COLLÈGE DE FRANCE
1530

Chaire de Physique de la Matière Condensée
Antoine Georges

De l'effet Hall quantique aux matériaux moirés

- Topologie et géométrie des matériaux quantiques -

Cours 2 – *Topologie et géométrie des états de Bloch*

Cycle 2025-2026
6 mai 2026

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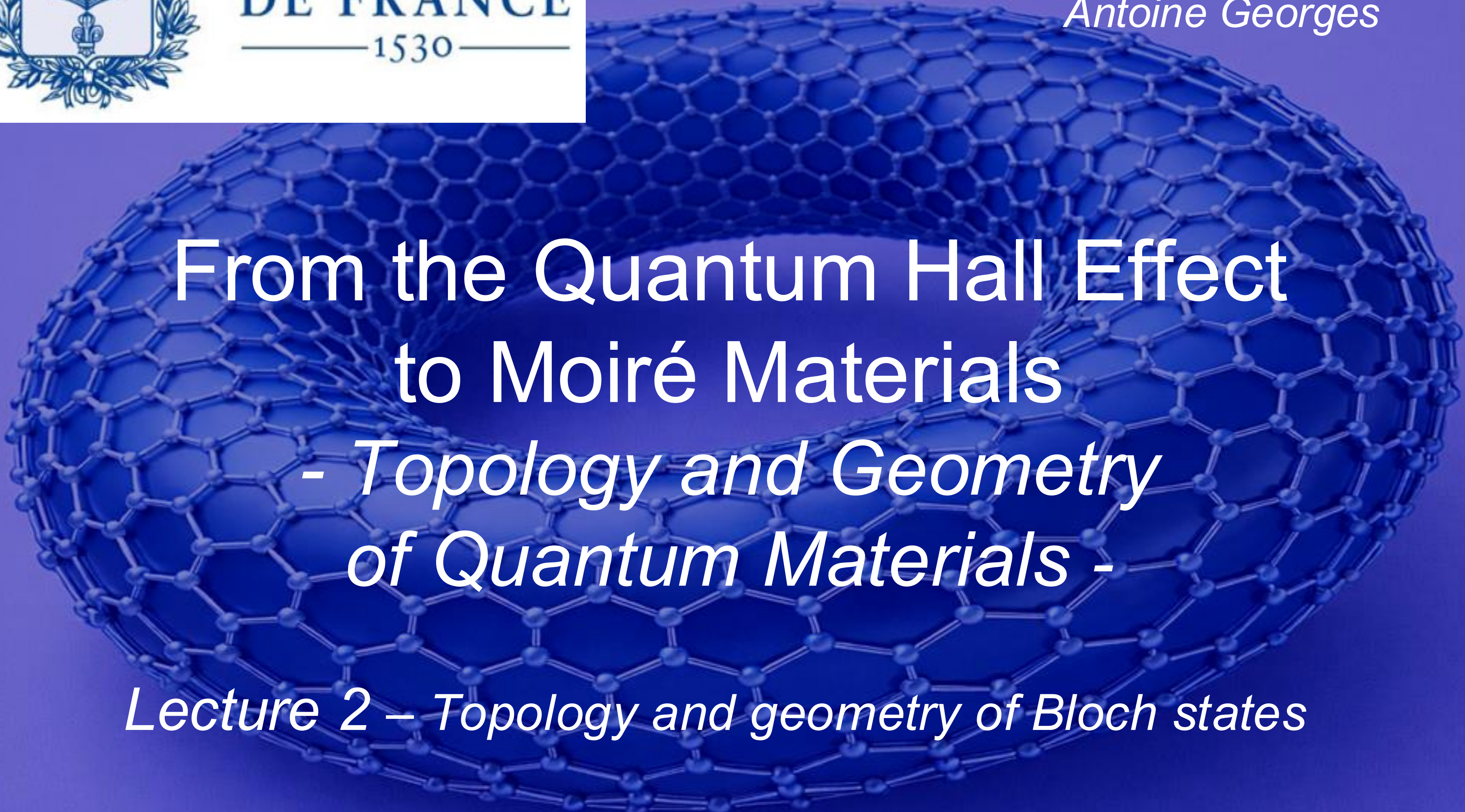

