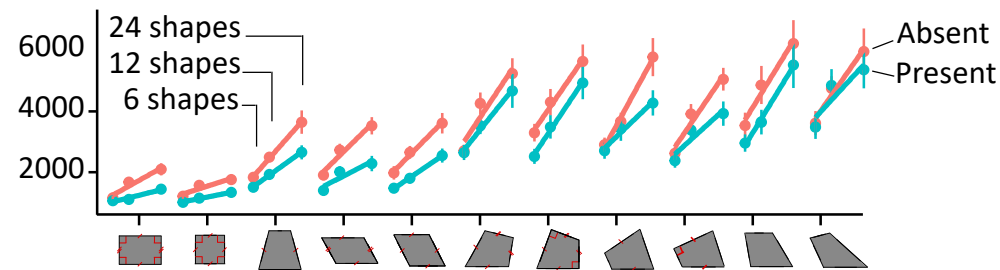
% error in visual search



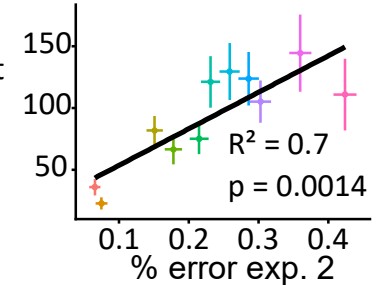
% error in visual search



Average response time (ms)



RT slope (ms/item)



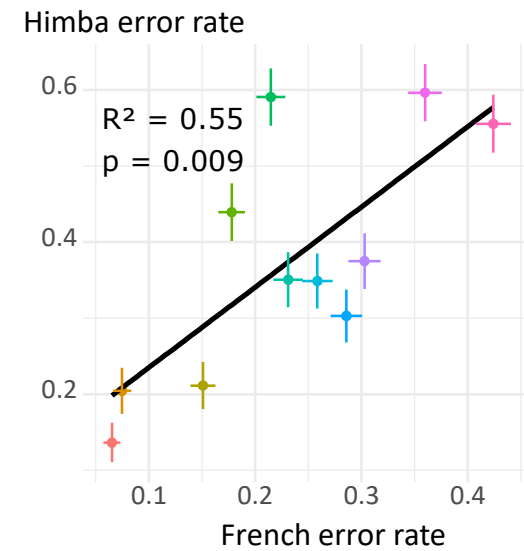
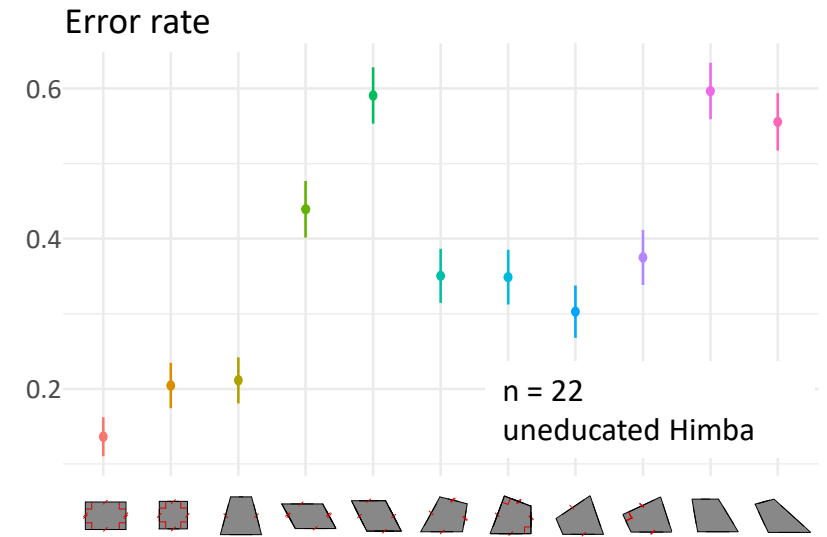
# Does the shape regularity effect depend on culture and education ?

## A study of adult Himba (with Serge Caparos)

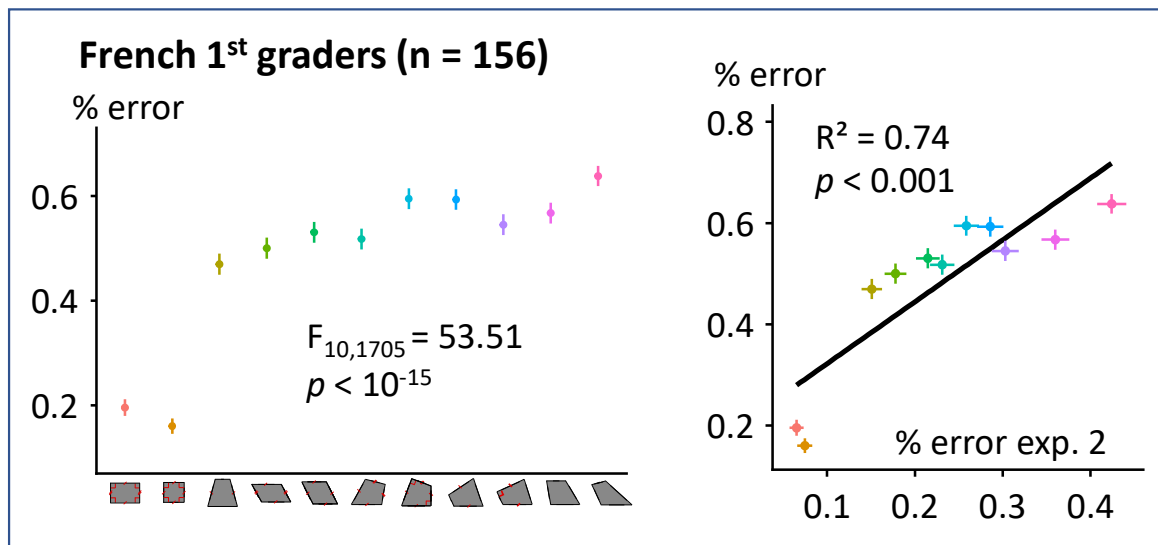
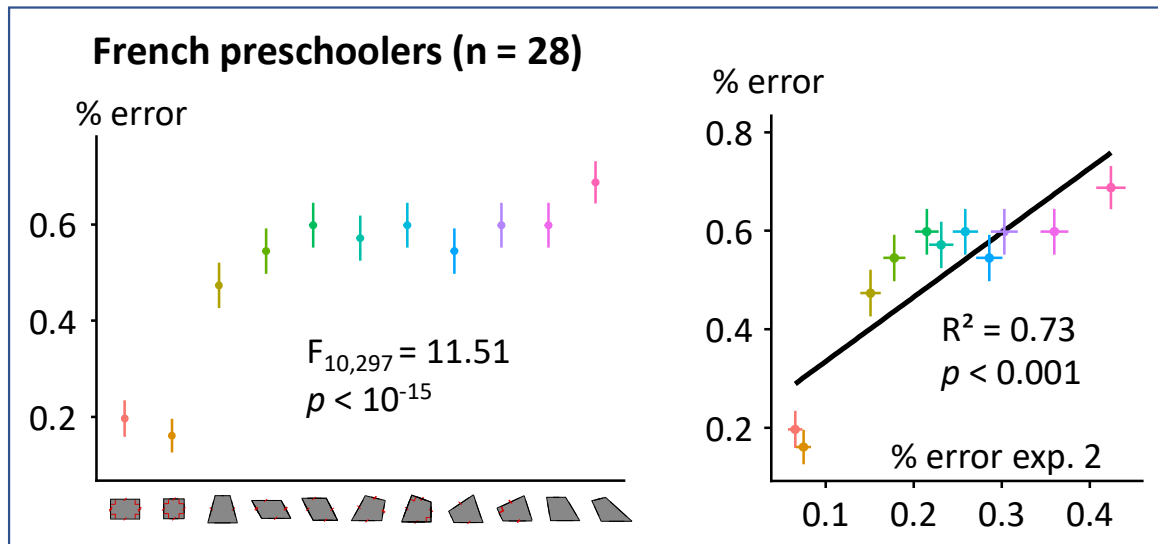
The Himba is a pastoral people from Northern Namibia, with little or no access to formal education, and a reduced language for geometry. They only have a word for circle, and phrases for square or triangle, e.g. “the angular thing with three angles”, otjinavikoro vitato (otjina=thing, vikoro=angles, vitato=3)



(Actual testing takes place in isolated tents)



# Does the shape regularity effect depend on education ?



## Data from French Preschoolers and 1<sup>st</sup> graders

We ran our test on 28 preschoolers (aged ~5) and 156 1<sup>st</sup> graders (aged ~6)

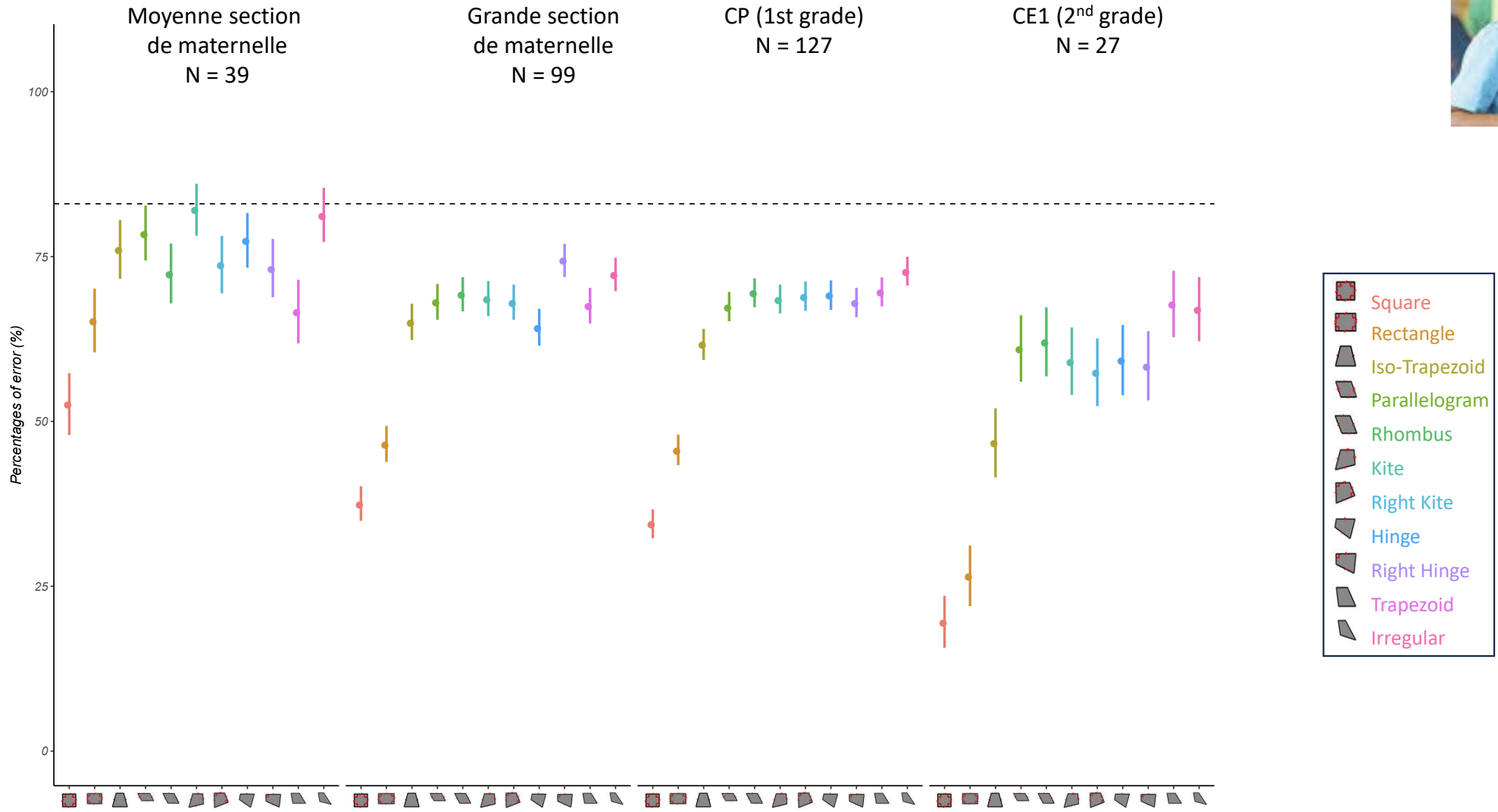
Error rate varied dramatically with shape, from 20% to 70% (chance is  $5/6=83\%$ )

This geometrical regularity effect is

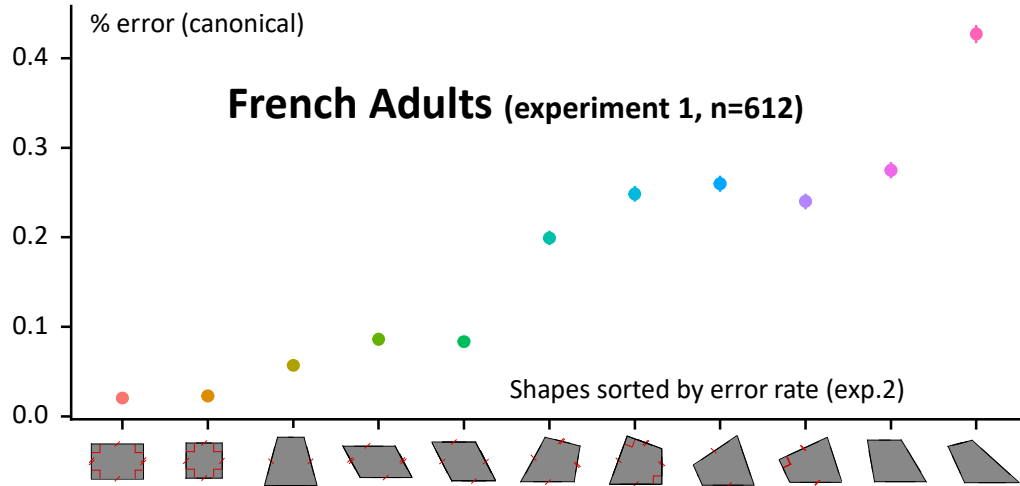
- Highly correlated with French adults
- Highly replicated in the two groups
- Not just driven by square and rectangle

# Evolution of the shape regularity effect with education

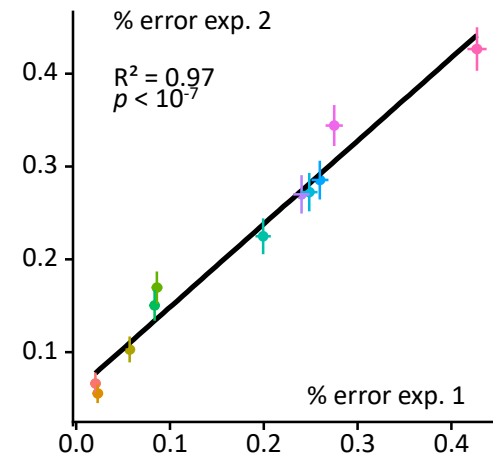
Replication and extension by Théo Morfoisse et al.



