# From the quantum Hall effect to moiré materials: topology and geometry of quantum materials
# Lectures 1-5 - Typos and Mistakes may remain

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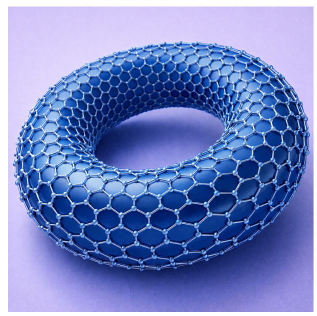
Topological concepts play a central role in condensed matter physics, enabling us to understand the robustness of certain phases of quantum matter and providing a unifying language to describe them. This series of lectures and seminars offers an introduction to these ideas through a journey from the quantum Hall effect to moiré materials. Two-dimensional materials and flat-band systems, which are at the heart of current research, are fertile platforms for exploring new topological and strongly correlated phases. The seminars complement the lectures by providing an overview of recent theoretical and experimental developments.



*Based on lecture notes at the Collège de France, May-June 2026. See recorded lectures and seminars on the website , also for the illustrating figures (not included in these notes).*

## CONTENTS



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# I. THE INTEGER QUANTUM HALL EFFECT

*Useful references*

- M. O. Goerbig, *Quantum Hall effects*, arXiv:0909.1998 — Les Houches Lectures
- David Tong, *The Quantum Hall effect*, TIFR Infosys lectures, arXiv:1606.06687
- Benoit Doucot, *Introduction to the theory of the integer quantum Hall effect, C.R.Physique 12, 323 (2011).*

## A. The classical Hall effect

### Setup

We consider a two-dimensional electron gas (2DEG) in the $(x,y)$ plane, subject to a perpendicular magnetic field

$$\mathbf{B} = B\,\hat{\mathbf{z}}.$$

The Lorentz force acting on an electron (charge $-e$) is

$$\mathbf{F} = -e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

The charge current density is

$$\mathbf{j} = -ne\mathbf{v},$$

where $n$ is the electron density.

We define the conductivity tensor through

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}.$$

By symmetry (isotropic system in the plane),

$$\sigma_{xx} = \sigma_{yy}, \qquad \sigma_{xy} = -\sigma_{yx}.$$

### Drude Equation of Motion

In the Drude model, the equation of motion including a relaxation time $\tau$ is

$$m\frac{d\mathbf{v}}{dt} = -e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) - \frac{m}{\tau}\mathbf{v}.$$

In steady state, $d\mathbf{v}/dt = 0$, so

$$\frac{m}{\tau}\mathbf{v} = -e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

Since

$$\mathbf{v} = (v_x, v_y, 0), \qquad \mathbf{v} \times \mathbf{B} = \begin{pmatrix} v_y B \\ -v_x B \\ 0 \end{pmatrix},$$

the steady-state equations become

$$\frac{m}{\tau} v_x = -e(E_x + B v_y), \tag{1.1}$$
$$\frac{m}{\tau} v_y = -e(E_y - B v_x). \tag{1.2}$$

Hall Geometry

In the standard Hall setup, current flows along $x$ and no current flows along $y$:

$$j_y = 0.$$

Since $j_y = -nev_y$, this implies

$$v_y = 0.$$

From the second equation:

$$0 = -e(E_y - Bv_x) \quad \Rightarrow \quad E_y = Bv_x.$$

From the first equation:

$$\frac{m}{\tau} v_x = -eE_x,$$

so

$$v_x = -\frac{e\tau}{m} E_x.$$

Thus the longitudinal current is

$$j_x = -nev_x = \frac{ne^2\tau}{m} E_x.$$

We identify the Drude conductivity:

$$\sigma_D = \frac{ne^2\tau}{m}.$$

### Resistivity Tensor

It is often simpler to compute the resistivity tensor defined by

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix}.$$

Using the previous relations one finds

$$\boxed{\rho_{xx} = \frac{m}{ne^2\tau} \equiv \frac{1}{\sigma_D}, \qquad \rho_{xy} = \frac{B}{ne}, \qquad \rho_{yx} = -\rho_{xy}.}$$

### Classical $B$-Dependence (Drude model)

– $\rho_{xx}$ is a constant independent of B.
– $\rho_{xy}$ depends on B linearly

### Hall coefficient

The Hall coefficient is defined as:

$$\boxed{R_H \equiv \frac{E_y}{Bj_x} = \frac{\rho_{yx}}{B} = -\frac{1}{ne}}$$

It is negative (positive) for electron-like (hole-like) carriers. I will avoid using this notation in the following to avoid confusion with the Hall resistance. One often also defines the Hall number:

$$n_H \equiv -\frac{1}{eR_H} = n$$

which (in the simplest case) yields the density of carriers

### Cyclotron Frequency

We introduce the cyclotron frequency

$$\boxed{\omega_c = \frac{eB}{m}}$$

In terms of $\sigma_D$ and $\omega_c\tau$, the resistivity tensor reads

$$\boldsymbol{\rho} = \frac{1}{\sigma_D} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}.$$

Conductivity Tensor

Inverting the resistivity tensor gives the conductivity tensor:

$$\boldsymbol{\sigma} = \frac{\sigma_D}{1 + (\omega_c \tau)^2} \begin{pmatrix} 1 & -\omega_c \tau \\ \omega_c \tau & 1 \end{pmatrix}.$$

Thus,

$$\sigma_{xx} = \frac{\sigma_D}{1 + (\omega_c \tau)^2}, \tag{1.3}$$

$$\sigma_{xy} = \frac{\sigma_D \, \omega_c \tau}{1 + (\omega_c \tau)^2}. \tag{1.4}$$

Dissipationless Limit

In the limit $\tau \to \infty$:

$$\rho_{xx} \to 0, \qquad \rho_{xy} = \frac{B}{ne}.$$

The conductivity tensor becomes purely off-diagonal:

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \sigma_{xy} \\ -\sigma_{xy} & 0 \end{pmatrix}.$$

Then

$$j_x = \sigma_{xy} E_y, \qquad j_y = -\sigma_{xy} E_x.$$

The current flows perpendicular to the applied electric field, as also clear directly from the equations of motion above. This corresponds to a dissipationless Hall response.