From the Quantum Hall Effect to Moiré Materials

- Topology and Geometry of Quantum Materials -

Lecture 2 – *Topology and geometry of Bloch states*

2025-2026 Lectures
May 6, 2026

Mercredi 6 mai

Cours 1 : L'effet Hall quantique entier

Cours 2 : Topologie et géométrie des états de Bloch

Mercredi 13 mai

Cours 3 *Chern Insulators, Haldane Model*

SÉMINAIRE :

Dmitri Efetov (LMU, Munich)

Engineering strong interactions and topology in moiré flat-bands

Lundi 18 mai [Date exceptionnelle]

SÉMINAIRE (11H00):

Nathan Goldman (LKB - CNRS, Collège de France et Université Libre de Bruxelles)

Correlated Topological Quantum Matter: From Abstract Invariants to Practical Topological Markers

Mercredi 20 mai

Cours 4 : L'effet Hall quantique fractionnaire

SÉMINAIRE :

Nicolas Regnault (Flatiron Institute et École Normale Supérieure)

Fractional Chern Insulators: toy models, moiré and new mysteries

Mercredi 20 mai

Cours 4 : L'effet Hall quantique fractionnaire

SÉMINAIRE :

Nicolas Regnault (Flatiron Institute et École Normale Supérieure)

Fractional Chern Insulators: toy models, moiré and new mysteries

Mercredi 27 mai

Cours 5 : Matériaux moirés - introduction

SÉMINAIRE :

Rebeca Ribeiro-Palau (C2N - Université Paris-Saclay)

Topological states in moiré materials

Mercredi 3 juin

Cours 6 : Géométrie quantique et supraconductivité

SÉMINAIRE :

Gwendal Fève (École Normale Supérieure)

Electron optics experiments in quantum Hall conductors: from single electrons to anyons

Mailing List

(Weekly announcement of lecture and seminar, etc.)

Send email to: listes-diffusion.cdf@college-de-france.fr
Subject line: subscribe chaire-pmc.ipcdf
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Website:

https://www.college-de-france.fr/site/antoine-georges/index.htm

Lectures are recorded
Videos are available on the CdF website and on YouTube

Outline – Lecture 2

- Bloch states in a periodic potential: reminders
- Topology of Bloch states: TKNN Thouless, Kohmoto, Nightingale, den Nijs PRL 49, 405 (1982)
- Transport and Hall conductance
- Experimental visualization of Berry curvature
- Graphene bandstructure

Bloch Theorem

d=1:

$$\psi_{k\nu}(x) = e^{ikx} u_{k\nu}(x) \;\; , \;\; u_{k\nu}(x+a) = u_{k\nu}(x)$$

$k \in [-\pi/a, +\pi/a]$ Brillouin zone

periodic function

More generally (arbitrary Bravais lattice):

$$\psi_{\mathbf{k}\nu}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}\nu}(\mathbf{r})$$

$$u_{\mathbf{k}\nu}(\mathbf{r}+\mathbf{T}) = u_{\mathbf{k}\nu}(\mathbf{r}) \;\; \forall \mathbf{T} \in L$$

k in Brillouin zone of reciprocal lattice

Proof uses [H,T]=0 and simultaneous diagonalization of translations.

Note: This property is the key behind the \`silicon age' !

Matrix diagonalization at each k

$$\boxed{\hat{H}_{\mathbf{k}} = e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{H}e^{i\mathbf{k}\cdot\mathbf{r}}} \qquad e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{p}e^{i\mathbf{k}\cdot\mathbf{r}} = \hat{p} + \hbar\mathbf{k}$$

$$\frac{1}{2m}\left(\hat{\mathbf{p}} + \hbar\mathbf{k}\right)u_{\mathbf{k}\nu}(\mathbf{r}) = u_{\mathbf{k}\nu}(\mathbf{r}) \;\;,\;\; \hat{\mathbf{p}} = \frac{\hbar}{i}\nabla_{\mathbf{r}}$$

$$u_{\mathbf{k}\nu}(\mathbf{r}) = \sum_{\mathbf{G}\in L^*} e^{i\mathbf{G}\cdot\mathbf{r}}\hat{u}_{\mathbf{G}}^{(\mathbf{k}\nu)}$$

$$\sum_{\mathbf{G}'} M_{\mathbf{G}\mathbf{G}'}^{(\mathbf{k})}\hat{u}_{\mathbf{G}'}^{(\mathbf{k}\nu)} = \varepsilon_{\mathbf{k}\nu}\hat{u}_{\mathbf{G}}^{(\mathbf{k}\nu)}$$

$$M_{\mathbf{G}\mathbf{G}'}^{(\mathbf{k})} = \frac{\hbar^2}{2m}\left(\mathbf{k} + \mathbf{G}\right)^2\delta_{\mathbf{G}\mathbf{G}'} + \hat{V}_{\mathbf{G}-\mathbf{G}'}$$