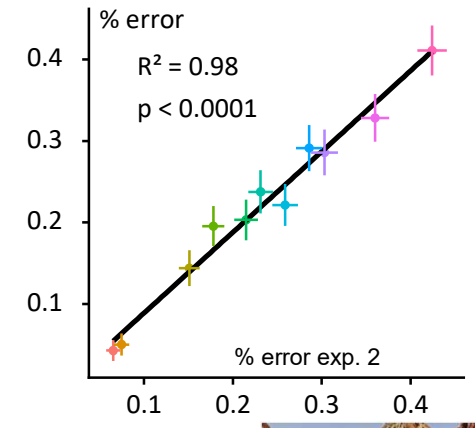
# The geometrical regularity effect: a human universal



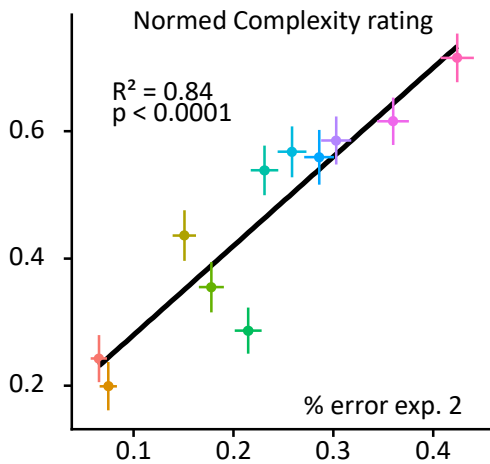
## Replication (experiment 2, n=117)



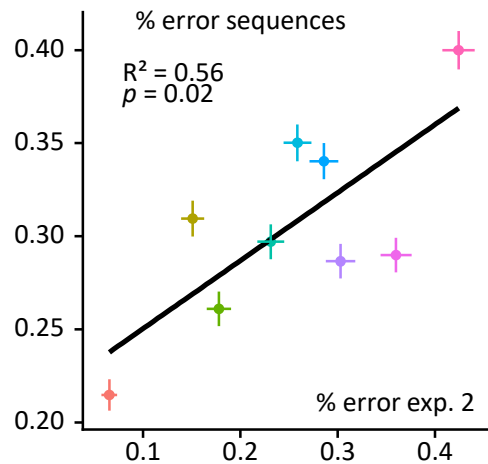
## Visual Search (n=10)



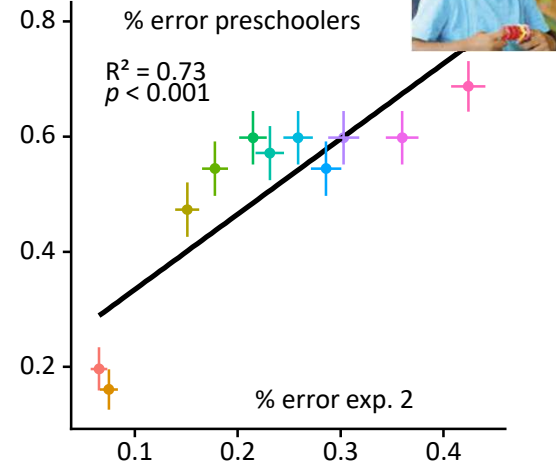
## Subjective ratings (n=48)



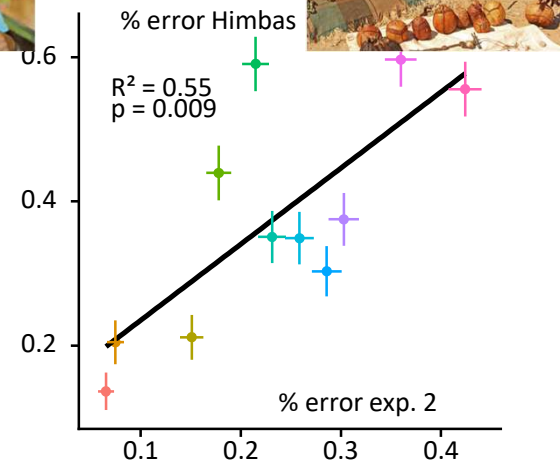
## Sequence format (n=16)



## Preschoolers (n=28)



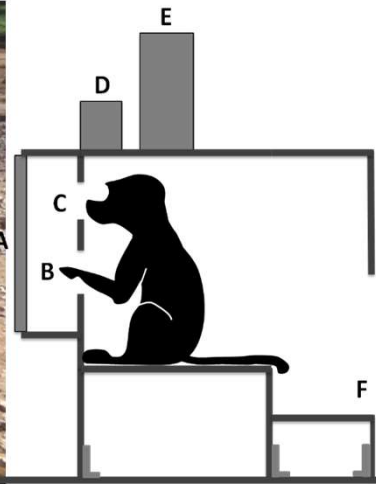
## Himba (n=22)



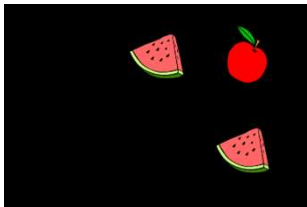


# Is the shape regularity effect present in non-human primates? A study in baboons (with Joël Fagot)

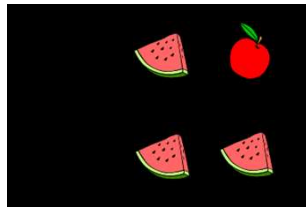
**Prediction:** baboons should fail to show the shape regularity effect



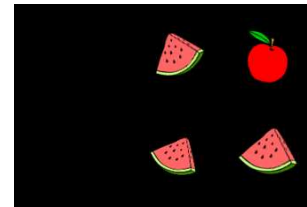
**Train 1**  
oddball  
3 choices  
fixed



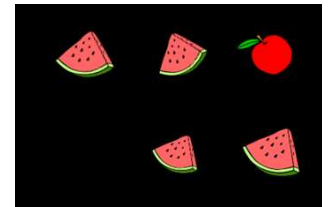
**Train 2**  
oddball  
4 choices  
fixed



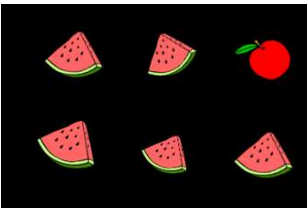
**Train 3**  
oddball  
4 choices  
variable



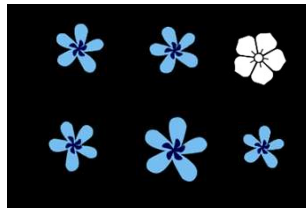
**Train 4**  
oddball  
5 choices  
variable



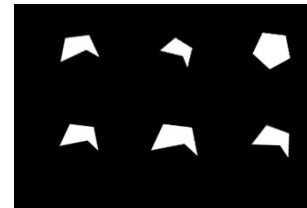
**Train 5**  
standard  
oddball,  
20 images



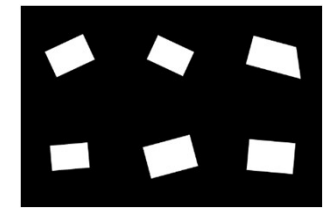
**Gen. 1**  
generalize  
20 new  
images



**Gen. 2**  
generalize  
6 easy  
polygons



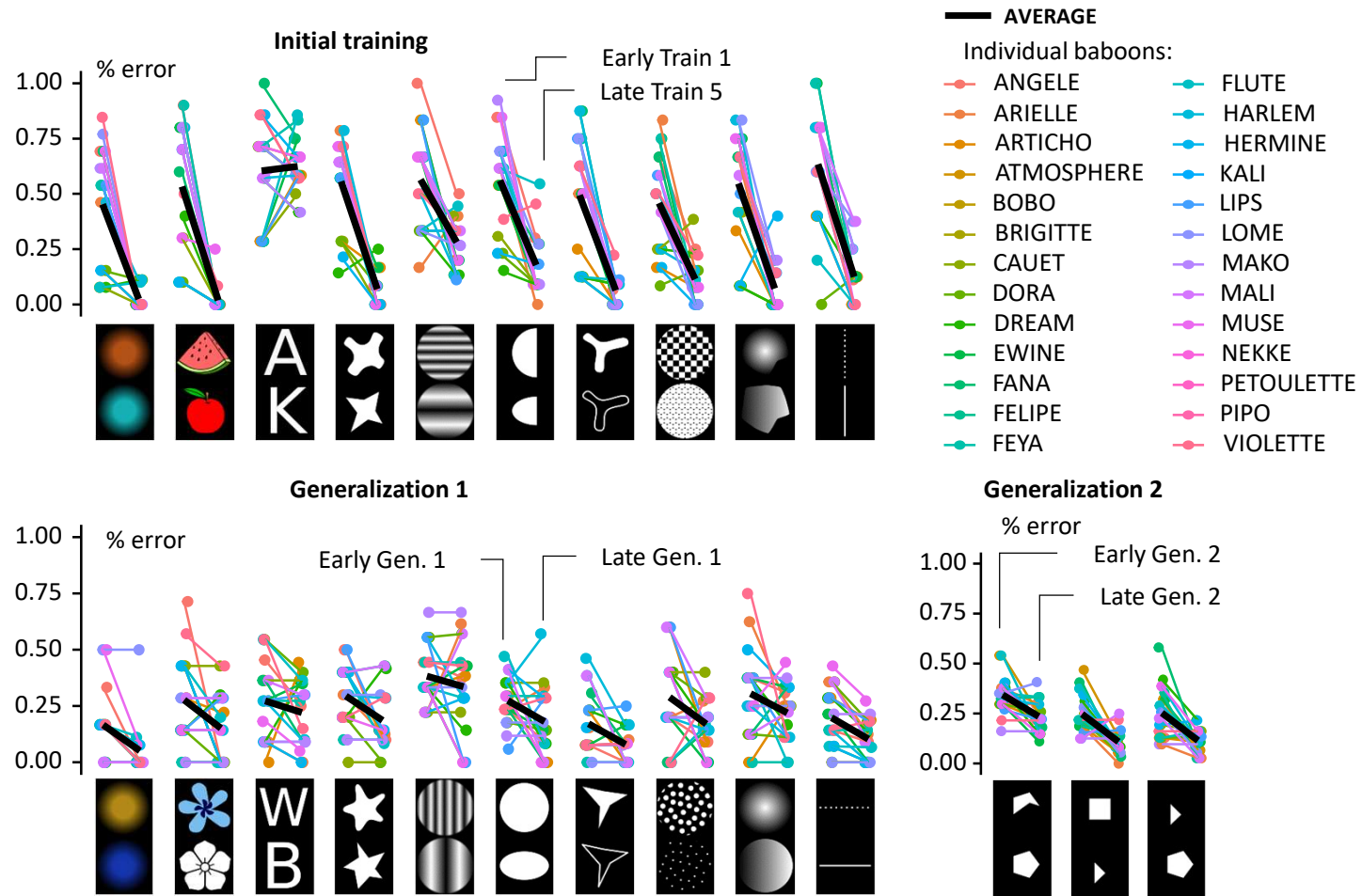
**Test**  
geometry  
test





# Baboons can easily be trained in the outlier task

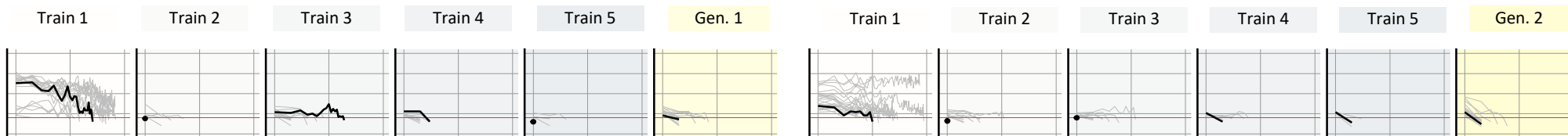
- N=26
- Avg 5200 trials for training
- Generalization avg 272/220 trials
- “Only” N=20 passed the training stage



## Detail: EWINE

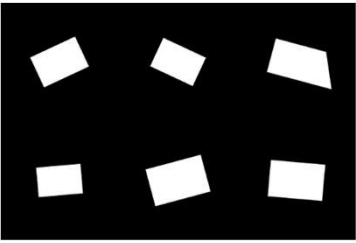
Session 1 : training and 1<sup>st</sup> generalization

Session 2 : training ; 2<sup>nd</sup> generalization and testing

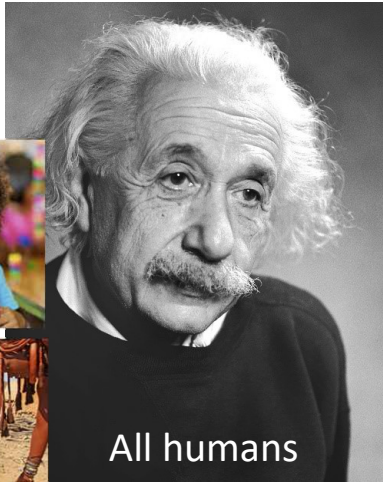


...?

# A major effect of shape regularity in humans, not baboons



Preschoolers (n=28)



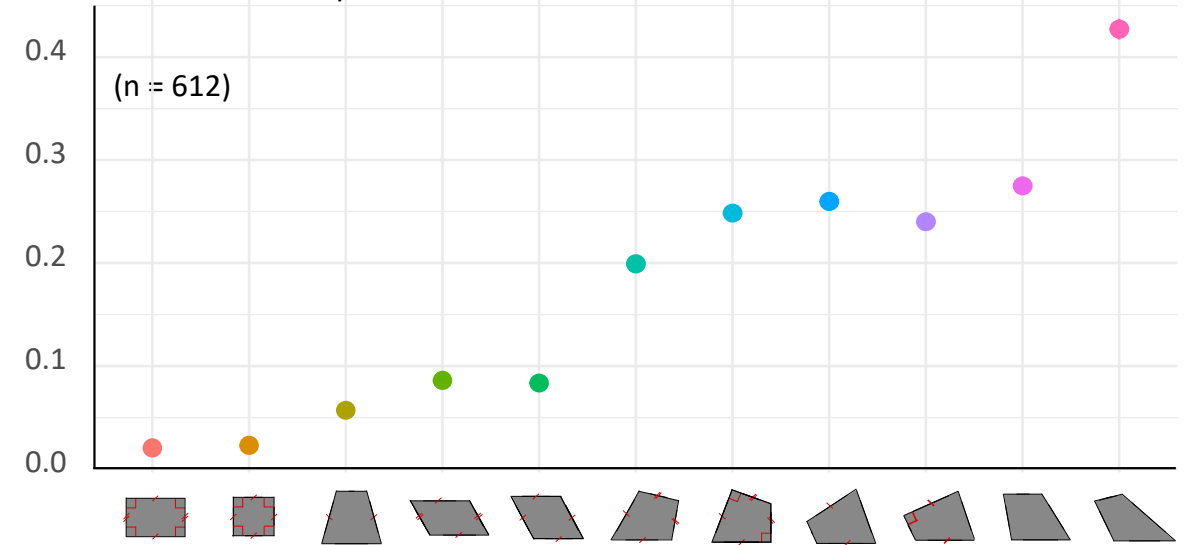
All humans

Himba (n=22)

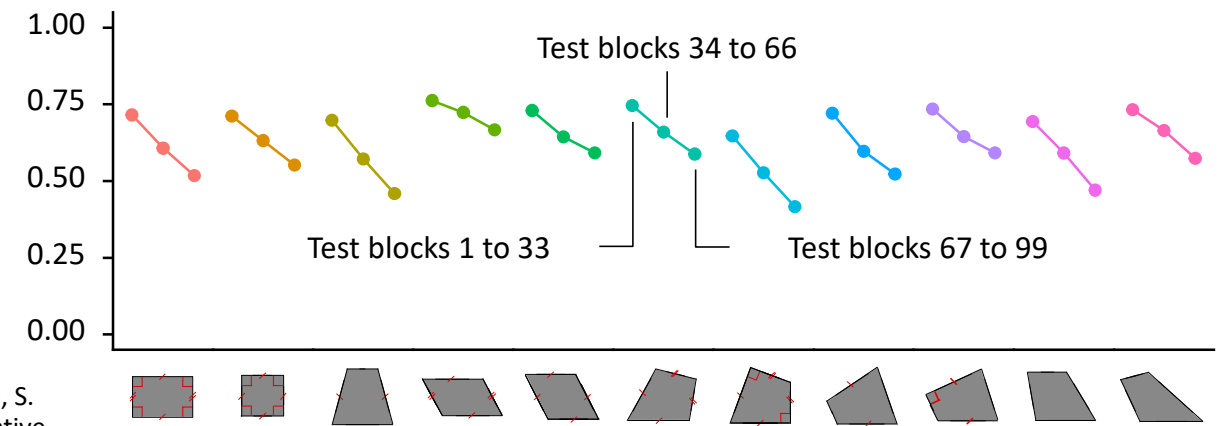


Baboons

Error Rate in humans)