### B. Landau Levels

#### 1. Setup, Spectrum

$$H = \frac{1}{2m} (\mathbf{p} + e\mathbf{A})^2 \quad , \ (q = -e) \tag{1.5}$$

Magnetic Field:

$$\mathbf{B} = \nabla \times \mathbf{A} = B\hat{z} \tag{1.6}$$

Cyclotron Frequency:

$$\omega_c = \frac{eB}{m} \tag{1.7}$$

Magnetic Length (I'll often drop the subscript 'B
Energy Spectrum (Landau Levels):

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \quad n = 0,1,2,\ldots \tag{1.8}$$

Degeneracy per Unit Area:

$$\frac{\mathcal{N}}{A} = \frac{eB}{h} = \frac{B}{\Phi_0} \tag{1.9}$$

Each state corresponds to one flux quantum
Filling fraction: ratio of number of electrons to number of states:

$$\nu = \frac{N}{\Phi/\Phi_0} = \frac{n_{2D}}{eB/h} = \frac{nh}{eB} = 2\pi\ell^2\, n = \frac{N}{A/2\pi\ell^2}$$

Each state occupies a surface $2\pi\ell^2$
Orders of magnitude for an electron in a field of 1 Tesla:

$$\omega_c \simeq 1.76\, s^{-1} \;\; , \; f_c = \frac{\omega_c}{2\pi} \simeq 28\,\mathrm{GHz} \;\; , \; \ell_B \simeq 26\,\mathrm{nm} \tag{1.10}$$

Key Properties:
– Highly degenerate energy levels
– Level spacing independent of momentum, set by $\hbar\omega_c$
– Foundation of quantum Hall physics

2. Landau Gauge

Gauge Choice:

$$\mathbf{A} = (0,Bx,0) \tag{1.11}$$

Hamiltonian:

$$H = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right] \tag{1.12}$$

Translation Invariance:

$$\psi(x,y) = e^{iky}\phi(x) \tag{1.13}$$

Effective Harmonic Oscillator:

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_c^2(x - x_0)^2 \tag{1.14}$$

Guiding Center:

$$x_0 = -\frac{p_y}{eB} = -\frac{\hbar k}{eB} = -\ell_B^2 k \tag{1.15}$$

Eigenstates:

$$\psi_{n,k}(x,y) = e^{iky}\,\phi_n(x - x_0) \sim e^{iky}\,H_n(x - x_0)e^{-(x-x_0)^2/2\ell_B^2} \tag{1.16}$$

– States labeled by continuous momentum $k$
– Degeneracy equals number of magnetic flux quanta

3. Symmetric Gauge

$$\mathbf{A} = \frac{B}{2}(-y,x,0) \qquad \mathbf{B} = \nabla \times \mathbf{A} = B\hat{z}$$

$$H = \frac{1}{2m}\left[\left(p_x - \frac{eB}{2}y\right)^2 + \left(p_y + \frac{eB}{2}x\right)^2\right]$$

$$\mathbf{\Pi} = \boldsymbol{p} - q\boldsymbol{A} = \boldsymbol{p} + e\boldsymbol{A}$$

$$\Pi_x = p_x - \frac{eB}{2}y, \qquad \Pi_y = p_y + \frac{eB}{2}x$$

$$[\Pi_x,\Pi_y] = \frac{eB}{2}\left([p_x,x] - [y,p_y]\right)$$

$$[x,p_x] = [y,p_y] = i\hbar$$

$$\Rightarrow [\Pi_x,\Pi_y] = -i\hbar eB$$

$$a = \frac{1}{\sqrt{2\hbar eB}}(\Pi_x - i\Pi_y), \qquad a^\dagger = \frac{1}{\sqrt{2\hbar eB}}(\Pi_x + i\Pi_y)$$

$$a^\dagger a = \frac{1}{2\hbar eB}\left(\Pi_x^2 + \Pi_y^2 - i[\Pi_x,\Pi_y]\right)$$

$$= \frac{1}{2\hbar eB}\left(\Pi_x^2 + \Pi_y^2 - \hbar eB\right)$$

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right)$$

$$[a,a^\dagger] = 1$$

Hence

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

$$a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle, \qquad a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

Degeneracy $\leftrightarrow$ angular momentum

$$\tilde{\Pi} = p + qA = p - eA$$

(NOT the true momentum; gauge dependent)

$$[\tilde{\Pi}_x,\tilde{\Pi}_y] = i\hbar eB$$

$$[\Pi_a,\tilde{\Pi}_b] \neq 0$$

in general, *except in the symmetric gauge*. Indeed:

$[\Pi_x,\tilde{\Pi}_x] = 2ie\hbar\,\partial_x A_x$ , $[\Pi_y,\tilde{\Pi}_y] = 2ie\hbar\,\partial_y A_y$ , $[\Pi_x,\tilde{\Pi}_y] = [\Pi_y,\tilde{\Pi}_x] = 2ie\hbar\,(\partial_x A_y + \partial_y A_x)$

In that gauge, it can be diagonalized simultaneously.

$$b = \frac{1}{\sqrt{2\hbar eB}}(\tilde{\Pi}_x + i\tilde{\Pi}_y), \qquad b^\dagger = \frac{1}{\sqrt{2\hbar eB}}(\tilde{\Pi}_x - i\tilde{\Pi}_y)$$

$$[b,b^\dagger] = -\frac{i\cdot 2}{2\hbar eB}[\tilde{\Pi}_x,\tilde{\Pi}_y] = 1$$

Eigenstates:

$$\frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}\left|0{,}0\right\rangle$$

$$E_n = \hbar\omega_c\left(n+\frac{1}{2}\right)$$

$m$ labels degeneracy.

*a. Lowest Landau level (LLL)*

$$a\left|n=0{,}m\right\rangle = 0 \quad \Rightarrow \quad \text{differential equation}$$

$$a = \frac{1}{\sqrt{2\hbar eB}}(\Pi_x - i\Pi_y)$$

$$= \frac{1}{\sqrt{2\hbar eB}}\left[p_x - ip_y + e(A_x - iA_y)\right]$$

$$= \frac{1}{\sqrt{2\hbar eB}}\left[-i\hbar(\partial_x - i\partial_y) - \frac{eB}{2}i(x-iy)\right]$$

Magnetic length:

$$\ell = \sqrt{\frac{\hbar}{eB}}$$

$$a = -i\sqrt{2}\left[\ell\frac{1}{2}(\partial_x - i\partial_y) + \frac{1}{4\ell}(x-iy)\right]$$

Intermezzo:

$$z = x - iy \quad \text{(note: sign!)}$$

Function:

$$f(z,\bar{z}) = f(x-iy,x+iy)$$

$$\partial_x f = \partial_z f + \partial_{\bar{z}} f$$

$$\partial_y f = -i\partial_z f + i\partial_{\bar{z}} f$$

$$(\partial_x - i\partial_y) f = 2\partial_z f$$

$$a = -i\sqrt{2}\left[\ell\partial_{\bar{z}} + \frac{1}{4\ell}z\right] \ , \ a^\dagger = i\sqrt{2}\left[-\ell\partial_z + \frac{1}{4\ell}\bar{z}\right]$$