G, G': reciprocal lattice vectors. In 1D: $G_n = n\,\dfrac{2\pi}{a}$

Properties of Bloch function

$$\langle u_m(\mathbf{k})|u_n(\mathbf{k})\rangle = \delta_{mn}$$

$$\langle u_{n\mathbf{k}}|u_{n\mathbf{k}'}\rangle \neq \delta_{\mathbf{k}\mathbf{k}'} \quad \langle \psi_{n\mathbf{k}}|\psi_{n'\mathbf{k}'}\rangle = \delta_{nn'}\delta_{\mathbf{k}\mathbf{k}'}$$

$$\boxed{\hat{H}_{\mathbf{k}+\mathbf{G}} = \hat{U}_{\mathbf{G}}\hat{H}_{\mathbf{k}}\hat{U}_{\mathbf{G}}^{-1}, \qquad \hat{U}_{\mathbf{G}} = e^{-i\mathbf{G}\cdot\mathbf{r}}}$$

$$\Rightarrow \boxed{\varepsilon_n(\mathbf{k}) = \varepsilon_n(\mathbf{k}+\mathbf{G})}$$

For a non-degenerate state:

$$|u_n(\mathbf{k}+\mathbf{G})\rangle = e^{i\phi(\mathbf{k})}\hat{U}_{\mathbf{G}}|u_{n\mathbf{k}}\rangle$$

Eigenvalues have BZ periodicity, not eigenfunctions

Comparing Bloch states at different k: Berry connection

Connexion:

$$\boldsymbol{A}_n(\boldsymbol{k}) = i\langle u_n(\boldsymbol{k})|\nabla_{\boldsymbol{k}} u_n(\boldsymbol{k})\rangle$$

Curvature:

$$F_{ab}^{(n)} = \partial_a A_b - \partial_b A_a = i\left[\langle \partial_a u_n|\partial_b u_n\rangle - \langle \partial_b u_n|\partial_a u_n\rangle\right]$$

Quantum Geometric Tensor:

$$ds^2 = 1 - |\langle u_n(\boldsymbol{k})|u_n(\boldsymbol{k}+d\boldsymbol{k})\rangle|^2$$

$$g_{ab}^{(n)}(\boldsymbol{k}) = \mathrm{Re}\left[\langle \partial_a u|\partial_b u\rangle - \langle \partial_a u|u\rangle\langle u|\partial_b u\rangle\right]$$

$$ds^2 = \sum_{ab} g_{ab}(\boldsymbol{k})\, dk_a\, dk_b$$

$$T_{ab}(\boldsymbol{k}) \equiv \langle \partial_a u|\partial_b u\rangle - \langle \partial_a u|u\rangle\langle u|\partial_b u\rangle$$

$$g = \mathrm{Re}\, T, \qquad F = -2\,\mathrm{Im}\, T, \qquad T = g - \frac{i}{2}F$$

$$T_{ab}^{(n)} = \sum_{m\neq n} \frac{\langle n|\partial_a \hat{H}_{\boldsymbol{k}}|m\rangle\langle m|\partial_b \hat{H}_{\boldsymbol{k}}|n\rangle}{(E_n - E_m)^2}$$

Transport

Velocity operator:

$$V_{\boldsymbol{k}}^{nn'} = \frac{1}{m}\left\langle \psi_{\boldsymbol{k}n} \middle| \frac{\hbar}{i}\nabla_r \middle| \psi_{\boldsymbol{k}n'} \right\rangle$$

$$V_{\boldsymbol{k}}^{nn'} = \frac{1}{\hbar}\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}|u_{\boldsymbol{k}n'}\rangle$$

$$V_{\boldsymbol{k}}^{nn} = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\varepsilon_n(\boldsymbol{k})$$

$$V_{\boldsymbol{k}}^{nn'} = \frac{1}{\hbar}(\varepsilon_{\boldsymbol{k}n'} - \varepsilon_{\boldsymbol{k}n})\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}} u_{\boldsymbol{k}n'}\rangle$$