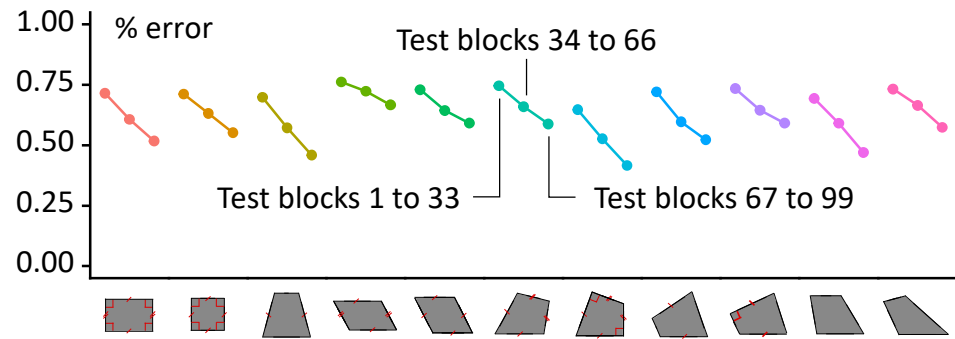


Error rate in baboons

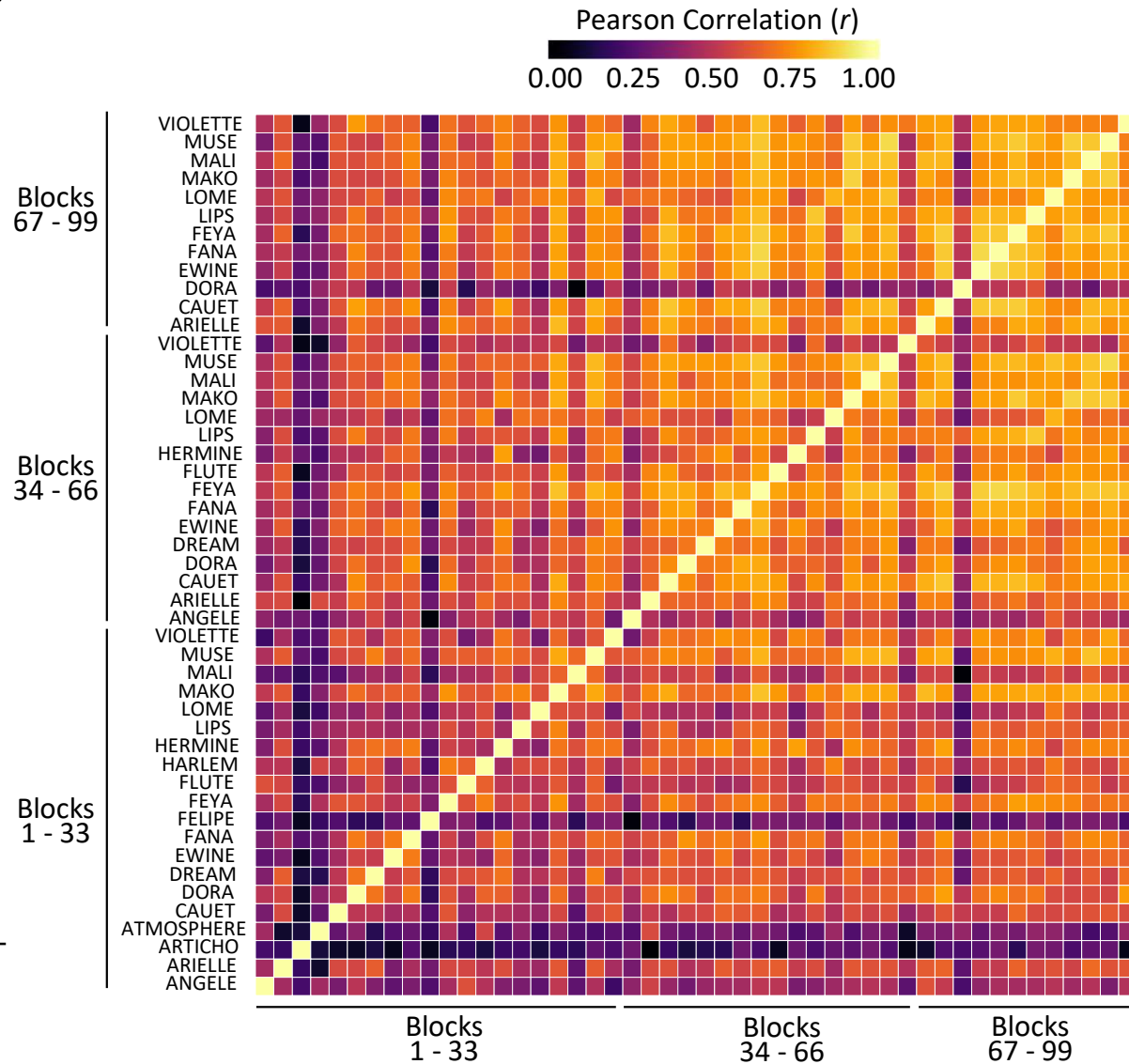
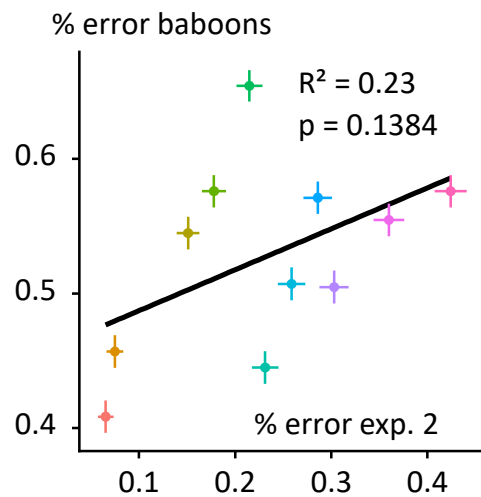


Sablé-Meyer, M., Fagot, J., Caparos, S., Kerkoerle, T. van, Amalric, M., & Dehaene, S. (2021). Sensitivity to geometric shape regularity in humans and baboons : A putative signature of human singularity. *PNAS*, 118(16).

# The shape regularity effect is absent in baboons

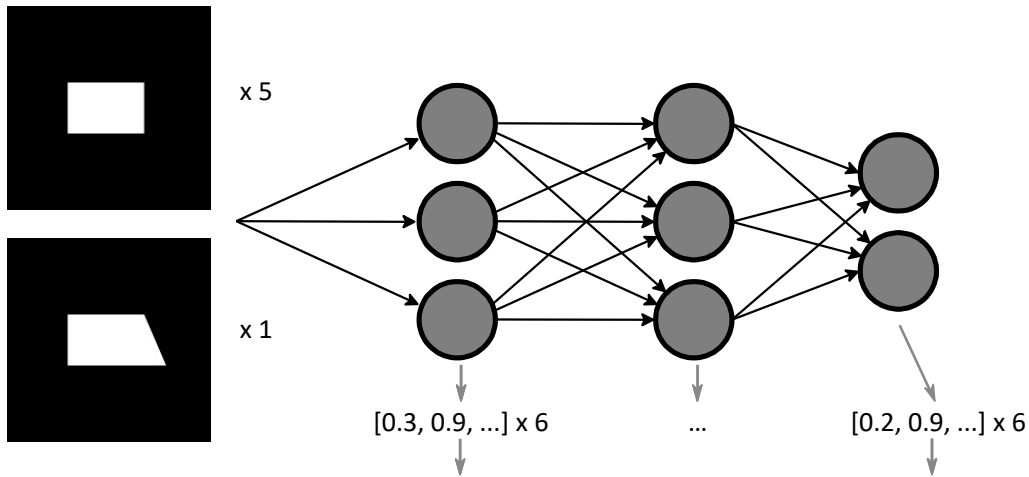


- Baboons are just a bit above chance with all shapes on the first block with quadrilaterals
- After 8800 trials, performance improves, but remains poor and uncorrelated with humans.
- Nevertheless, there is a striking consistency of the baboon pattern across time and individuals



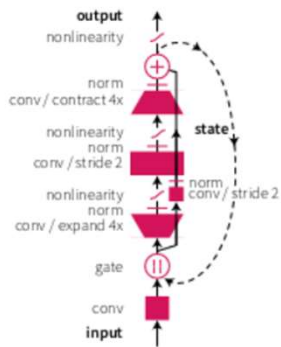


# Model 1: shape perception by a convolutional neural network (CNN)

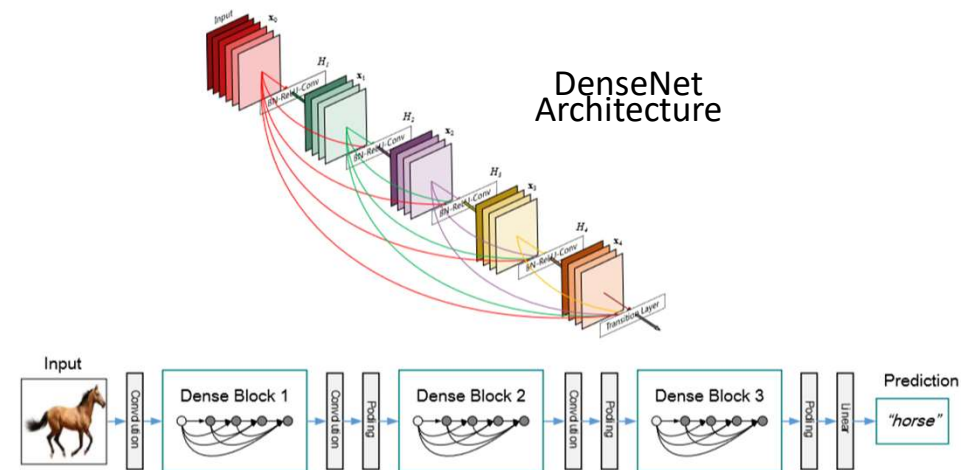
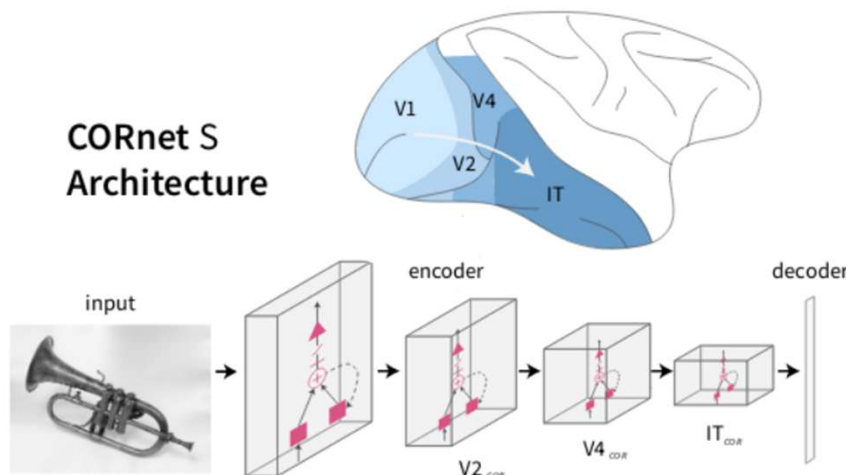


- We presented our stimuli to CoreNet-S, a model trained to categorize natural images and which provides a good match to human performance and inferotemporal neuronal recordings.
- A similar experiment was done with two other CNNs, DenseNet and ResNet, with similar results

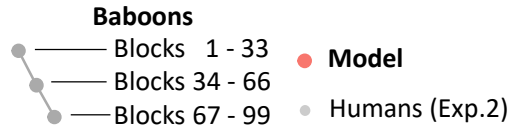
**Outlier =**  
**Vector most different from the mean of the others**  
**(in a given layer)**



Training

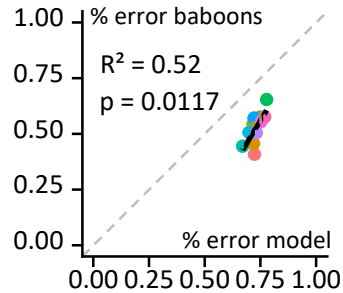
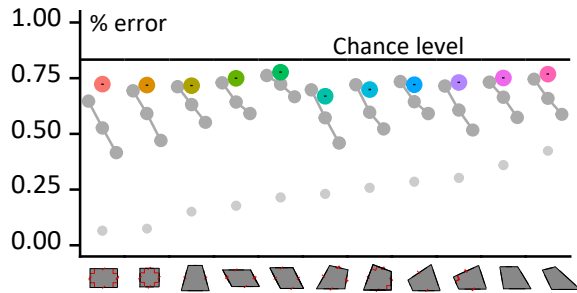


# The neural network model exhibits no geometric regularity effect... ...but predicts baboon behavior

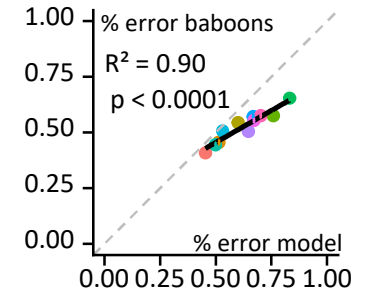
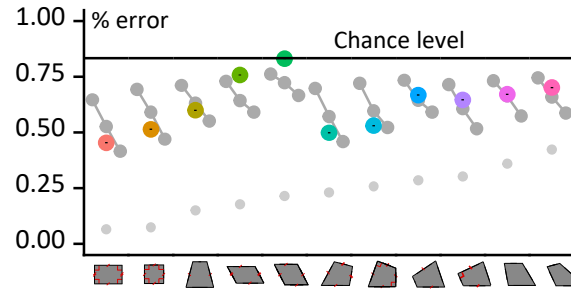


**Baboons' late performance**  
(blocks 81-99)  
versus **Model**

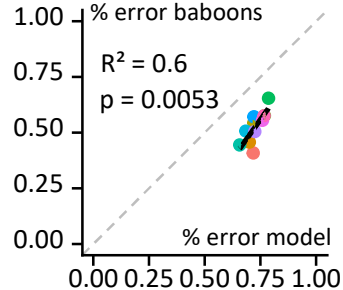
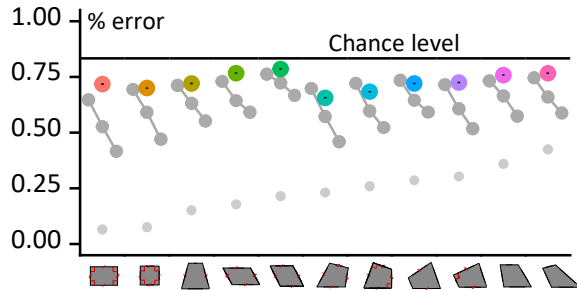
**CorNet S V1**



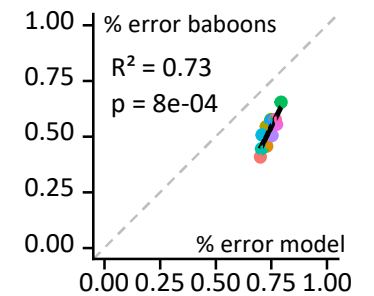
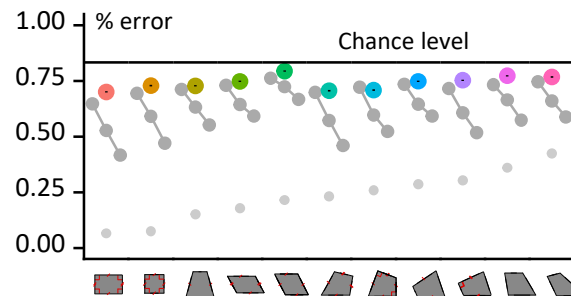
**CorNet S IT**



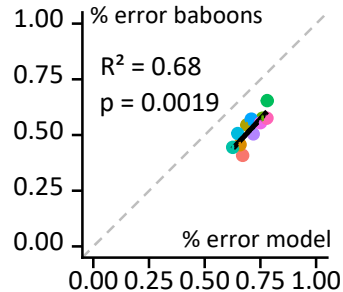
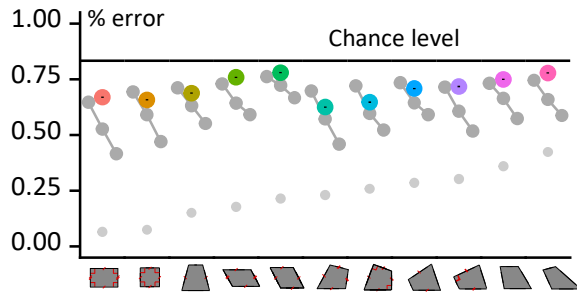
**CorNet S V2**