LLL equation:

$$\ell\partial_{\bar{z}} f + \frac{1}{4\ell} z f = 0$$

$$\frac{1}{f}\partial_{\bar{z}} f = -\frac{z}{4\ell^2}$$

$$\ln f = -\frac{z\bar{z}}{4\ell^2} + g(z)$$

$$f(z,\bar{z}) = g(z)\, e^{-z\bar{z}/(4\ell^2)}$$

$$b = -i\sqrt{2}\left(\ell\partial_z + \frac{1}{4\ell}\bar{z}\right)$$

$$b^\dagger = -i\sqrt{2}\left(\ell\partial_{\bar{z}} - \frac{1}{4\ell}z\right)$$

Angular momentum:

$$L_z = i\hbar(x\partial_y - y\partial_x) = \hbar(z\partial_z - \bar{z}\partial_{\bar{z}})$$

Basis functions in LLL:

$$\left(\frac{z}{\ell}\right)^m e^{-|z|^2/4\ell^2}$$

$$L_z = m\hbar$$

State localized on ring of radius:

$$r = \sqrt{2m}\,\ell$$

Degeneracy:

$$\mathcal{N} = \sum_{m;m\leq R^2/2\ell^2} 1 = \frac{R^2}{2\ell^2}$$

$$\frac{\mathcal{N}}{A} = \frac{1}{2\pi\ell^2} = \frac{B}{\Phi_0}$$

#### 4. Physical Interpretation

Guiding Center Coordinates:

$$X = x - \frac{1}{eB}p_y, \quad Y = y + \frac{1}{eB}p_x \tag{1.17}$$

Non-commutativity:

$$[X,Y] = i\ell_B^2 \tag{1.18}$$

Separation of Motion:
– Cyclotron motion (quantized $\rightarrow$ Landau levels)
– Guiding center motion (source of degeneracy)
Total Degeneracy:

$$N_\phi = \frac{BA}{\Phi_0}, \quad \Phi_0 = \frac{h}{e} \tag{1.19}$$

Physical Picture:
– Electron executes circular cyclotron orbits
– Orbit center moves freely $\rightarrow$ degeneracy

### C. Integer Quantum Hall effect: phenomenology

### D. IQHE: Topological aspects

This section requires familiarity with the Berry phase and Berry connection, which are reviewed in Appendix A.

#### Definition of currents

Total current:

$$\boldsymbol{J} = -e\sum_{i=1}^{N}\boldsymbol{v}_i$$

Current density in $d$-dimensions:

$$\boldsymbol{j}(\boldsymbol{r}) = -e\, n_d(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r})$$

$$\boldsymbol{J} = \int d^d r\, \boldsymbol{j}(\boldsymbol{r})$$

Intensity of current:

$$I = \int \boldsymbol{j}\cdot d\boldsymbol{S}_\perp$$

where $d\boldsymbol{S}_\perp$ is perpendicular to the direction.
Dimensionality:

$$[J] \sim Ne\frac{L}{t}$$

$$[j] \sim \frac{1}{L^d}Ne\frac{L}{t} \sim e\frac{N}{t}L^{1-d}$$

$$[I] \sim L^{d-1}[j] \sim e\frac{N}{t} \quad \text{Ampère} = \frac{\text{charge}}{\text{time}}$$

Conductivity:

$$j = \sigma E$$

$$[\sigma] \sim \frac{j}{E} \sim e\frac{N}{t}L^{1-d}\left(\frac{\Delta U}{L}\right)^{-1} \sim \frac{[I]}{[\Delta U]}L^{2-d} \sim L^{2-d}\,\Omega^{-1}$$

#### Two dimensions

$$I_x = \int dy\, j_x \sim L_y j_x$$

$$J_x = \int dxdy\, j_x \sim L_x L_y j_x = A j_x$$

| | |
|---|---|
| $\boldsymbol{j} = -ne\boldsymbol{v}$ | $[e]L^{1-d}t^{-1}$ |
| $\boldsymbol{J} = \int d^d r \boldsymbol{j}$ | $[e]Lt^{-1}$ |
| $I = \int da_\perp^{d-1} \boldsymbol{j}$ | $[e]t^{-1}$ |
| $\boldsymbol{j} = \sigma \boldsymbol{E}$ | $[\sigma] = [\frac{e^2}{h}]L^{2-d}$ |
| $I = GU\ (U = RI)$ | $[G] = [\frac{e^2}{h}]$ |

TAB. I – *Summary of definitions and dimensionalities (d dimensions)*

$$J_x = L_y I_x$$

$$I = G\Delta U \quad \text{conductance}$$

$$j = \sigma E$$

$G$ and $\sigma$ have both dimensions $\Omega^{-1}$ in $d = 2$.

**Kubo formula (cf. Mahan)**

$$\sigma_{\alpha\beta}(\boldsymbol{q} = 0,\omega) = \frac{iD}{\omega}\delta_{\alpha\beta} + \frac{1}{\omega V}\int_0^\infty dt\, e^{i\omega t} \int d^d r d^d r' \left\langle [j_\alpha(\boldsymbol{r},t), j_\beta(\boldsymbol{r}',0)]\right\rangle$$

This formula assumes time translation invariance but not spatial.
Or, in terms of total currents in $d = 2$:

$$\mathrm{Re}\,\sigma_{\alpha\beta}(q = 0,\omega) = \frac{1}{A}\frac{1}{\omega}\int_0^\infty dt\, e^{i\omega t} \langle [\hat{J}_\alpha(t), \hat{J}_\beta(0)]\rangle$$

$$A = L_x L_y$$

We then insert a complete set of states (Lehman representation) and carefully take the $\omega \to 0$ limit for the transverse component (see appendix). We finally obtain the dc Hall conductivity as (restoring $\hbar$):

$$\boxed{\sigma_{xy} = -i\hbar \frac{1}{A}\sum_{n\neq 0} \frac{J_{0n}^x J_{n0}^y - J_{n0}^x J_{0n}^y}{(E_n - E_0)^2} \quad , \quad J_{0n}^a \equiv \langle \Psi_0 | \hat{J}^a | \Psi_n \rangle}$$

**IQHE on a torus**

Gauge invariance requires:

$$BL_xL_y = n\frac{h}{e}, \quad n \in \mathbb{Z}$$

Proof from magnetic translations.