Kubo:

$$\sigma_{xy} = -\frac{i\hbar}{V}\sum_{\boldsymbol{k}}\sum_{n} f(\boldsymbol{k}) \sum_{m\neq n} \frac{j^x_{nm}(\boldsymbol{k})\, j^y_{mn}(\boldsymbol{k}) - (x \leftrightarrow y)}{(\varepsilon_{n\boldsymbol{k}} - \varepsilon_{m\boldsymbol{k}})^2}$$

Hall conductance (TKNN):

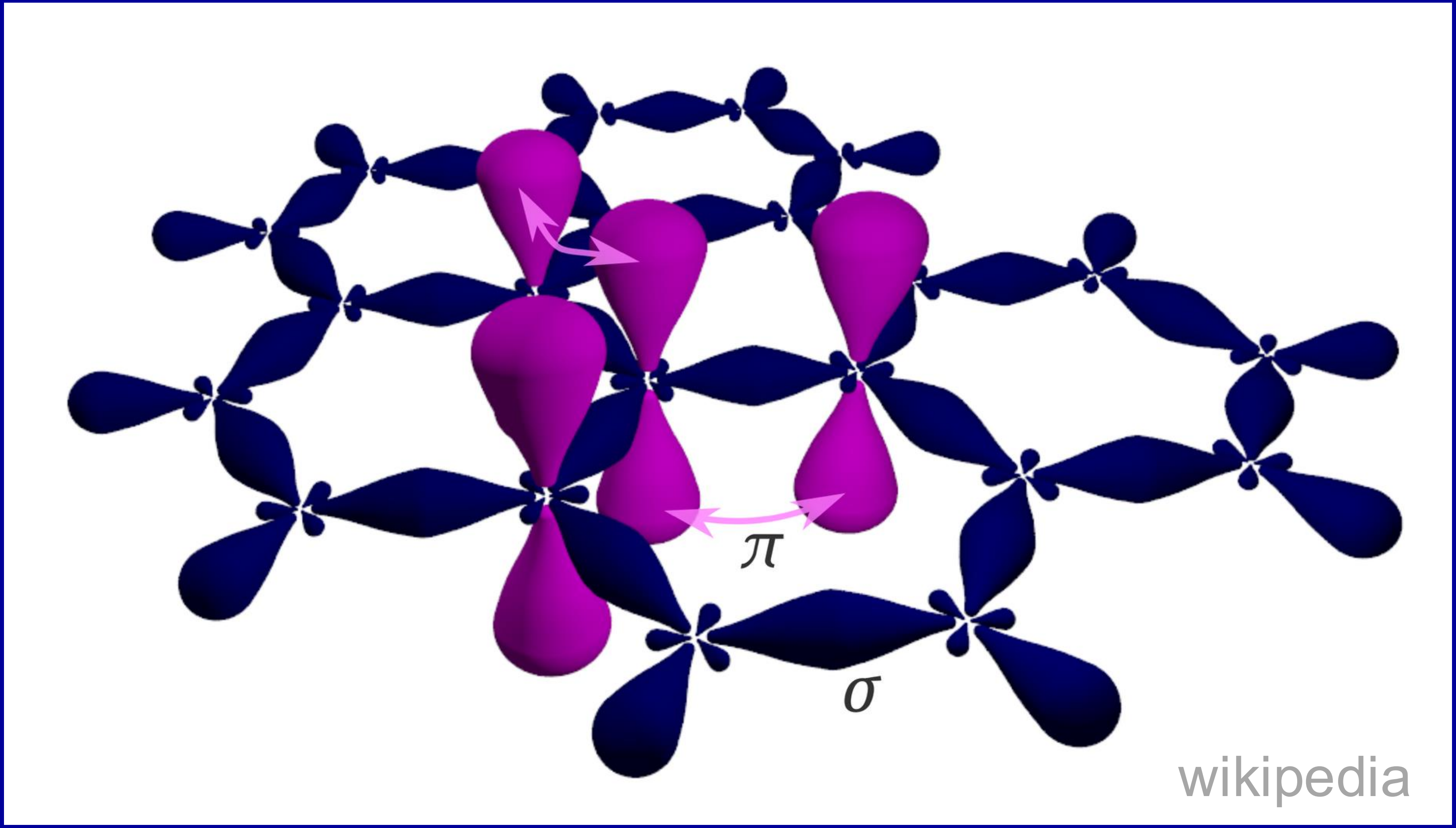
$$\sigma_{xy} = -\frac{e^2}{\hbar}\, i \int \frac{d^2k}{(2\pi)^2}\left[T^{(n)}_{xy} - T^{(n)}_{yx}\right]$$