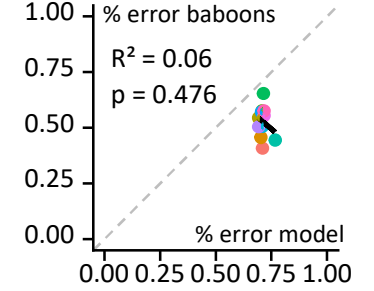
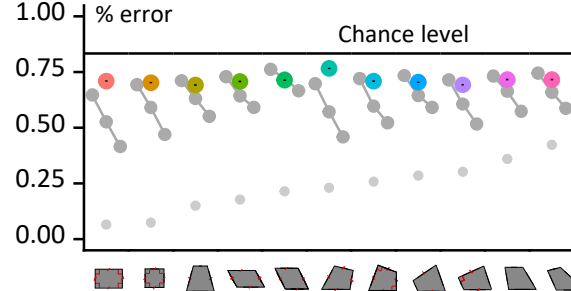
**area outlier**



**CorNet S V4**



**perimeter outlier**

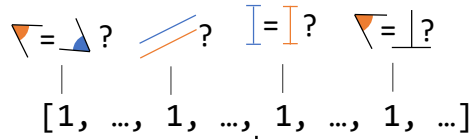




# Model 2: A symbolic model with discrete geometrical properties

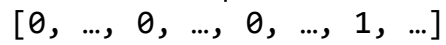


→ List internal properties



Compute  $\mathcal{L}_1$ -norm distance to predict complexity

→ List internal properties

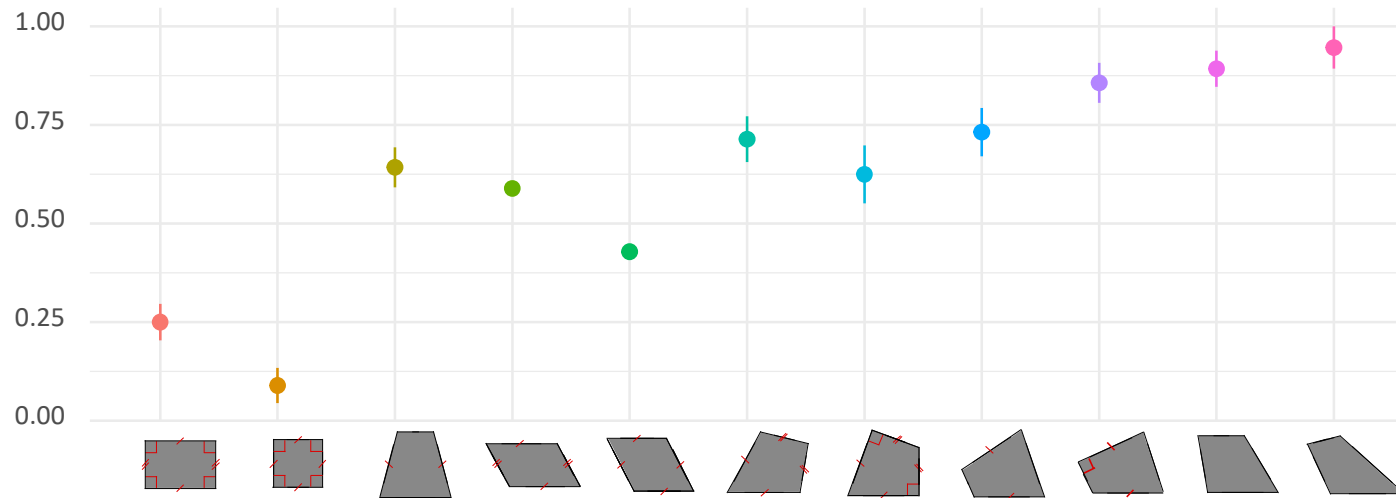


The symbolic model lists the discrete properties of the shapes (within a certain tolerance level)

- Equal angles
- Parallelisms
- Equal lengths
- Right angles

The ease of outlier detection is predicted by the number of properties that differ.

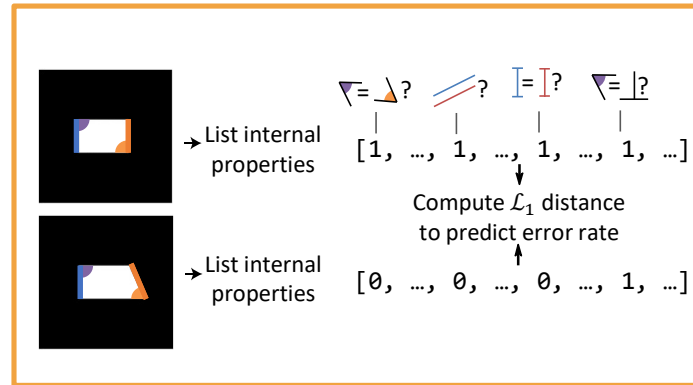
This model nicely predicts the shape regularity effect:



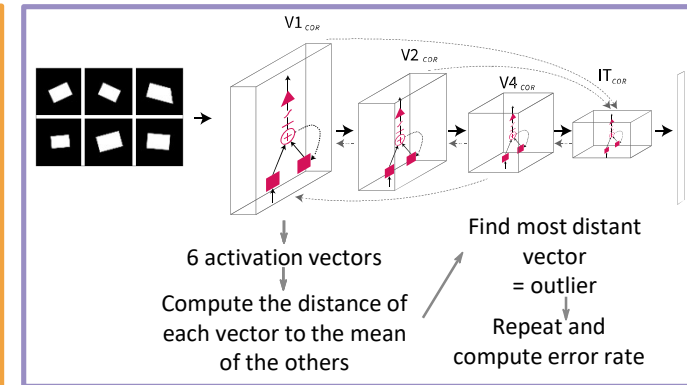
# A double dissociation between humans and baboons

- Baboon behavior is captured by a convolutional neural network model of the ventral visual pathway.
- A symbolic model captures human behavior, but fails to predict of the baboon data even at the individual level
- Himba and preschoolers rely on a mixture of the two strategies

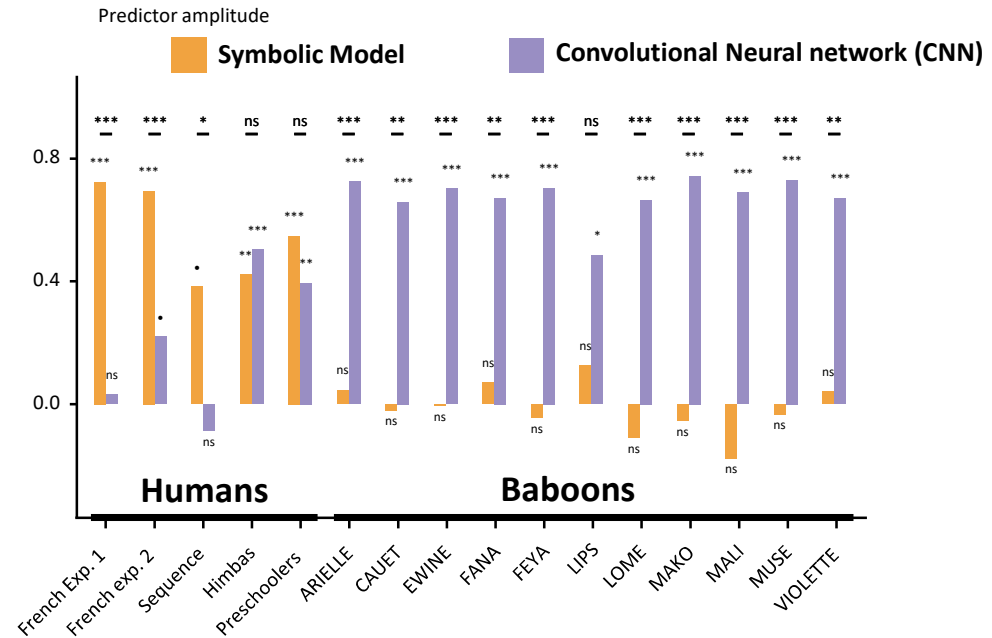
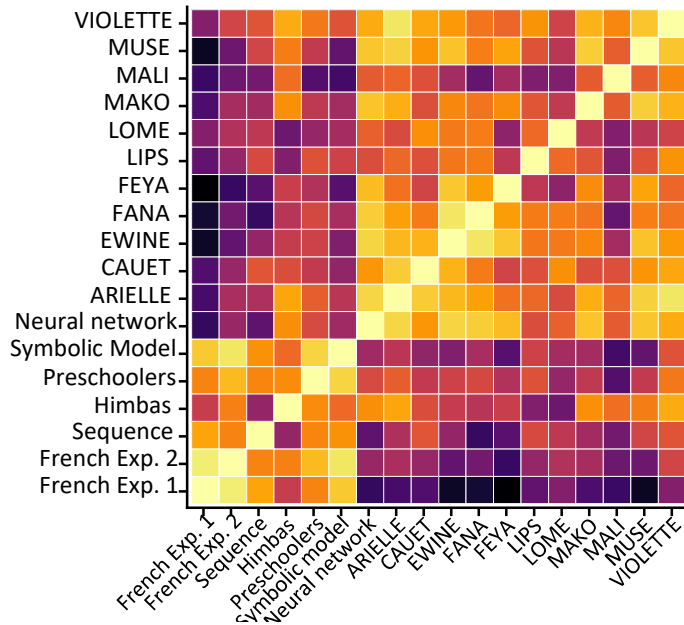
## Symbolic model



## Convolutional neural network (CNN)



Pearson Correlation ( $r$ ) 0.00 0.25 0.50 0.75 1.00

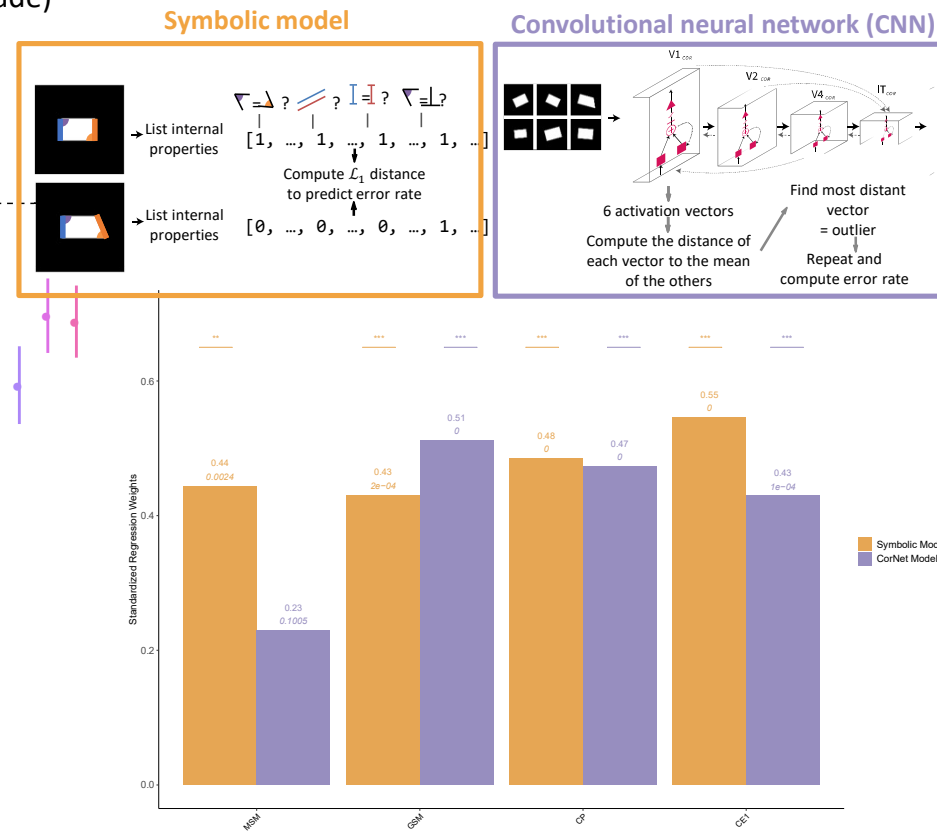
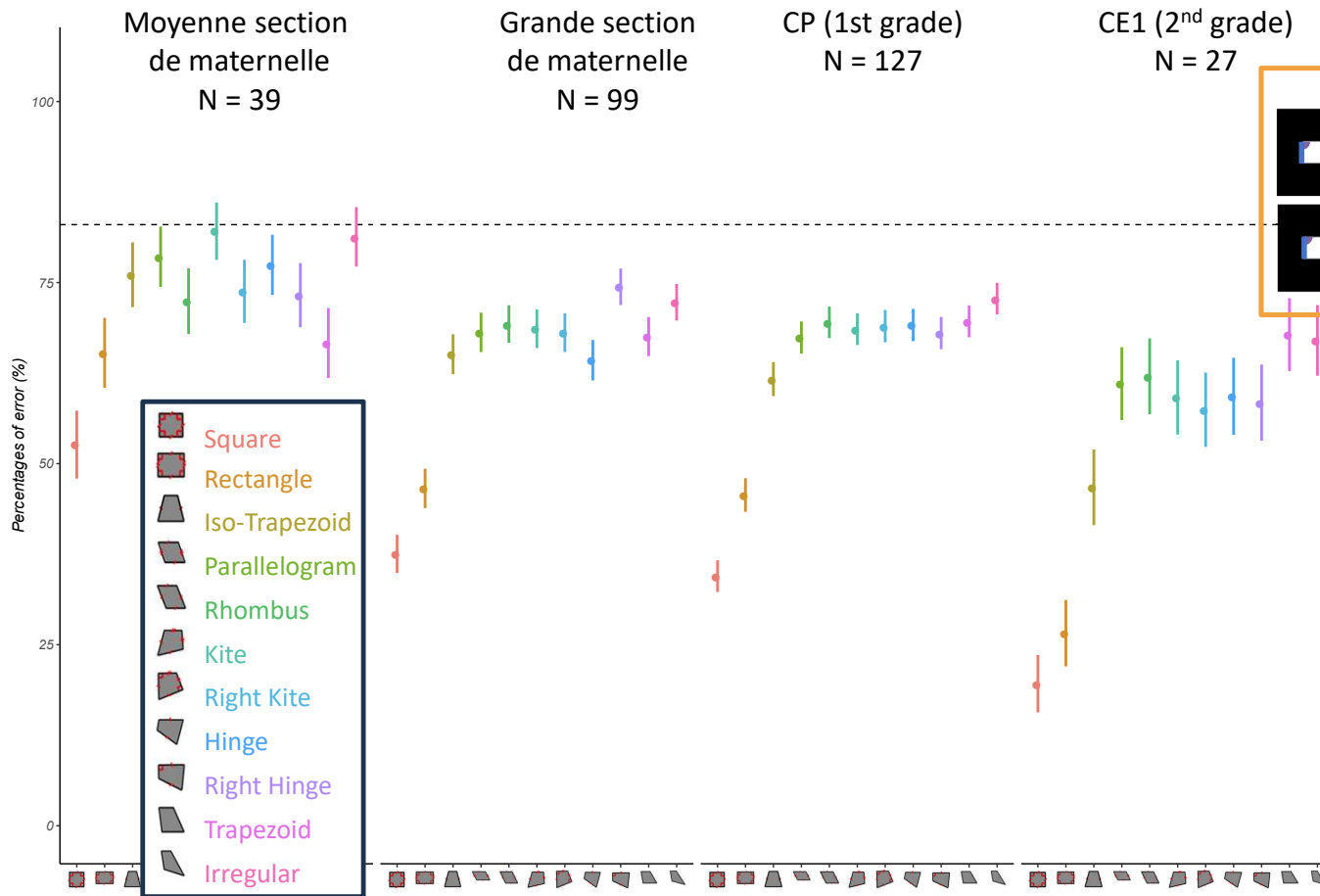


# Evolution of the shape regularity effect with education

Research by Théo Morfisse et al.



The symbolic model is already significant in the youngest children, but gains in goodness-of-fit with education.



# Could experience explain the human pattern?

## Training in an urban environment or “carpentered world”?

Could the urban environment in which humans live influence their geometric judgements?

We do find that the Himba exhibit a smaller geometrical regularity effect.

In our previous research, the Mundurucu were “less Euclidean” (Izard et al.).

Some visual illusions, such as the Ebbinghaus illusion, are affected by cultural context.