Add fluxes $\phi_x$,$\phi_y$ through the two cycles of the torus. This creates twisted boundary conditions with twist angles given by the Aharonov-Bohm effect:

$$\theta_a = 2\pi\frac{\phi_a}{\phi_0}$$

Vector potential corresponding to $\phi_x$:

$$\oint \boldsymbol{A}\cdot d\boldsymbol{\ell} = A_xL_x = \phi_x \Rightarrow A_x = \frac{\phi_x}{L_x}$$

Hamiltonian becomes (Landau gauge):

$$\hat{H} = \sum_i \hat{h}_k(i) + \sum_i V(x_i) + \hat{H}_{\text{int}}$$

$$\hat{h}_k = \frac{1}{2m}\left(\hat{p}_x + e\frac{\phi_x}{L_x}\right)^2 + \frac{1}{2m}\left(\hat{p}_y + eBx + e\frac{\phi_y}{L_y}\right)^2$$

$$\delta\hat{H} = -\frac{\delta\phi_x}{L_x}\hat{J}_x - \frac{\delta\phi_y}{L_y}\hat{J}_y$$

Hence, the currents operators are given by the change of the hamiltonian under a twist:

$$\boxed{\hat{J}_a = -L_a\,\frac{\delta\hat{H}}{\delta\phi_a}}$$

Perturbation of the ground state:

$$\delta|\psi_0\rangle = \sum_{n\neq 0}\frac{\langle\psi_n|\delta\hat{H}|\psi_0\rangle}{E_0 - E_n}|\psi_n\rangle$$

$$\frac{\partial}{\partial\phi_\alpha}|\psi_0\rangle = -\frac{1}{L_\alpha}\sum_{n\neq 0}\frac{\langle\psi_n|\hat{J}_\alpha|\psi_0\rangle}{E_n - E_0}|\psi_n\rangle$$

$$\left\langle \frac{\partial \psi_0}{\partial \phi_y} \middle| \frac{\partial \psi_0}{\partial \phi_x} \right\rangle = \frac{1}{L_x L_y} \sum_{n \neq 0} \frac{\langle \psi_0 | \hat{J}_y | \psi_n \rangle \langle \psi_n | \hat{J}_x | \psi_0 \rangle}{(E_n - E_0)^2}$$

Kubo:

$$\sigma_{xy} = -i\hbar \frac{1}{L_x L_y} \sum_{n \neq 0} \frac{J^x_{0n} J^y_{n0} - J^x_{n0} J^y_{0n}}{(E_n - E_0)^2}$$

$$\sigma_{xy} = -i\hbar \left[ \langle \partial_x \psi_0 | \partial_y \psi_0 \rangle - \langle \partial_y \psi_0 | \partial_x \psi_0 \rangle \right]$$

The fluxes form a torus corresponding to angles:

$$\theta_\alpha = 2\pi \frac{\phi_\alpha}{\phi_0}, \quad \theta_\alpha \in [0,2\pi]$$

Since $\hbar(2\pi/\phi_0)^2 = e^2/\hbar$ we obtain:

$$\boxed{\sigma_{xy} = -i\frac{e^2}{\hbar} \left[ \langle \frac{\partial \psi_0}{\partial \theta_x} \middle| \frac{\partial \psi_0}{\partial \theta_y} \rangle - (x \leftrightarrow y) \right] \; = \; -\frac{e^2}{\hbar} F_{xy}(\theta)}$$

Alternatively:

$$\sigma_{xy} = -i\frac{e^2}{\hbar} \sum_{n \neq 0} \frac{H^x_{0n} H^y_{n0} - H^y_{n0} H^x_{0n}}{(E_n - E_0)^2} \;\; , \;\; H^\alpha \equiv \frac{\partial \hat{H}}{\partial \theta_\alpha}$$

**Hall conductance and many-body Chern number**

$$\sigma_{xy}(\theta) = -\frac{e^2}{\hbar} F_{xy}(\theta)$$

To show quantization, we integrate over the oriented measure $d\theta_x \wedge d\theta_y$. This requires a proof (cf. Hastings and Michalakis, Commun. Math Phys 334, 433 (2015)) - loosely speaking we can appeal to 'independence on boundary conditions':

$$\sigma_{xy} = -\int \frac{d\theta_x \wedge d\theta_y}{(2\pi)^2} \frac{e^2}{\hbar} F_{xy}(\theta_x,\theta_y)$$

$$\boxed{\sigma_{xy} = -\frac{e^2}{h} \int \frac{d\theta_x \wedge d\theta_y}{2\pi} F_{xy}(\theta_x,\theta_y)}$$

This is the Chern number $C_1$ associated with the fiber bundle:

$$\boxed{\sigma_{xy} = -\frac{e^2}{h} C_1}$$

Note that this is the many-body Chern number, associated with the $U(1)$ fiber bundle whose fiber is the *many-body* ground-state wave-function.

### E. Microscopic picture: edge states, disorder

## II. TOPOLOGY AND GEOMETRY OF BLOCH STATES

*Useful references for Sections II, III, IV and V*

– J.Cayssol and J.-N. Fuchs, *Topological and gemoetrical aspects of band theory*, J.Phys.Mater 4, 034007 (2021).
– D.Carpentier, *Topology of bands in solids: from insulators to Dirac matter* Séminaire Poincaré XVIII 87
– M.Fruchart and D.Carpentier *An introduction to topological insulators* C.R.Physique 14, 779 (2013).
– B.A. Bernevig and T.L. Hughes *Topological insulators and topological superconductors*, Princeton University Press (2013)

Warning: In contrast to the previous section, where the index $'n'$ referred to a many-body state, it stands here for a single-particle state (Bloch band). It may be better to use $\nu$,$\nu'$ to label Bloch bands, but I'll keep $n$,$n'$ below...

### A. Bloch States: key properties

$$\hat{H}\psi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = \varepsilon\psi(\mathbf{r}) \tag{2.1}$$

$$V(\mathbf{r}+\mathbf{T}) = V(\mathbf{r}), \qquad \mathbf{T} = \sum_i n_i \mathbf{a}_i \tag{2.2}$$

with $\mathbf{T}$ a translation of the Bravais lattice.
Reciprocal lattice:

$$\mathbf{a}_i\cdot\mathbf{b}_j = 2\pi\delta_{ij}, \qquad \mathbf{G} = \sum_i m_i \mathbf{b}_i \tag{2.3}$$

In $d=2$:

$$\mathbf{b}_1 = 2\pi\frac{R_{\pi/2}\,\mathbf{a}_2}{\mathbf{a}_1\cdot R_{\pi/2}\mathbf{a}_2}, \qquad \mathbf{b}_2 = 2\pi\frac{R_{\pi/2}\,\mathbf{a}_1}{\mathbf{a}_2\cdot R_{\pi/2}\mathbf{a}_1} \tag{2.4}$$

with $R_{\pi/2}$ the rotation by $\pi/2$.