$$\sigma^{(n)}_{xy} = -\frac{e^2}{h}\, C_n$$

$$C_n = \int \frac{dk_x \wedge dk_y}{2\pi}\, F^{(n)}_{xy}(\boldsymbol{k})$$

Graphene

C: 1s2 2s2 2p2; 2s,2px,2py→ sp2 hybrids (sigma-bonds)



p_z → pi-bonding

Fig: J.Cayssol and JN Fuchs
JPhysMater 4 034007 (2021)

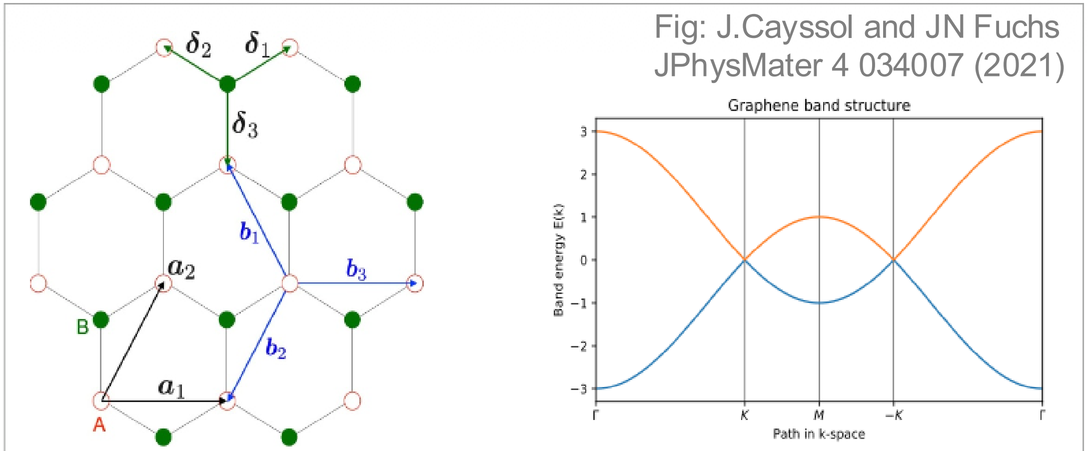
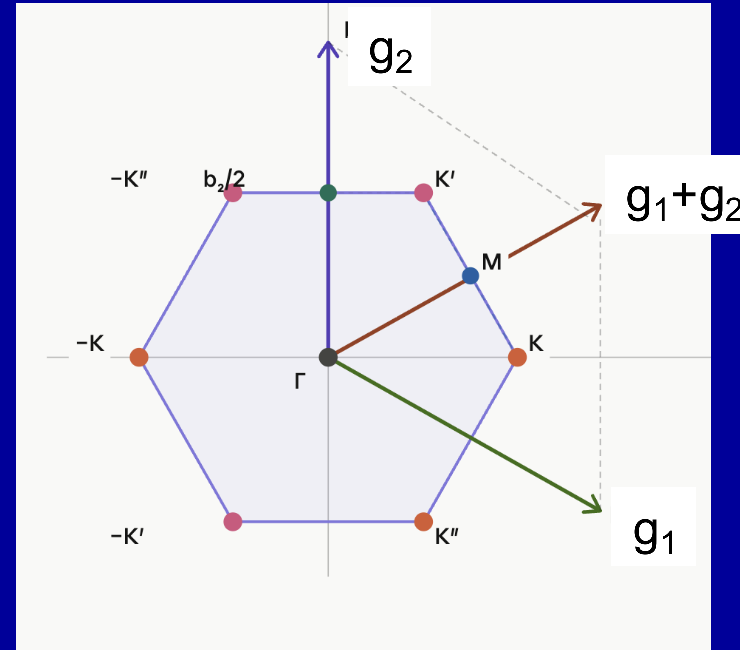
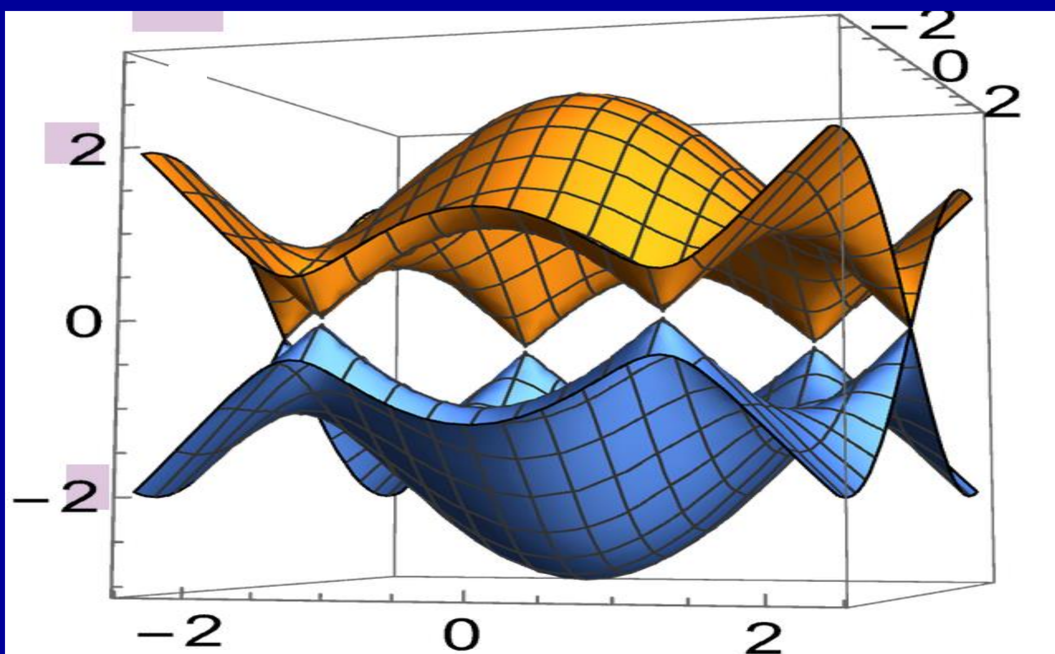