Bremner, A. J., Doherty, M. J., Caparos, S., de Fockert, J., Linnell, K. J., & Davidoff, J. (2016). Effects of Culture and the Urban Environment on the Development of the Ebbinghaus Illusion. *Child Development*, 87(3), 962-981.

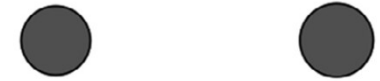
<https://doi.org/10.1111/cdev.12511>

However :

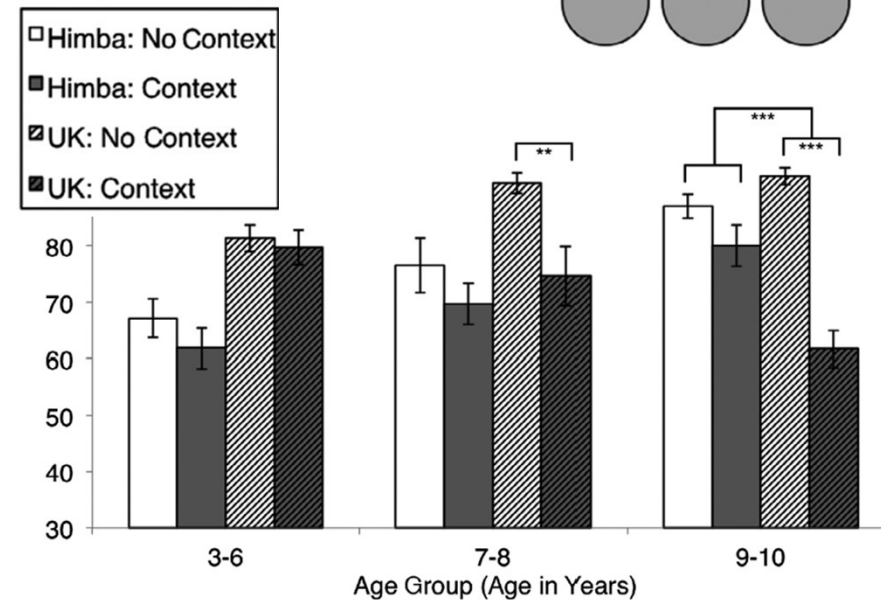
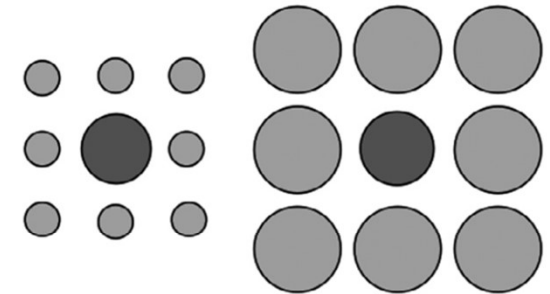
- the baboons live a world which is arguably more “carpentered” than the Himba, yet they have opposite results.
- The preschoolers that we tested are younger than the age of 7-8 at which Bremner et al. find that the cultural effect appears.



(A) No Context



(B) Misleading Context



# Could experience explain the human pattern?

Would more training with geometrical shapes suffice to induce the geometrical regularity effect ?

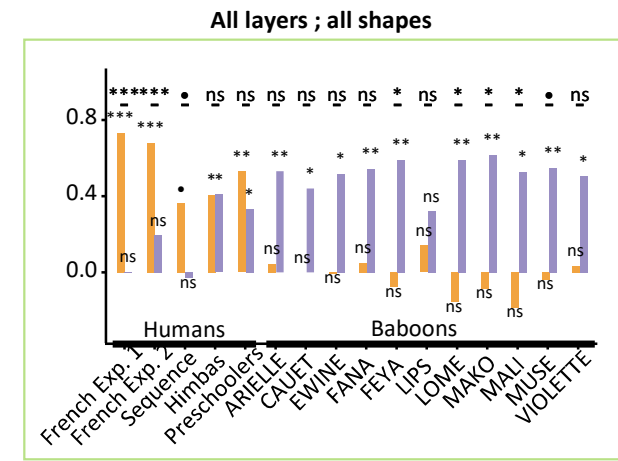
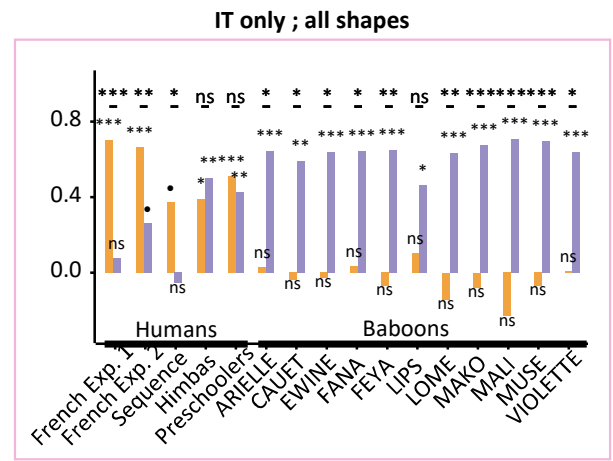
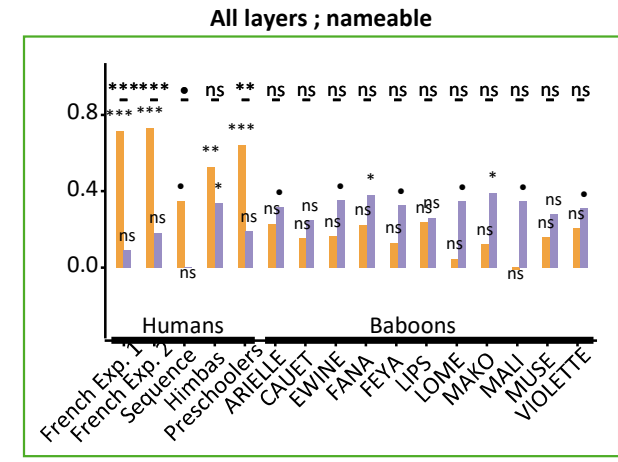
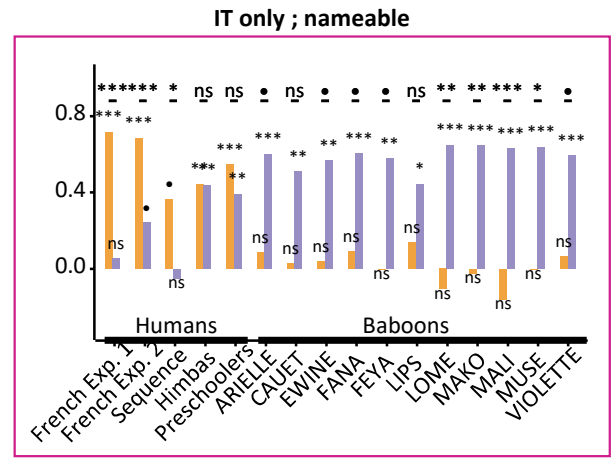
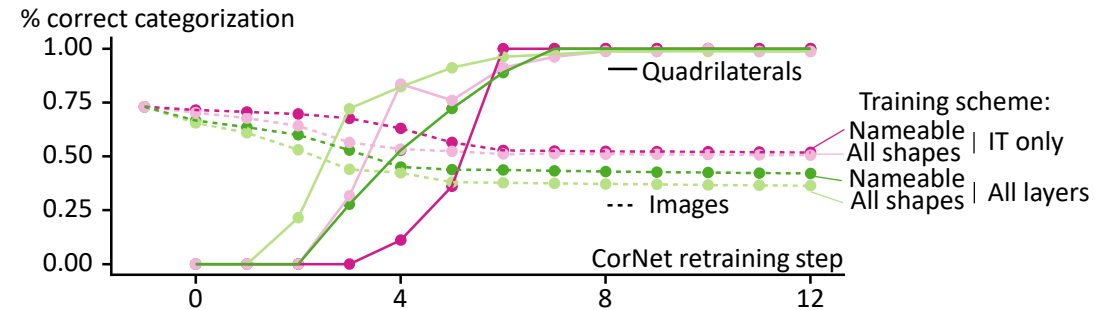
It is true that we are comparing very different populations: adults and children with considerable experience (at least with tools, body paintings, etc), versus baboons in a restricted environment.

However :

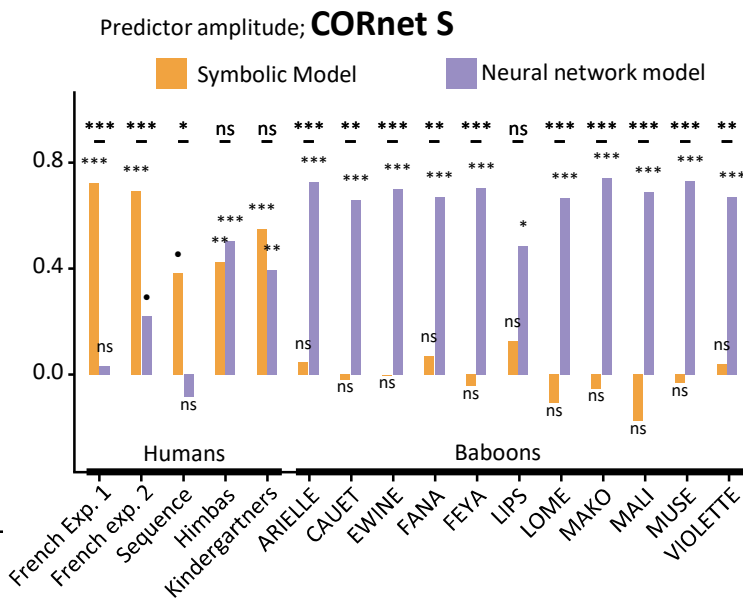
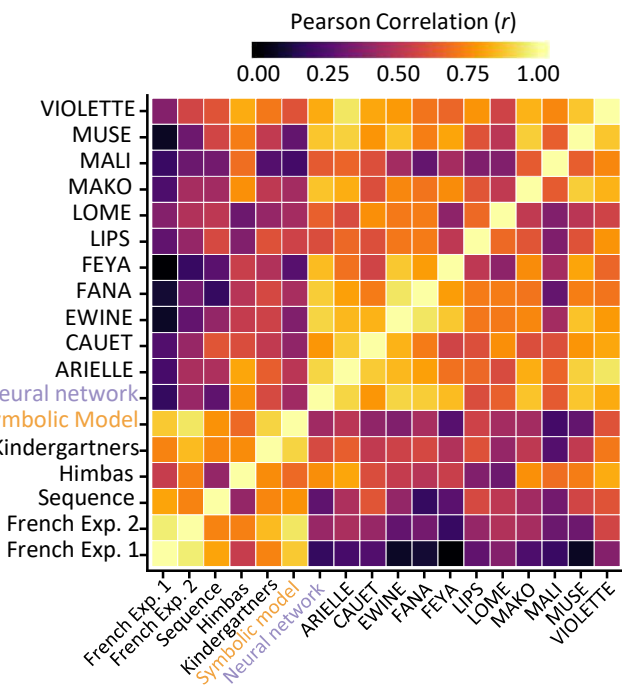
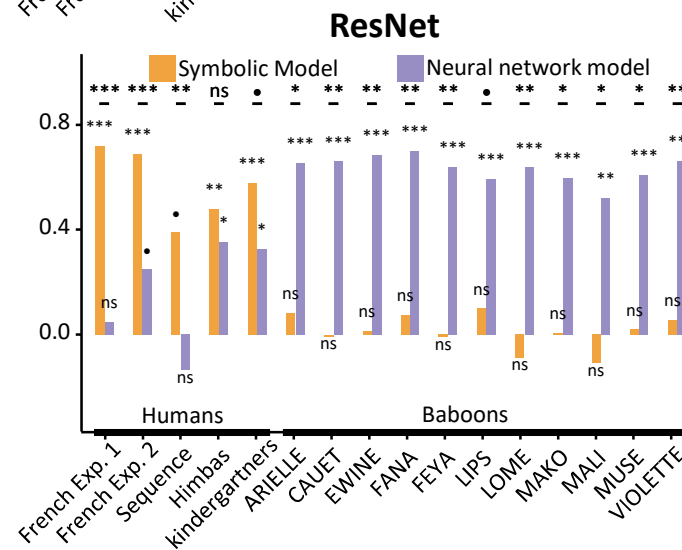
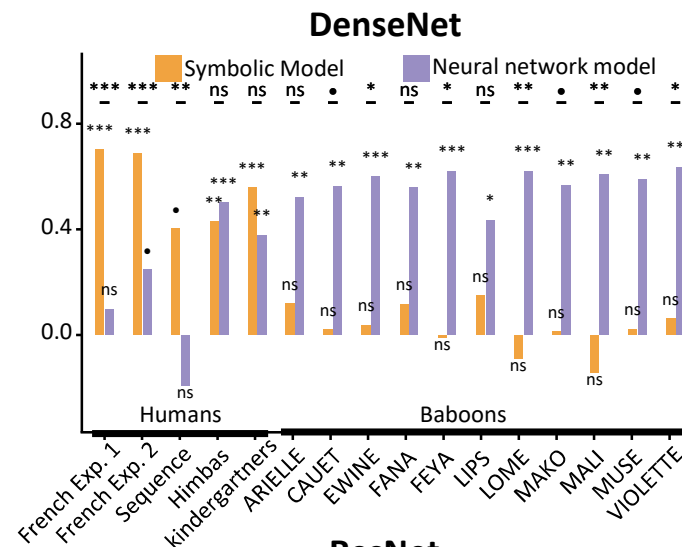
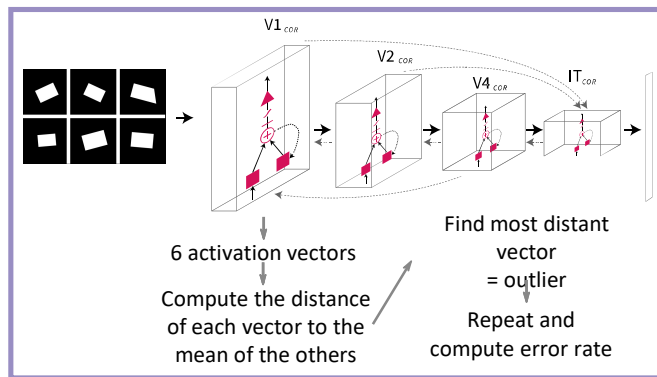
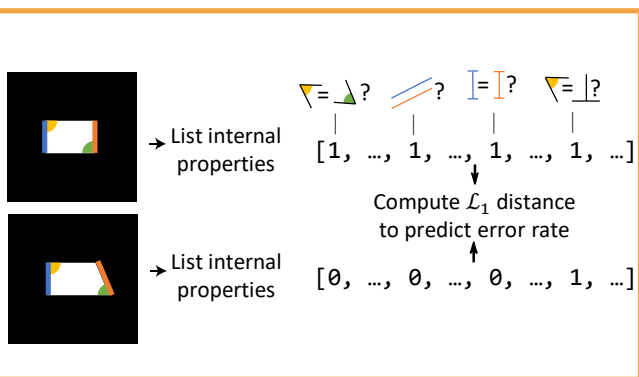
- The baboons received thousands of training trials.
- We trained the network to label our geometrical shapes with additional output units:
  - Either all 11 shapes, or just the shapes with names
  - Either by updating the entire network, or by changing the last layer (IT only)

The network reaches perfect scores on novel displays of those shapes, but predictions are unchanged.

We believe that success in this task is due to a specific architecture, which is present in humans, but absent in non-human primates and CNNs.