*Bloch theorem*

$$\boxed{\psi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{n\mathbf{k}}(\mathbf{r})\ \ ,\ u_{n\mathbf{k}}(\mathbf{r}+\mathbf{T}) = u_{n\mathbf{k}}(\mathbf{r})} \tag{2.5}$$

with $\mathbf{k}\in$ BZ the Brillouin zone (unit cell) of the reciprocal lattice (not uniquely defined).

Unitary transformation:

$$\hat{H}_{\mathbf{k}} = e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{H} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.6}$$

Note - using $\hat{p} = \frac{\hbar}{i}\nabla$ and acting on any function:

$$e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{p} e^{i\mathbf{k}\cdot\mathbf{r}} = \hat{p} + \hbar\mathbf{k}$$

$$\hat{H}_{\mathbf{k}} = \frac{1}{2m}(\mathbf{p} + \hbar\mathbf{k})^2 + V(\mathbf{r}) \tag{2.7}$$

$$\hat{H}_{\mathbf{k}}|u_{n\mathbf{k}}\rangle = \varepsilon_n(\mathbf{k})|u_{n\mathbf{k}}\rangle \tag{2.8}$$

Since $u_{nk}$ is periodic, it has a Fourier series on the reciprocal lattice:

$$u_n(\mathbf{k}) = \sum_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}}\, u_{nk}^{G} \tag{2.9}$$

and the Fourier modes are obtained by diagonalizing the following (infinite) matrix at each k-point:

$$M_k^{GG'} = \frac{\hbar^2}{2m}\,(\mathbf{k} + \mathbf{G})^2\,\delta_{\mathbf{G}\mathbf{G}'} + V_{\mathbf{G}-\mathbf{G}'} \;\;,\;\; \sum_{\mathbf{G}'} M_k^{GG'} u_n(\mathbf{k})^{\mathbf{G}'} = \varepsilon_n(\mathbf{k}) u_n(\mathbf{k})^{\mathbf{G}}$$

*Orthogonality properties* The Bloch functions $u_n$, in contrast to the $\psi_n$, do *not* form an orthonormal basis. They satisfy:

$$\langle u_m(\mathbf{k})|u_n(\mathbf{k})\rangle = \delta_{mn} \tag{2.10}$$

but in general:

$$\langle u_{n\mathbf{k}}|u_{n\mathbf{k}'}\rangle \neq \delta_{\mathbf{k}\mathbf{k}'} \tag{2.11}$$

In contrast:

$$\langle \psi_{n\mathbf{k}}|\psi_{n'\mathbf{k}'}\rangle = \delta_{nn'}\delta_{\mathbf{k}\mathbf{k}'}$$

*Periodicity in Reciprocal Space* The Bloch eigenvalues are periodic in reciprocal space:

$$\boxed{\varepsilon_n(\mathbf{k}) = \varepsilon_n(\mathbf{k} + \mathbf{G})}$$

To prove this, we note that:

$$\hat{H}_{\mathbf{k}+\mathbf{G}} = \hat{U}_{\mathbf{G}} \hat{H}_{\mathbf{k}} \hat{U}_{\mathbf{G}}^{-1}, \qquad \hat{U}_{\mathbf{G}} = e^{-i\mathbf{G}\cdot\mathbf{r}} \tag{2.12}$$

Hence, $\hat{H}_{\mathbf{k}+\mathbf{G}}$ and $\hat{H}_{\mathbf{k}}$ have the same eigenvalues, and proper labeling insures the periodicity (with possible ambiguities at degeneracy points).

If the $\varepsilon_n(\mathbf{k})$ is non-degenerate, it follows that:

$$|u_n(\mathbf{k}+\mathbf{G})\rangle = e^{i\phi(\mathbf{k})} \hat{U}_{\mathbf{G}} |u_{n\mathbf{k}}\rangle \tag{2.13}$$

The two eigenvectors are equivalent *up to a phase*. For a band with non-zero Chern number, there is no smooth choice of this phase in all the BZ.

### B. Comparing Bloch states: Berry connection

To compare eigenstates at nearby $\boldsymbol{k}$-points, consider the overlap:

$$\begin{aligned} \langle u_n(\boldsymbol{k}) | u_n(\boldsymbol{k}+d\boldsymbol{k})\rangle &\approx \langle u_n(\boldsymbol{k})|(1 + d\boldsymbol{k}\cdot\nabla_{\boldsymbol{k}})|u_n(\boldsymbol{k})\rangle \\ &= 1 + d\boldsymbol{k}\cdot\langle u_n(\boldsymbol{k})|\nabla_{\boldsymbol{k}} u_n(\boldsymbol{k})\rangle \\ &\approx e^{-id\boldsymbol{k}\cdot\boldsymbol{A}_n(\boldsymbol{k})} \end{aligned} \tag{2.14}$$

where we define the **Berry connection**:

$$\boxed{\boldsymbol{A}_n(\boldsymbol{k}) = i\langle u_n(\boldsymbol{k})|\nabla_{\boldsymbol{k}} u_n(\boldsymbol{k})\rangle} \tag{2.15}$$

This is the Berry connection associated with the $U(1)$ fiber bundle whose base space is the BZ torus and whose fiber is generated by $|u_n(\boldsymbol{k})\rangle$ for the $n$-th band.

To linear order in $d\boldsymbol{k}$, only the phase of $|u_n(\boldsymbol{k})\rangle$ changes, and this change is given by the Berry connection.

*a. Berry Curvature* Using $\partial_a \equiv \partial/\partial k_a$, the Berry curvature tensor is:

$$F_{ab}^{(n)} = \partial_a A_b - \partial_b A_a = i\left[\langle \partial_a u_n|\partial_b u_n\rangle - \langle \partial_b u_n|\partial_a u_n\rangle\right] \tag{2.16}$$

Gauge freedom:

The phase of $|u_n(\boldsymbol{k})\rangle$ is arbitrary: the state

$$|\tilde{u}_n(\boldsymbol{k})\rangle = e^{i\varphi(\boldsymbol{k})}|u_n(\boldsymbol{k})\rangle \tag{2.17}$$

is an equally valid Bloch function. Under this gauge transformation:

$$\tilde{A}_n(\boldsymbol{k}) = i\langle\tilde{u}|\nabla_{\boldsymbol{k}}\tilde{u}\rangle = A_n(\boldsymbol{k}) - \nabla_{\boldsymbol{k}}\varphi \tag{2.18}$$

The Berry connection transforms like a gauge field. $F$ and $g$ (defined below) are **gauge invariant**.