Figure 14. (Left) Graphene honeycomb lattice structure. Red open (green filled) dots for A (B) sublattice. The basis vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ generate the Bravais lattice. The vectors $\boldsymbol{\delta}_\alpha$ $(\alpha = 1, 2, 3)$ are connecting a given site to its three nearest neighbors. The vectors $\boldsymbol{b}_i$ and their opposite $-\boldsymbol{b}_i$ $(i = 1, 2, 3)$ connect a given site to its six second-nearest neighbors. The distance between two sites is $a = 0.142$ nm and the surface of the unit cell is $A_{cell} = 3\sqrt{3}a^2/2$. (Right) Section of the electronic energy dispersion $E(\boldsymbol{k}) = \pm|\boldsymbol{d}(\boldsymbol{k})|$ of graphene for $k_y = 0$, showing the two Dirac points at $\boldsymbol{k} = \pm\boldsymbol{K}$. The first BZ has an hexagonal shape. High symmetry [illegible]ts in this BZ are: the center Γ, two inequivalent corners of the hexagon K and K' and the mid-point on the boundary [illegible] M.

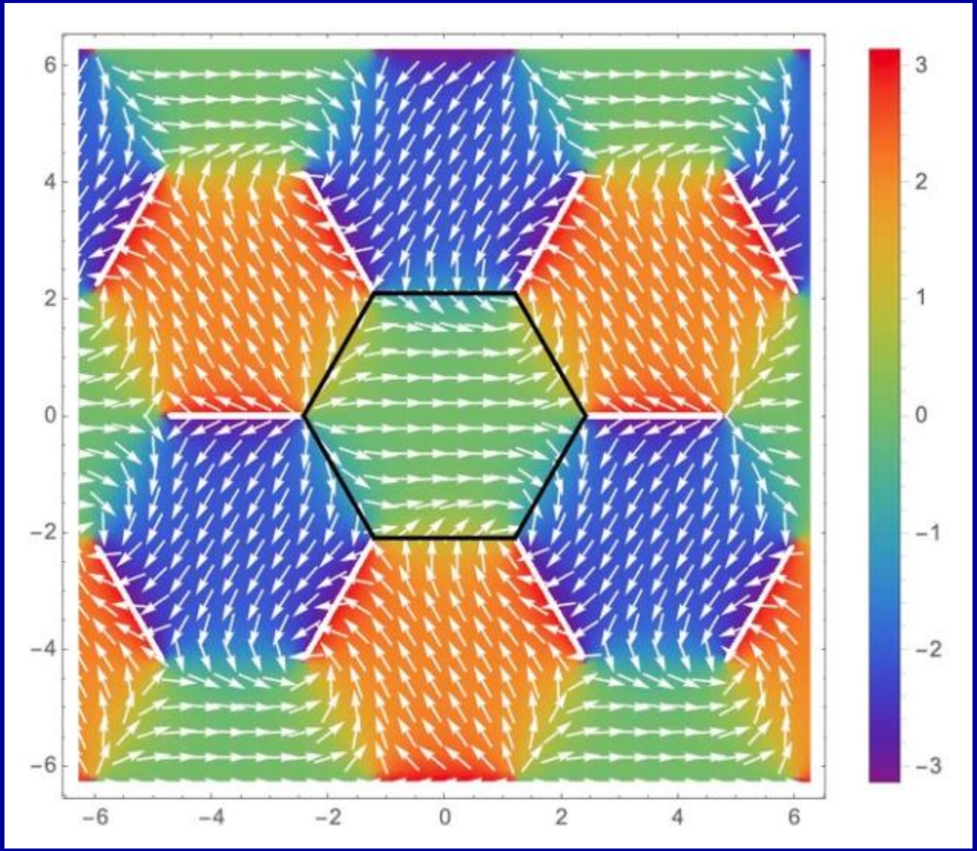




Point	k_x	k_y
K	$\frac{4\pi}{3\sqrt{3}\,a}$	0
M	$\frac{\pi}{\sqrt{3}\,a}$	$\frac{\pi}{3a}$
K'	$\frac{2\pi}{3\sqrt{3}\,a}$	$\frac{2\pi}{3a}$

$$K' = -K + (g_1 + g_2)$$

Phase representation of H(k) for graphene



$$\boldsymbol{n}(\boldsymbol{k}) = \frac{\boldsymbol{d}(\boldsymbol{k})}{\|\boldsymbol{d}(\boldsymbol{k})\|} = \begin{pmatrix} \cos\varphi_k \sin\theta_k \\ \sin\varphi_k \sin\theta_k \\ \cos\theta_k \end{pmatrix}$$

Graphene:

$$\theta_k = \pi/2$$

$$\cos\varphi_k = \frac{\sum_\alpha \cos(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha)}{\|\boldsymbol{d}\|}$$

Phase phi(k) in reciprocal space

Note the triple periodicity

Singularities at K,K' (BZ corners)