# The double dissociation (symbolic vs connectionist) is robust to the underlying CNN



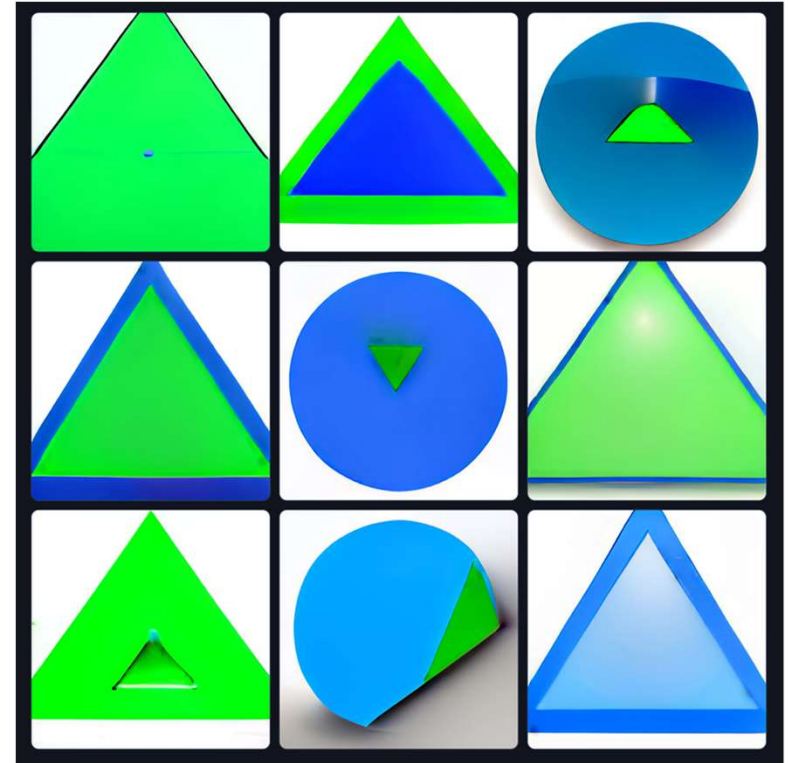


# Can artificial neural networks capture human geometrical skills?

Imagine Darth Vader in a 19th century dress



Imagine a green triangle to the left of a blue circle.



Current AI often has a problem with representing nested logical and geometrical structures

Generated with Dall-e 2 (Ramesh et al., 2022)  
With thanks to Théo Desbordes

# Current AI systems often fail at intuitive mathematical tasks



neural net guesses memes  
@ResNeXtGuesser

Image prediction: ping-pong ball  
Confidence: 99.99%

**ChatGPT3 on fractions :**

Me: what is larger, two fourths or one half?

ChatGPT :

One half is larger than two fourths.

Two fourths can be written as  $\frac{2}{4}$ , which is equal to  $\frac{1}{2}$ . Therefore, one half is equal to two fourths.

Me: didn't you just contradict yourself?

ChatGPT: I apologize, I made a mistake in my previous answer. Two fourths ( $\frac{2}{4}$ ) is smaller than one half ( $\frac{1}{2}$ ).

Me : Are you sure?

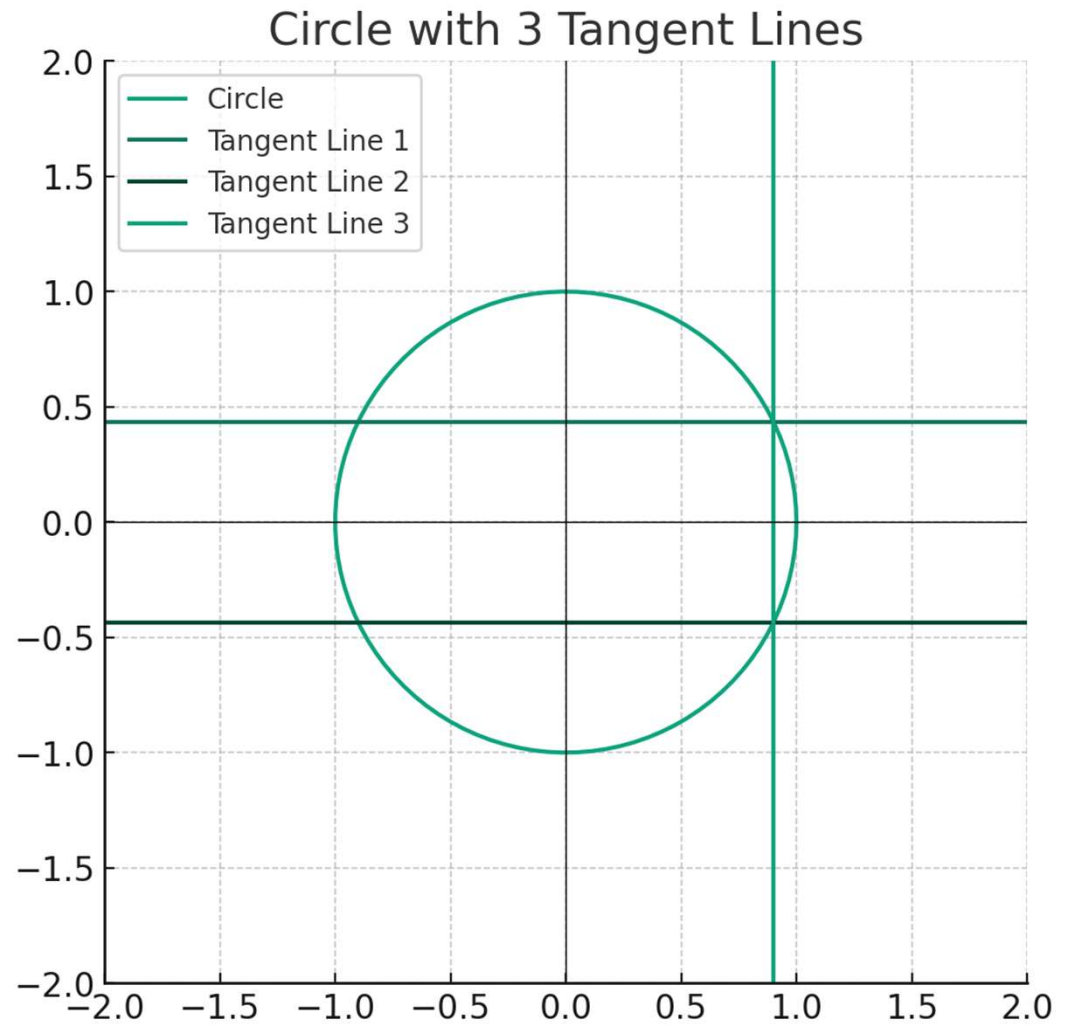
ChatGPT: Yes, I am sure. Two fourths ( $\frac{2}{4}$ ) is equal to  $\frac{1}{2}$ . Since  $\frac{1}{2}$  is larger than  $\frac{2}{4}$ , it follows that one half is larger than two fourths.

# Current AI systems often fail at intuitive mathematical tasks

## ChatGPT4 on geometry :

Me: Can you draw a circle and 3 tangent lines?

ChatGPT4: Here is a visual representation of a circle with three tangent lines. The circle is centered at the origin with a radius of 1. The tangent lines are approximated to touch the circle near the top, bottom, and right side



# Current AI systems often fail at intuitive mathematical tasks

## ChatGPT4 on geometry :

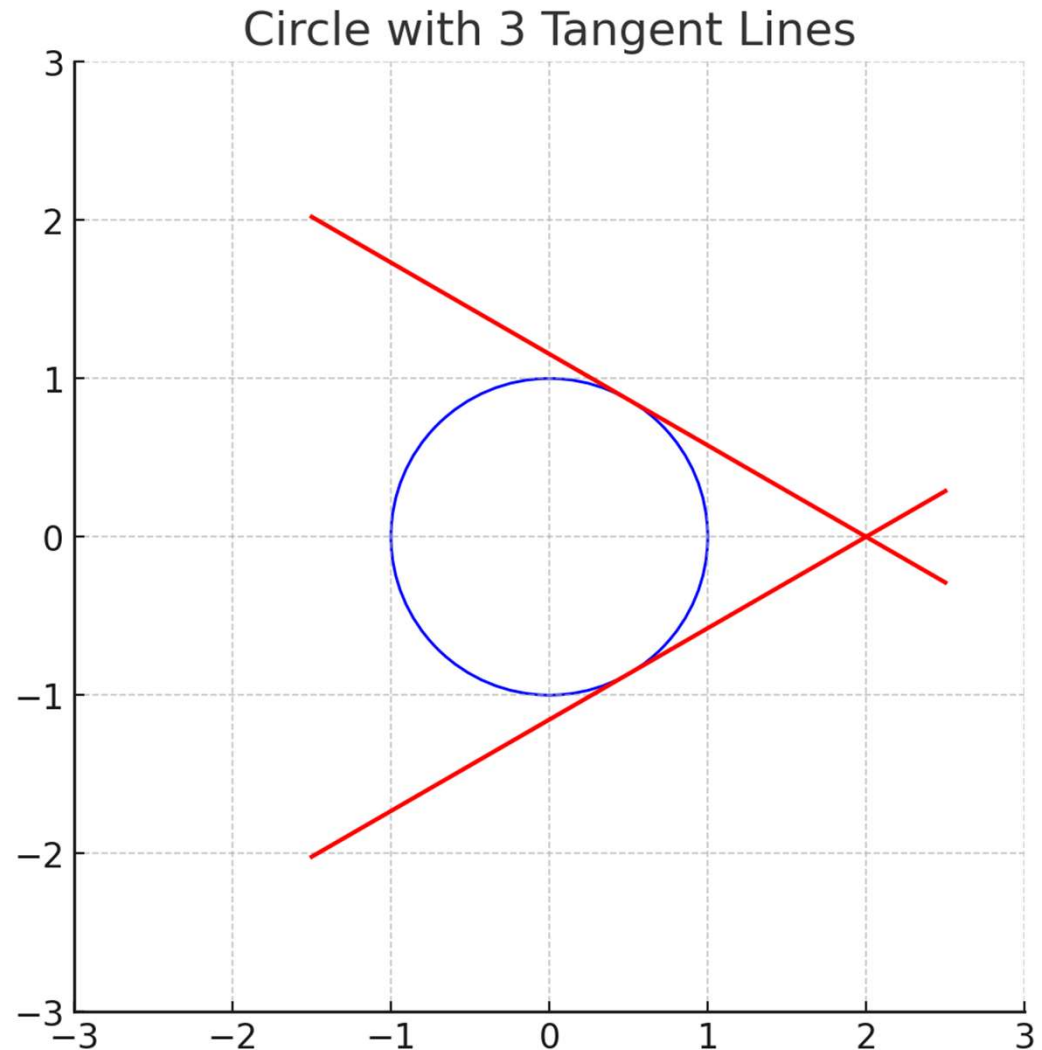
Me: Can you draw a circle and 3 tangent lines?

ChatGPT4: Here is a visual representation of a circle with three tangent lines. The circle is centered at the origin with a radius of 1. The tangent lines are approximated to touch the circle near the top, bottom, and right side

Me: Can you try again? The three lines have to be tangent to the circle

ChatGPT4: Here's an updated diagram showing a circle with three tangent lines. The lines are drawn at angles of 0, 60, and -60 degrees to the horizontal, tangential to the circle at these points.

Conclusion: ChatGPT4 cannot count, doesn't "see" math, and merely parrots facts.



# Can advanced artificial neural networks solve our geometric tasks?

Campbell, D., Kumar, S., Giallanza, T., Griffiths, T. L., & Cohen, J. D. (2024). Human-Like Geometric Abstraction in Large Pre-trained Neural Networks (arXiv:2402.04203). arXiv. <https://doi.org/10.48550/arXiv.2402.04203>

3 recent AI vision models are challenged with our test :

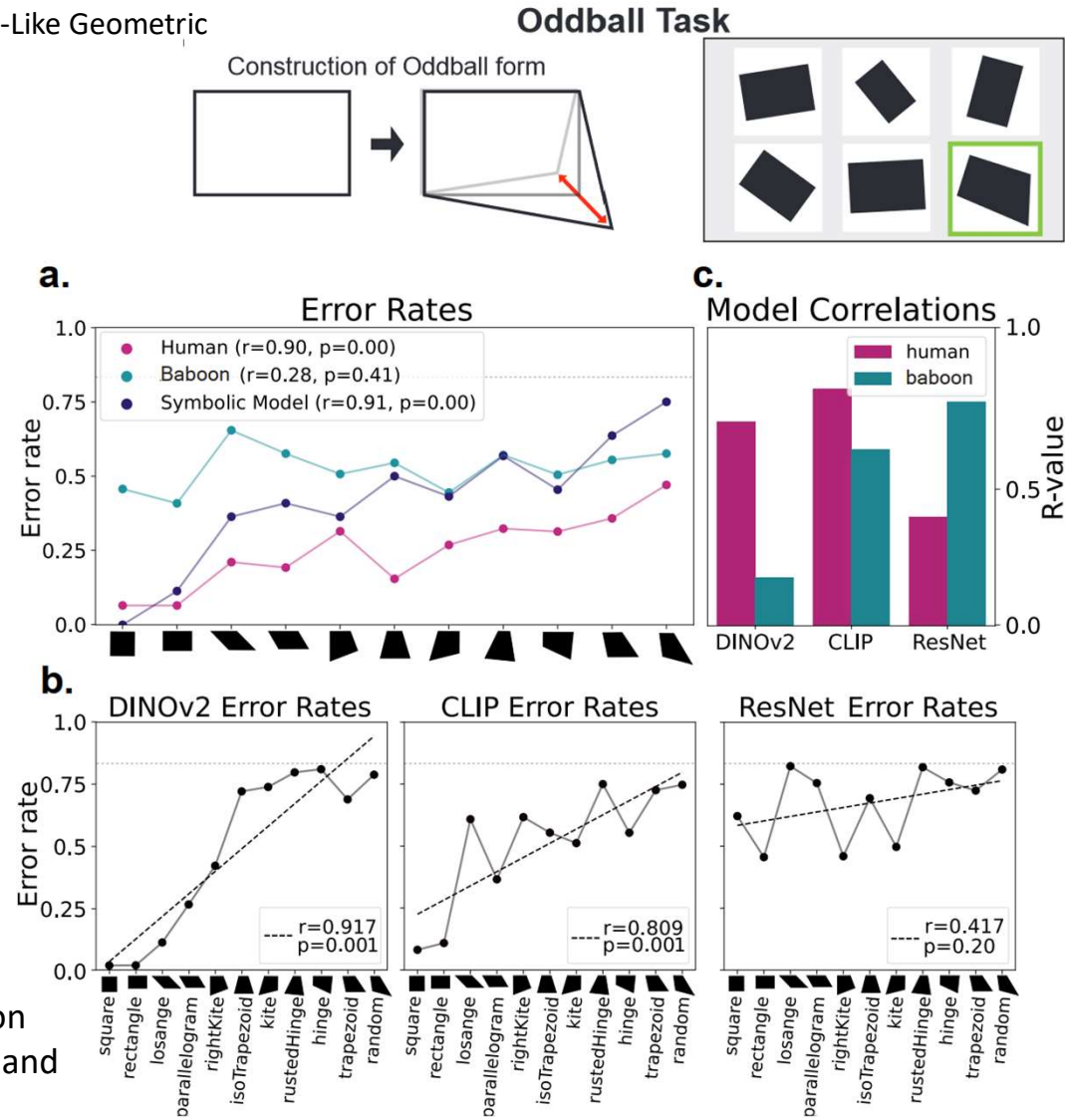
- ResNet : a convolutional neural network
- CLIP : a transformer for vision and language
- DinoV2 : a large, 1-billion parameter vision transformer, trained both to identify image similarity up to affine augmentations, and to complete image patches.

Like us, they extract the embeddings evoked by each of the six shapes (in the last layer?), and define the outlier as the one which is maximally different from the mean of the others.

They replicate our findings: ResNet, a simple convolutional network, does not predict the human geometrical regularity effect – but only the baboon data.

However, they also find that DinoV2 and, to a lesser extent, CLIP can predict human behavior relatively well (though not with perfect linearity, as the symbolic model does).

Interestingly, CLIP is in between, captures a bit of both human and baboon data, and resembles the profile of human preschoolers (it treats squares and rectangles as radically different).





# Testing the proposed dual-route model of geometric processing

Mathias Sablé-Meyer et al, *submitted*

Idea = use **representational similarity analysis** : which shapes does a given brain area treat as « similar » ?

Cichy, R. M., Pantazis, D., & Oliva, A. (2014). Resolving human object recognition in space and time. *Nature Neuroscience*, 17(3), Article 3.  
<https://doi.org/10.1038/nn.3635>

Kriegeskorte, N., Mur, M., & Bandettini, P. A. (2008). Representational similarity analysis-connecting the branches of systems neuroscience. *Frontiers in systems neuroscience*, 2, 4.