### C. Quantum Geometric Tensor (QGT)

*a. Definition via Infidelity* To next order in $d\boldsymbol{k}$, consider the infidelity (distance metric):

$$ds^2 = 1 - |\langle u_n(\boldsymbol{k})|u_n(\boldsymbol{k}+d\boldsymbol{k})\rangle|^2 \tag{2.19}$$

A simple calculation yields:

$$\boxed{g_{ab}^{(n)}(\boldsymbol{k}) = \mathrm{Re}\left[\langle\partial_a u|\partial_b u\rangle - \langle\partial_a u|u\rangle\langle u|\partial_b u\rangle\right]} \tag{2.20}$$

$$\boxed{ds^2 = \sum_{ab} g_{ab}(\boldsymbol{k})\, dk_a\, dk_b} \tag{2.21}$$

*b. Tensor Decomposition* The **Quantum Geometric Tensor** is defined as:

$$T_{ab}(\boldsymbol{k}) \equiv \langle\partial_a u|\partial_b u\rangle - \langle\partial_a u|u\rangle\langle u|\partial_b u\rangle \tag{2.22}$$

Its real and imaginary parts give:

$$g = \mathrm{Re}\, T, \qquad F = -2\,\mathrm{Im}\, T, \qquad T = g - \frac{i}{2}F \tag{2.23}$$

Equivalently, introducing the projector $\hat{P}_n(\boldsymbol{k}) = |u_n(\boldsymbol{k})\rangle\langle u_n(\boldsymbol{k})|$ onto band $n$:

$$T_{ab}(\boldsymbol{k}) = \langle\partial_a u|(1 - |u_n\rangle\langle u_n|)|\partial_b u\rangle = \langle\partial_a u|(1-\hat{P}_n)|\partial_b u\rangle \tag{2.24}$$

*c. Derivation of g* Setting $\delta_a = dk_a$, expand:

$$|u(\boldsymbol{k}+d\boldsymbol{k})\rangle = \left(1 + \delta_a\partial_a + \tfrac{1}{2}\delta_a\delta_b\partial_a\partial_b + \cdots\right)|u(\boldsymbol{k})\rangle \tag{2.25}$$

Using $\langle u|\partial_a\partial_b u\rangle = -\langle\partial_a u|\partial_b u\rangle$ (integration by parts) and $\langle u|u\rangle = 1$, which implies:

$$0 = \langle\partial_a u|u\rangle + \langle u|\partial_a u\rangle \implies \langle u|\partial_a u\rangle \in i\mathbb{R} \tag{2.26}$$

Expanding $|\langle u(\boldsymbol{k})|u(\boldsymbol{k}+d\boldsymbol{k})\rangle|^2$ to $\mathcal{O}(\delta^2)$:

$$\begin{aligned}|\langle u(\boldsymbol{k})|u(\boldsymbol{k}+d\boldsymbol{k})\rangle|^2 = 1 &+ \delta_a[\langle u|\partial_a u\rangle + \langle\partial_a u|u\rangle]\\ &+ \delta_a\delta_b[\langle\partial_a u|u\rangle\langle u|\partial_b u\rangle - \langle\partial_a u|\partial_b u\rangle]\end{aligned} \tag{2.27}$$

The $\mathcal{O}(\delta)$ term vanishes, giving:

$$ds^2 = \mathrm{Re}[\langle\partial_a u|\partial_b u\rangle - \langle\partial_a u|u\rangle\langle u|\partial_b u\rangle] \tag{2.28}$$

(the second term is already real).

### D. Properties of the Berry Curvature and QGT

*a. Perturbation Theory Expression* From first-order perturbation theory:

$$|\partial_a u_n\rangle_\perp \equiv (1 - |u_n\rangle\langle u_n|)|\partial_a u_n\rangle = \sum_{m\neq n} \frac{\langle m|\partial_a \hat{H}_{\boldsymbol{k}}|n\rangle}{E_n - E_m}|m\rangle \tag{2.29}$$

Substituting into the QGT:

$$\boxed{T_{ab}^{(n)} = \sum_{m\neq n} \frac{\langle n|\partial_a \hat{H}_{\boldsymbol{k}}|m\rangle\langle m|\partial_b \hat{H}_{\boldsymbol{k}}|n\rangle}{(E_n - E_m)^2}} \tag{2.30}$$

*b. Alternative Expression* Differentiating the eigenvalue equation $H(\boldsymbol{k})|u_m(\boldsymbol{k})\rangle = E_m(\boldsymbol{k})|u_m(\boldsymbol{k})\rangle$ with respect to $k_a$ and projecting onto $\langle n|$ ($n \neq m$):

$$\langle n|\partial_a H|m\rangle + E_n\langle n|\partial_a m\rangle = E_m\langle n|\partial_a m\rangle \tag{2.31}$$

$$\frac{\langle n|\partial_a H|m\rangle}{E_m - E_n} = \langle n|\partial_a m\rangle \tag{2.32}$$

This yields the alternative form:

$$\boxed{T_{ab}(\boldsymbol{k}) = -\sum_{m\neq n} \langle u_n|\partial_a u_m\rangle\langle u_m|\partial_b u_n\rangle} \tag{2.33}$$

*c. Periodicity of $T_{ab}$* From the periodicity relation $|u_n(\boldsymbol{k}+\boldsymbol{G})\rangle = e^{i\varphi_n(\boldsymbol{k})}|u_n(\boldsymbol{k})\rangle$, for $m \neq n$ the phases cancel:

$$\langle u_m(\boldsymbol{k}+\boldsymbol{G})|\partial_b u_n(\boldsymbol{k}+\boldsymbol{G})\rangle = e^{i(\varphi_n - \varphi_m)}\langle u_m(\boldsymbol{k})|\partial_b u_n(\boldsymbol{k})\rangle \tag{2.34}$$

Hence:

$$T_{ab}(\boldsymbol{k}+\boldsymbol{G}) = T_{ab}(\boldsymbol{k}) \tag{2.35}$$

### E. Properties of the Berry Curvature $F$

– **Sum rule:**

$$\sum_n F_{ab}^{(n)}(\boldsymbol{k}) = 0 \tag{2.36}$$

– **Time Reversal Symmetry (TRS):**

$$F_{ab}^{(n)}(-\boldsymbol{k}) = -F_{ab}^{(n)}(\boldsymbol{k}) \tag{2.37}$$

– **Inversion symmetry:**

$$F_{ab}(-\boldsymbol{k}) = F_{ab}(\boldsymbol{k}) \tag{2.38}$$

– **TRS + Inversion** $\Rightarrow F = 0$ for all $\boldsymbol{k}$.