Fig: J.Cayssol and JN Fuchs
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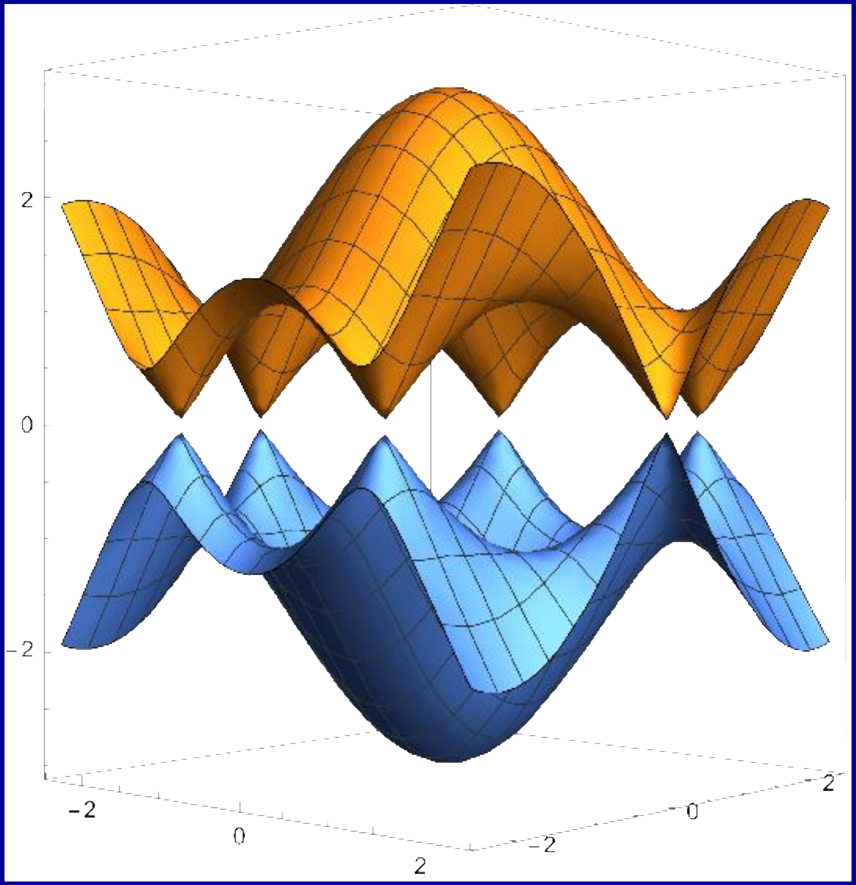
Bands of hBN (staggered mass)



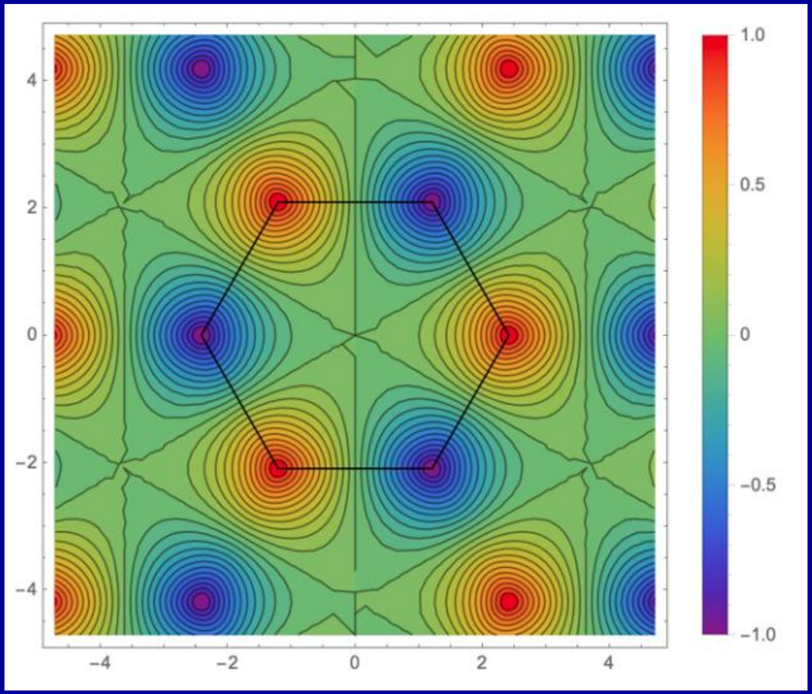
$$\mathbf{k} = \pm\mathbf{K} + \mathbf{q}$$

$$\varepsilon_{\pm} = \pm\sqrt{v_F^2 q^2 + m^2}$$

Massive relativistic particle

Berry curvature of hBN

$$F^{\pm}_{xy}(\boldsymbol{k}) = \pm a^2 \frac{\sqrt{3}t^2 M}{|E(\boldsymbol{k})|^3} \sin\left(\boldsymbol{k}.\frac{\boldsymbol{\delta}_2 - \boldsymbol{\delta}_3}{2}\right) \sin\left(\boldsymbol{k}.\frac{\boldsymbol{\delta}_3 - \boldsymbol{\delta}_1}{2}\right) \sin\left(\boldsymbol{k}.\frac{\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2}{2}\right)$$



M:
Staggered
(`Semenoff')
mass

Fig: J.Cayssol and JN Fuchs
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K and K = -K have opposite signs of F (`vorticity')
The integrated Berry curvature over the BZ vanishes
In contrast to H(k), F(k) has the periodicity of the R.L.