Behavior



Functional MRI  
in adults and in children



Magneto-encephalography  
in adults

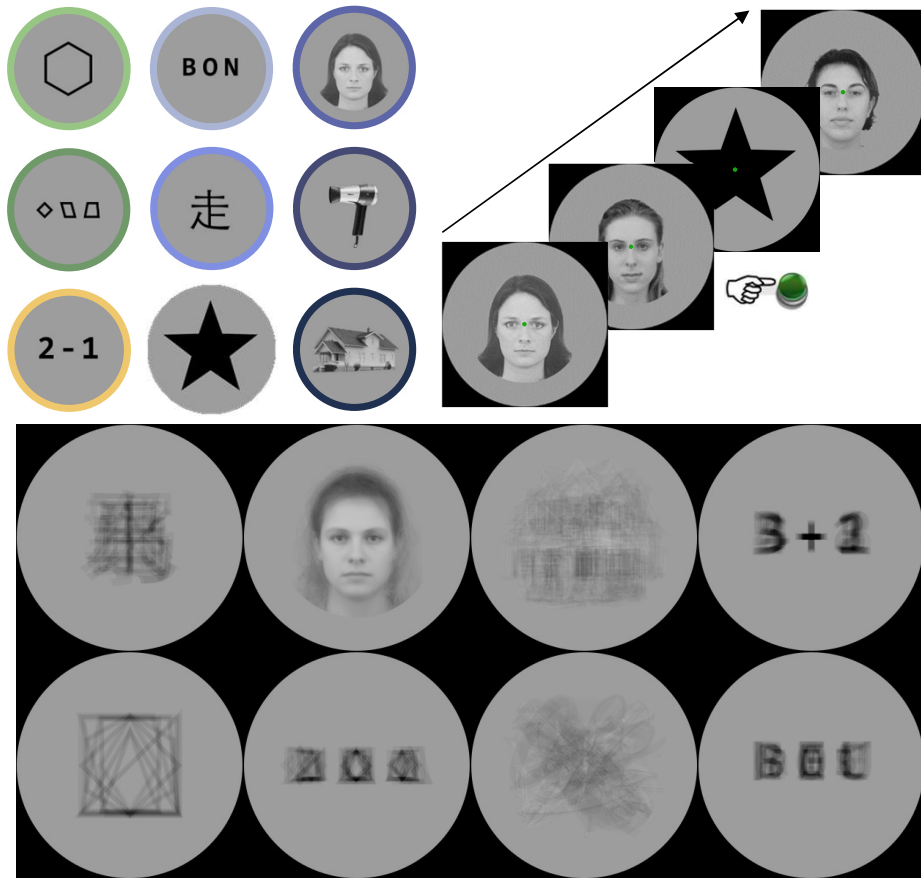




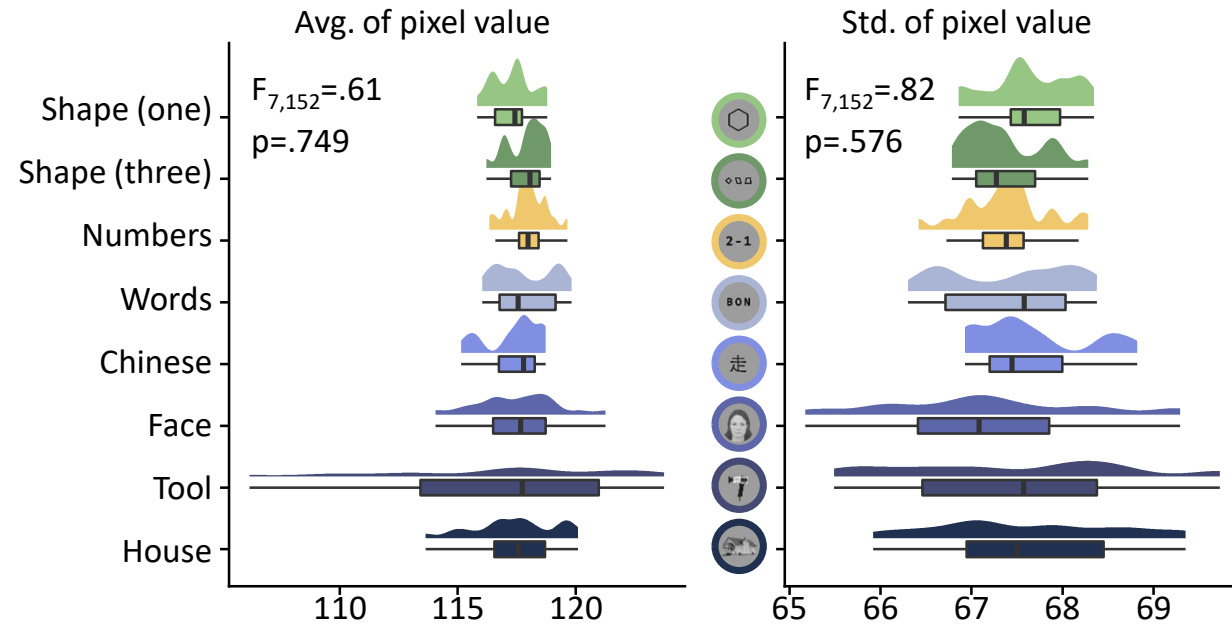


# Testing the predicted double dissociation using 3T fMRI

Results from a category localizer



- N=20 Adults plus N=20 first graders
- Tasks:
  - **Category localizer**
  - Number line task
  - Geometry task
- Preprocessed with fmriprep, analysis with nilearn, 2<sup>nd</sup> level smoothing 8mm, cluster obtained with non-parametric permutation tests

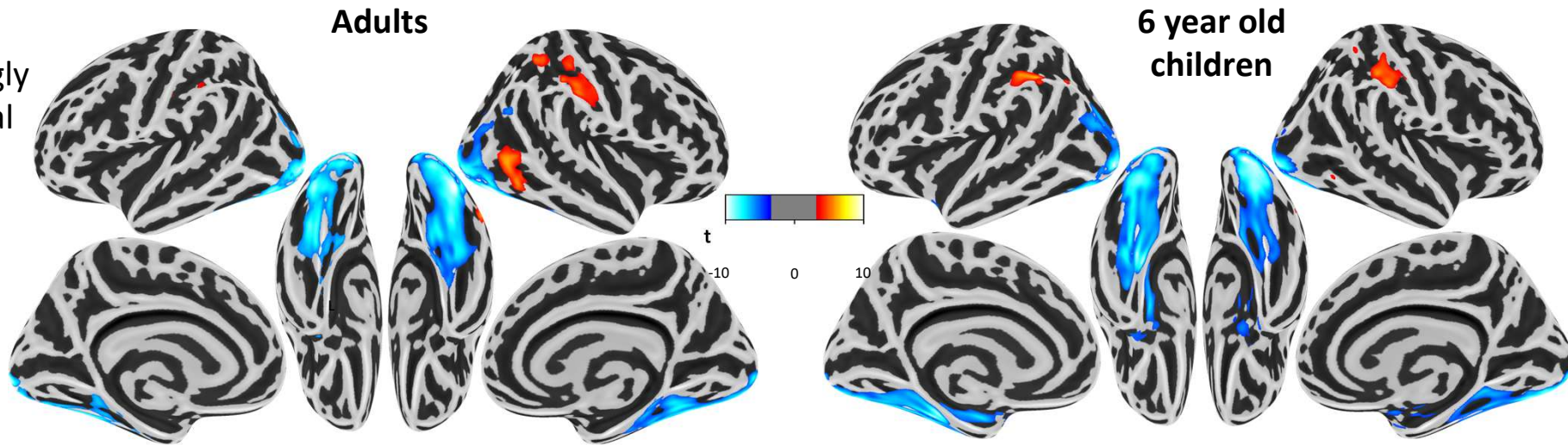
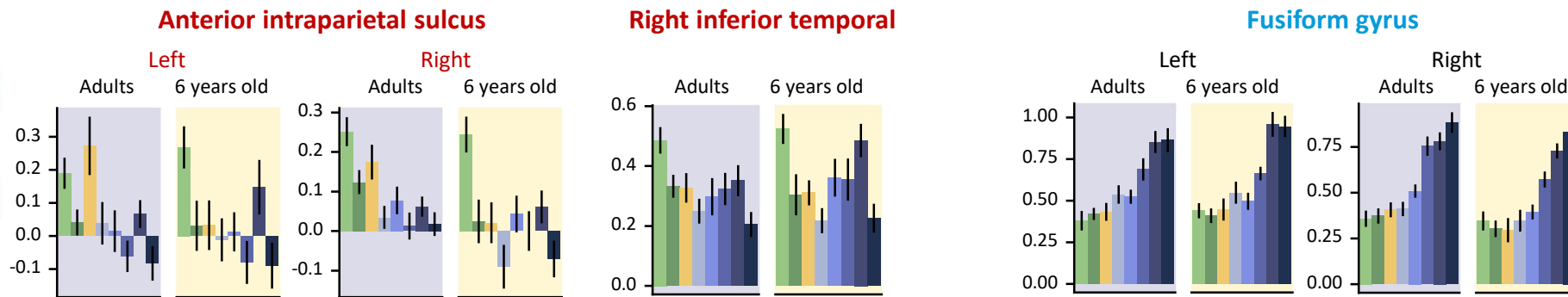


# The dorsal math-responsive network is involved in geometric shape recognition

fMRI indicates that geometric shapes induce only a weak activation of the ventral visual pathway.



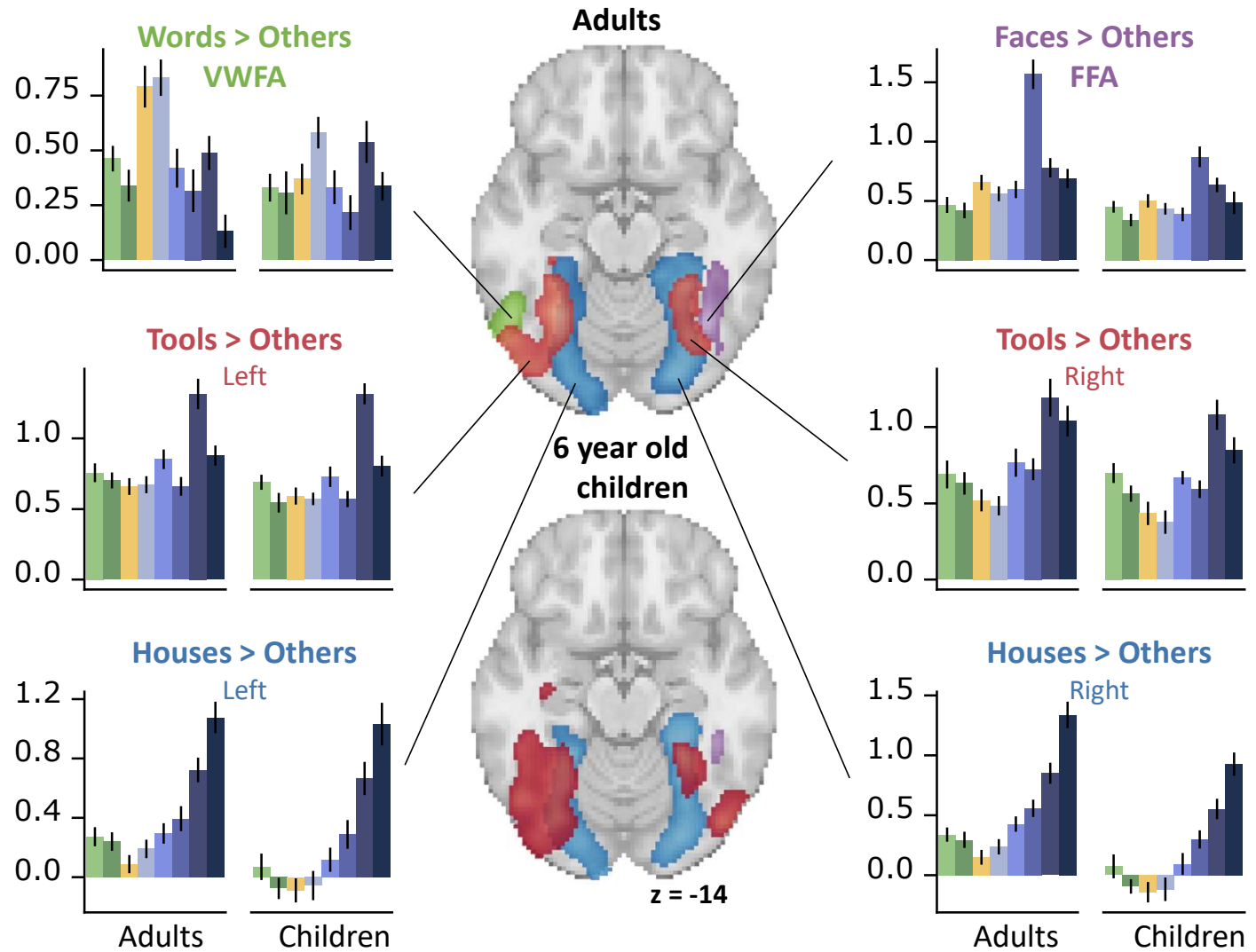
Instead, they strongly activate the bilateral intraparietal and right inferior temporal math-responsive network.

# Various ventral visual regions are not particularly fond of geometric shapes

By studying the fMRI response of subject-specific ROIs, we can reject several plausible alternative theories about geometric shapes.

- They are not similar to **other learned symbols that convey linguistic meaning** (no strong activation in the VWFA)
- Although they differ by **graded values of angles and lengths**, they are not encoded in the FFA, a region sensitive to graded differences between faces.
- Neither do they activate regions interested in **man-made objects**
- Nor in those interested in the **encoding of places and spatial navigation** (parahippocampal place area, PPA)



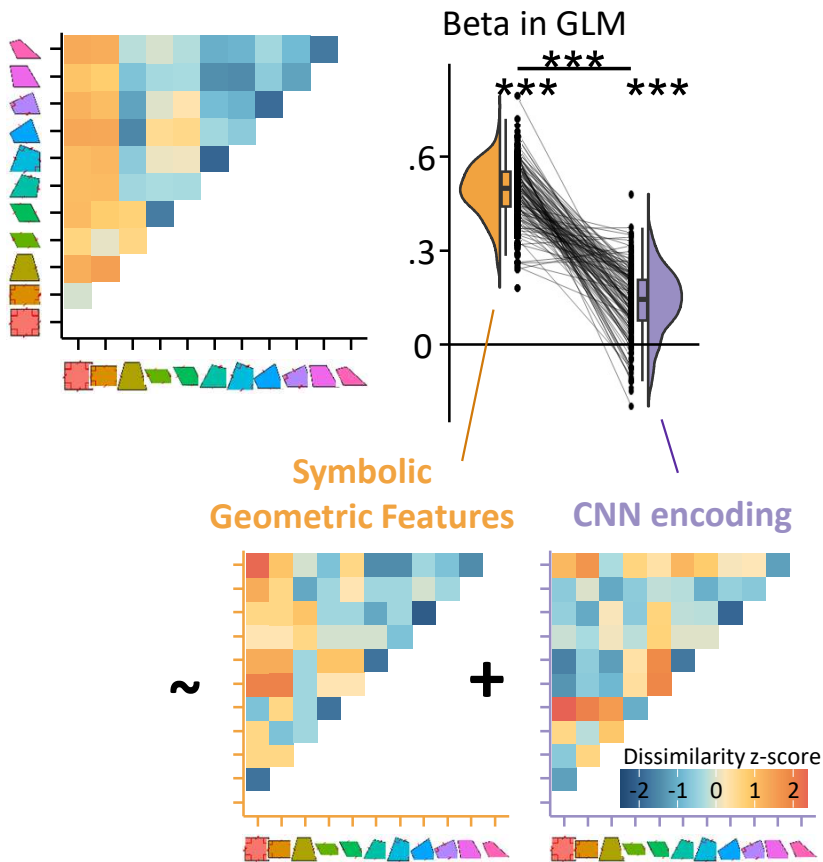
# Representational similarity supports the existence of a distinct symbolic geometry code

Mathias Sablé-Meyer obtained similarity matrices from both behavior and 3T fMRI.