### F. Hall conductance of Bloch bands (TKNN)

#### 1. Matrix Elements of the Velocity Operator

The velocity matrix elements between Bloch states are:

$$\begin{aligned} V_{\boldsymbol{k}}^{nn'} &= \left\langle \psi_{\boldsymbol{k}n} \left| \frac{\hat{\boldsymbol{p}}}{m} \right| \psi_{\boldsymbol{k}n'} \right\rangle = \frac{1}{m} \left\langle \psi_{\boldsymbol{k}n} \left| \frac{\hbar}{i}\nabla_r \right| \psi_{\boldsymbol{k}n'} \right\rangle \\ &= \frac{1}{m}\langle u_{\boldsymbol{k}n}|\hat{\boldsymbol{p}} + \hbar\boldsymbol{k}|u_{\boldsymbol{k}n'}\rangle \end{aligned} \tag{2.39}$$

Recalling that:

$$\hat{H}_{\boldsymbol{k}} = \frac{1}{2m}(\hat{\boldsymbol{p}} + \hbar\boldsymbol{k})^2 + V(\boldsymbol{r}) \implies \nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}} = \frac{\hbar}{m}(\hat{\boldsymbol{p}} + \hbar\boldsymbol{k}) \tag{2.40}$$

Hence:

$$\boxed{V_{\boldsymbol{k}}^{nn'} = \frac{1}{\hbar}\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}|u_{\boldsymbol{k}n'}\rangle} \tag{2.41}$$

*Diagonal Elements: Group Velocity* Starting from $\hat{H}_{\boldsymbol{k}}|u_{\boldsymbol{k}n}\rangle = \varepsilon_{\boldsymbol{k}n}|u_{\boldsymbol{k}n}\rangle$, differentiate with respect to $\boldsymbol{k}$:

$$\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}|u_{\boldsymbol{k}n}\rangle + \hat{H}_{\boldsymbol{k}}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n}\rangle = \nabla_{\boldsymbol{k}}\varepsilon_{\boldsymbol{k}n}|u_{\boldsymbol{k}n}\rangle + \varepsilon_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n}\rangle \tag{2.42}$$

Projecting onto $\langle u_{\boldsymbol{k}n}|$ gives the group velocity:

$$\boxed{V_{\boldsymbol{k}}^{nn} = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\varepsilon_n(\boldsymbol{k})} \tag{2.43}$$

*Off-diagonal Elements* For $n \neq n'$, projecting the differentiated eigenvalue equation onto $\langle u_{\boldsymbol{k}n}|$:

$$\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}|u_{\boldsymbol{k}n'}\rangle + \varepsilon_{\boldsymbol{k}n}\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n'}\rangle = \varepsilon_{\boldsymbol{k}n'}\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n'}\rangle \tag{2.44}$$

Hence, for $n \neq n'$:

$$\boxed{V_{\boldsymbol{k}}^{nn'} = \frac{1}{\hbar}(\varepsilon_{\boldsymbol{k}n'} - \varepsilon_{\boldsymbol{k}n})\langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n'}\rangle} \tag{2.45}$$

where we have used $\langle u_{\boldsymbol{k}n}|u_{\boldsymbol{k}n'}\rangle = \delta_{nn'}$, which implies:

$$\langle \nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n}|u_{\boldsymbol{k}n'}\rangle + \langle u_{\boldsymbol{k}n}|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}n'}\rangle = 0 \tag{2.46}$$

2. Response to an Electric Field

No applied magnetic field here.

It is preferable to use the vector potential $\boldsymbol{A}(t)$ in order to avoid the ill-defined position operator $\hat{r}$:

$$\boldsymbol{A}(t) = -\boldsymbol{E}\cdot t, \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{2.47}$$

We assume an adiabatic evolution (semi-classical approximation), with time-dependent Hamiltonian:

$$H_{\boldsymbol{k}}(t) = \frac{1}{2m}\left[\hat{\boldsymbol{p}} + \hbar\boldsymbol{k} + e\boldsymbol{A}(t)\right]^2 + V(\boldsymbol{r}) \tag{2.48}$$

Hence the quasi-momentum $\tilde{\boldsymbol{k}}$:

$$\tilde{\boldsymbol{k}}(t) = \boldsymbol{k} - \frac{e}{\hbar}\boldsymbol{E}\,t \tag{2.49}$$

sweeps through the Brillouin zone. For the longitudinal response, this leads to Bloch oscillations in the dissipationless limit.

Since we are interested in the transverse response, we do not need to introduce dissipation.

3. Hall conductance

The Kubo formula in terms of single-particle eigenstates gives the Hall conductivity:

$$\sigma_{xy} = -\frac{i\hbar}{V}\sum_{\boldsymbol{k}}\sum_{n} f(\boldsymbol{k}) \sum_{m\neq n} \frac{j^x_{nm}(\boldsymbol{k})\, j^y_{mn}(\boldsymbol{k}) - (x \leftrightarrow y)}{(\varepsilon_{n\boldsymbol{k}} - \varepsilon_{m\boldsymbol{k}})^2} \tag{2.50}$$

with the current matrix elements:

$$j_{nm} = -e\, V^{\boldsymbol{k}}_{nm} = -\frac{e}{\hbar}\langle n\boldsymbol{k}|\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}|m\boldsymbol{k}\rangle \tag{2.51}$$

Hence, with $\partial_x H \equiv \partial H/\partial k_x$:

$$\sigma_{xy} = -\frac{e^2}{\hbar}\, i \int \frac{d^2k}{(2\pi)^2}\left[T^{(n)}_{xy} - T^{(n)}_{yx}\right] \tag{2.52}$$

Using $T_{xy} - T_{yx} = T - \bar{T} = 2i\,\mathrm{Im}\,T = -iF$:

$$\boxed{\sigma_{xy} = -\frac{e^2}{\hbar}\int \frac{d^dk}{(2\pi)^2}\sum_n f_n(\boldsymbol{k})\, F^{(n)}_{xy}(\boldsymbol{k})} \tag{2.53}$$

For a filled band:

$$\boxed{\sigma_{xy}^{(n)} = -\frac{e^2}{h}\,C_n} \tag{2.54}$$

where $C_n$ is the Chern number of band $n$:

$$C_n = \int \frac{dk_x \wedge dk_y}{2\pi}\,F_{xy}^{(n)}(\boldsymbol{k})$$

Importantly, this is the Chern number associated with a single-particle eigenstate (contrast to the topological analysis of the IQHE in the previous section).