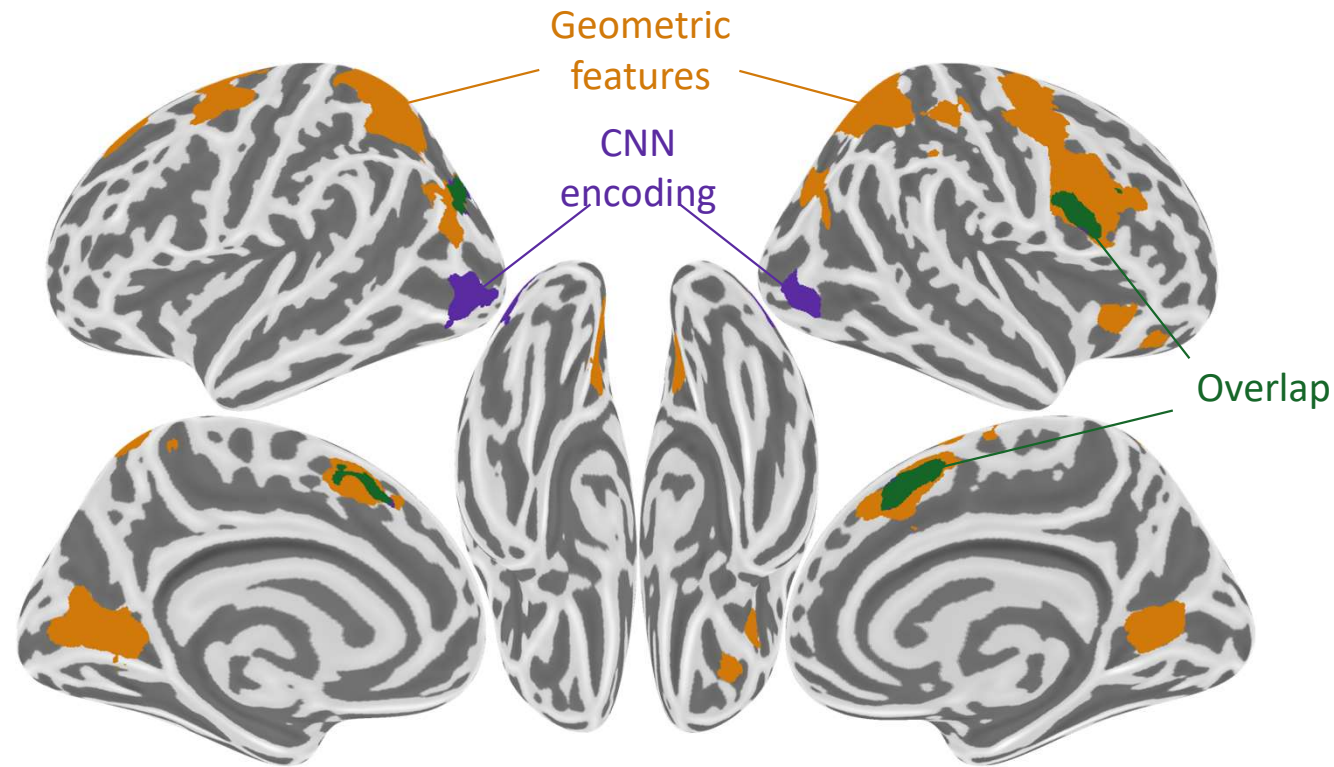
Behavior was dominated by symbolic geometric features, but also influenced by visual similarity (as modeled by a CNN).

fMRI was also dominated by symbolic geometry, with CNN encoding capturing lateral occipital activity.

## Human similarity ratings



## Searchlight RSA of fMRI data

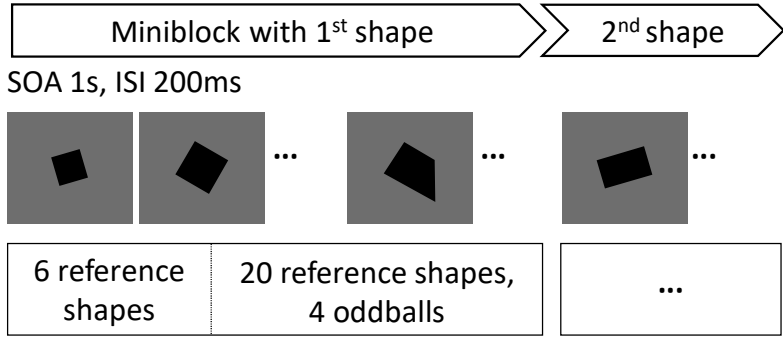




## MEG recordings : passive shape encoding and novelty effects



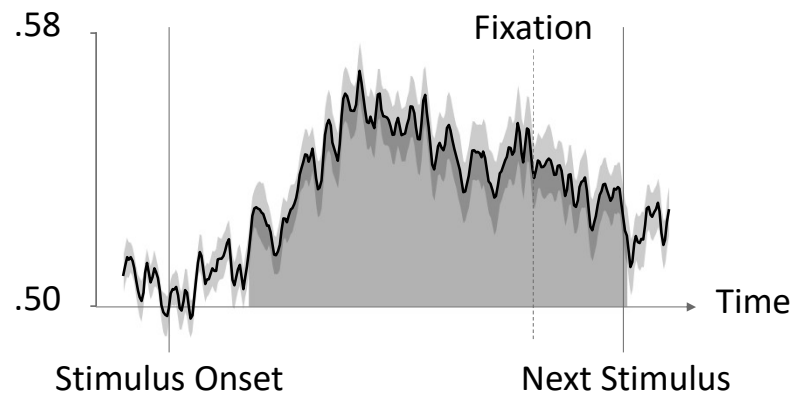




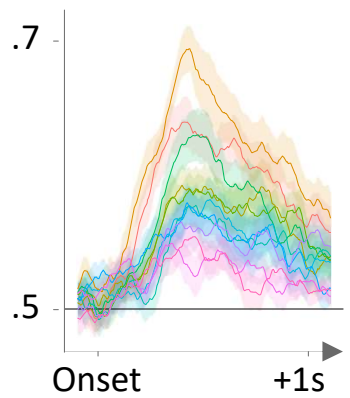
# Replicating the geometric regularity effect in a passive task in MEG



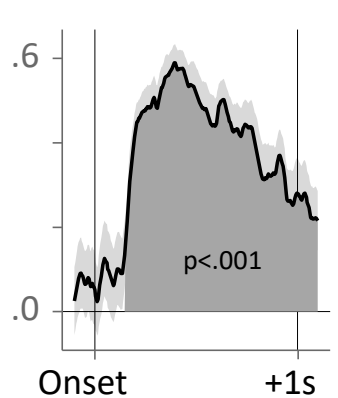
Decoding ROC AUC: reference vs. oddball



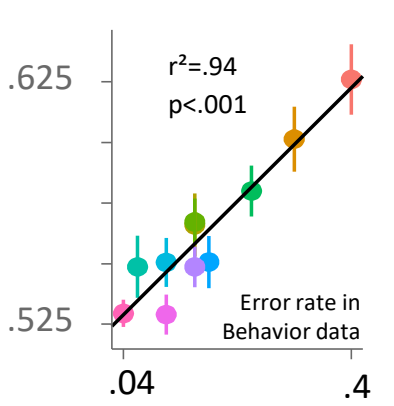
Decoding Performance Per shape



Correlation w/ Symbolic model

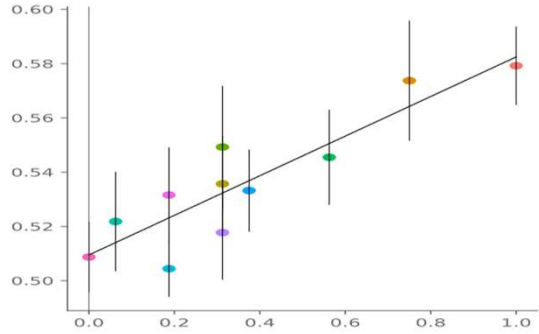
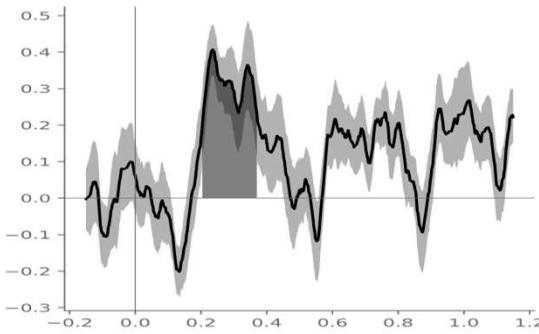
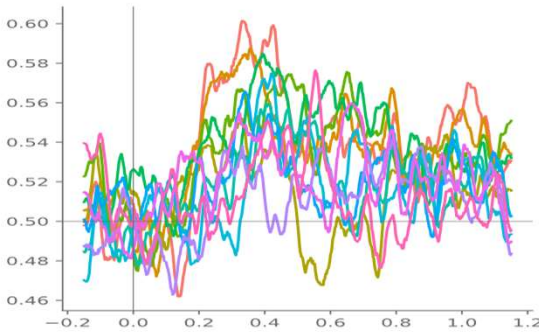
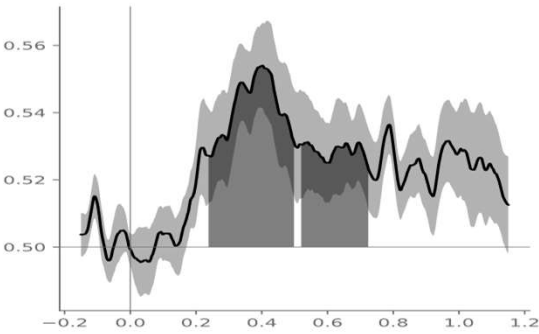


Cluster's avg. Correlation

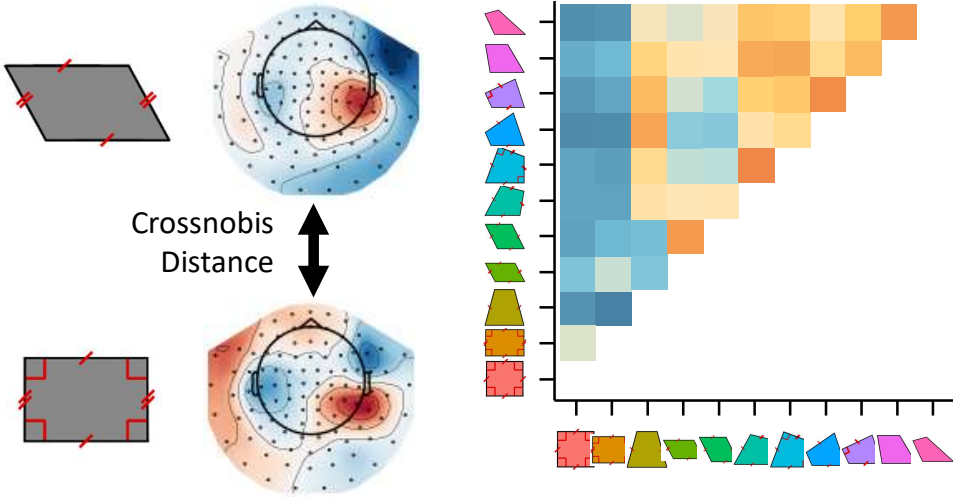


- Square
- Rectangle
- Iso. Trapezoid
- Parallelogram
- Rhombus
- Kite
- Right Kite
- Hinge
- RightHinge
- Trapezoid
- Irregular

Ongoing replication with intracranial recordings in epileptic patients (N=16, Work with Yunzhe Liu)

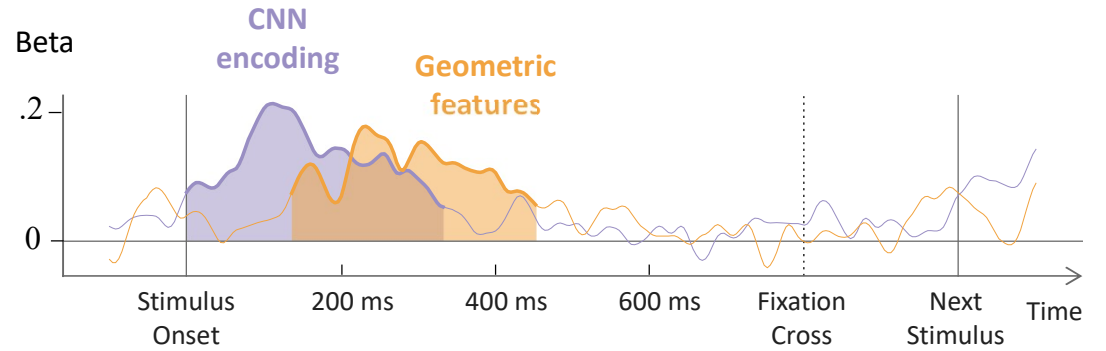


Human MEG data at time t



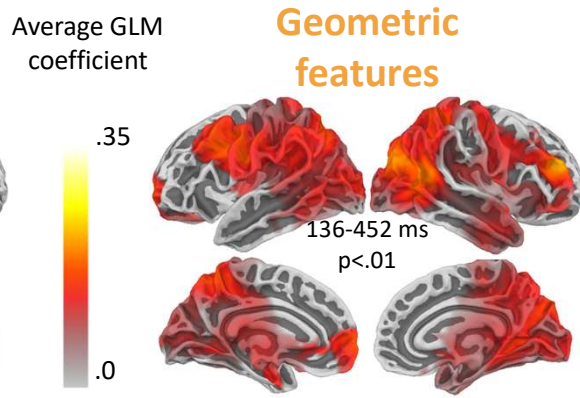
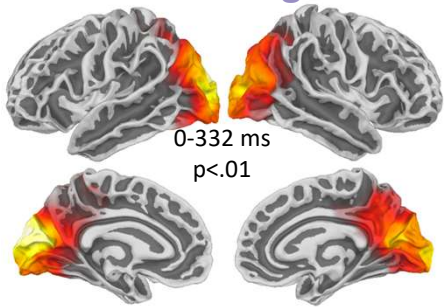
# MEG data support a two-stage process of shape processing (RSA analysis)

Regression of representational similarity on the two predictors

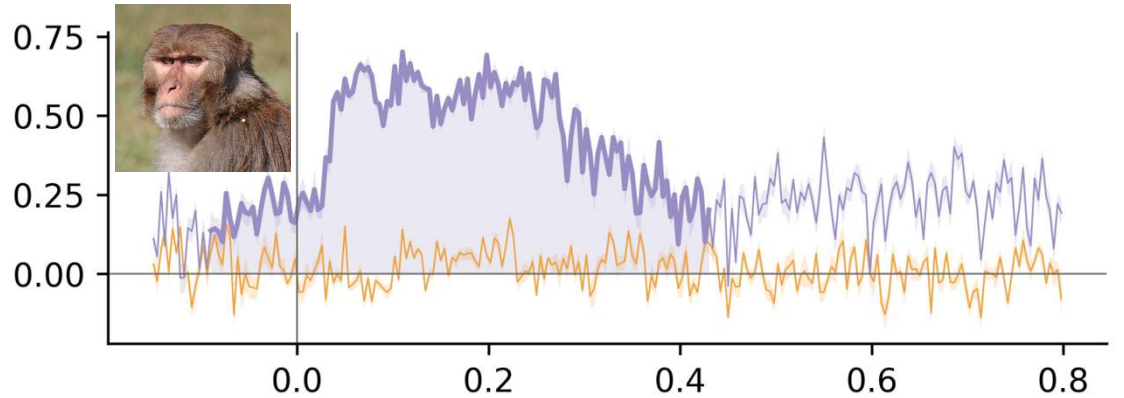


Ongoing replication with intracranial recordings in epileptic patients

**CNN encoding**



Macaque IT data (N=1; highly preliminary; work with Paolo Papale)





# Conclusions

Even a task as simple as encoding a square suffices to evidence a uniquely human knack for geometry.

The **geometric regularity effect**, a **human universal**, is absent in baboons.

There are **two pathways for the human perception of geometric shapes** – one well-captured by CNNs and shared with non-human primates, and another symbolic and possibly unique to humans.

The symbolic pathway emerges around 5-6 years of age in children  
→ What is the role of experience?

This work includes a **severe critique of artificial neural networks**: they do not capture human geometric abilities  
→ How to model the language of shapes?