4. Semi-classical Interpretation

With an applied electric field but $\boldsymbol{B} = \boldsymbol{0}$, the equations of motion for the centre of mass and velocity of a wave-packet are:

$$\dot{\boldsymbol{r}} = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\varepsilon_{\boldsymbol{k}} - \dot{\boldsymbol{k}} \times \boldsymbol{F} \tag{2.55}$$

$$\hbar\dot{\boldsymbol{k}} = -eE \qquad \text{(dissipationless)} \tag{2.56}$$

The second term in the first equation is the **Karplus–Luttinger anomalous term**, representing an effective $B$-field in $\boldsymbol{k}$-space.

For a filled band, the current is:

$$\boldsymbol{j} = -e\int \frac{d^2k}{(2\pi)^2}\dot{\boldsymbol{r}} = -e\int \frac{d^2k}{(2\pi)^2}\left[\frac{1}{\hbar}\nabla_{\boldsymbol{k}}\varepsilon_{\boldsymbol{k}} + \frac{e}{\hbar}\boldsymbol{E} \times \boldsymbol{F}\right] \tag{2.57}$$

The first term vanishes for a filled band. With $\boldsymbol{E} = E_y\,\hat{y}$ and using:

$$\int \frac{d^2k}{2\pi}F_n = C_n\,\hat{z}, \qquad \boldsymbol{E} \times \boldsymbol{F} = C_n\,\hat{x} \tag{2.58}$$

one recovers:

$$\sigma_{xy}^{(n)} = -\frac{e^2}{h}\,C_n \tag{2.59}$$

*Equations of Motion with Dissipation* We have considered the equations of motion in the absence of dissipation, which is valid for the transverse response. Dissipation can be added by including a phenomenological friction force in the second equation:

$$\hbar\dot{k}_a = -eE_a - \frac{1}{\tau}\sum_b M_{ab}(\boldsymbol{k})\,\dot{r}_b \tag{2.60}$$

where the effective mass tensor is:

$$\left(M^{-1}\right)_{ab} = \frac{1}{\hbar^2}\frac{\partial^2\varepsilon_n}{\partial k_a\,\partial k_b} \tag{2.61}$$

# III. GRAPHENE: BANDSTRUCTURE

Carbon is 1s$^2$2s$^2$2p$^2$. The 1s electrons form core levels. The 2s and 2p$_{x,y}$ orbitals in graphene form bond-centered sp2 hybrids. Consider two neighboring carbon atoms. Sigma-hybridisation of the sp2 hybrids lead to 3 bonding and 3 antibonding bands. The 3 bonding bands each accomodate 2 electrons from the two atoms, hence 6 electrons in total out of the 8 valence ones coming from two neighboring C atoms in the motif ($\sigma$-bonds). We are left with one electron per C atom in the $p_z$ orbital, which $\pi-$ bonds with its neighbors. The following is the tight-binding description of this $p_z$ band.

## A. Graphene: Lattice Structure

The Bravais lattice of graphene is triangular. A possible choice for the primitive lattice vectors is:

$$\boldsymbol{a}_1 = \sqrt{3}\,a\,\boldsymbol{e}_x, \qquad \boldsymbol{a}_2 = a\left(\frac{\sqrt{3}}{2}\boldsymbol{e}_x + \frac{3}{2}\boldsymbol{e}_y\right) \tag{3.1}$$

with $\|\boldsymbol{a}_1\| = \|\boldsymbol{a}_2\| = \sqrt{3}\,a$.

Each A site is connected to its three B neighbours by the vectors:

$$\boldsymbol{\delta}_1 = \frac{a}{2}\left(\sqrt{3}\,\boldsymbol{e}_x + \boldsymbol{e}_y\right), \quad \boldsymbol{\delta}_2 = \frac{a}{2}\left(-\sqrt{3}\boldsymbol{e}_x + \boldsymbol{e}_y\right), \quad \boldsymbol{\delta}_3 = -a\,\boldsymbol{e}_y \tag{3.2}$$

### Reciprocal Lattice

The reciprocal lattice vectors are:

$$\boldsymbol{g}_1 = \frac{2\pi}{a}\left[\frac{1}{\sqrt{3}}\boldsymbol{e}_x - \frac{1}{3}\boldsymbol{e}_y\right], \qquad \boldsymbol{g}_2 = \frac{2\pi}{a}\,\frac{2}{3}\,\boldsymbol{e}_y \tag{3.3}$$

with $\|\boldsymbol{g}_1\| = \|\boldsymbol{g}_2\| = \frac{4\pi}{3a}$ and $\boldsymbol{g}_i \cdot \boldsymbol{a}_j = 2\pi\delta_{ij}$.

## B. Brillouin Zone and High-Symmetry Points

Instead of the parallelogram built on $\boldsymbol{g}_1$,$\boldsymbol{g}_2$, the conventional choice of BZ is hexagonal. The inequivalent Dirac points are:

$$K = \frac{4\pi}{3\sqrt{3}\,a}\,\boldsymbol{e}_x \tag{3.4}$$

$-K$ is its time-reversed partner. The second inequivalent corner is:

$$K' = \frac{2\pi}{3\sqrt{3}\,a}\,\boldsymbol{e}_x + \frac{2\pi}{3a}\,\boldsymbol{e}_y \tag{3.5}$$

with:

$$\|K'\| = \frac{2\pi}{3a}\left(\frac{1}{3}+1\right)^{1/2} = \frac{4\pi}{3\sqrt{3}\,a} \tag{3.6}$$

One can verify:

$$K' = -K + (\boldsymbol{g}_1 + \boldsymbol{g}_2) \tag{3.7}$$

so $K'$ and $-K$ are equivalent (related by a reciprocal lattice vector), while $K$ and $-K$ are inequivalent time-reversal partners.

The $M$-point (midpoint of a BZ edge) is:

$$M = \frac{1}{2}(K + K') = \frac{\pi}{\sqrt{3}\,a}\,\boldsymbol{e}_x + \frac{\pi}{3a}\,\boldsymbol{e}_y = \frac{1}{2}(\boldsymbol{g}_1 + \boldsymbol{g}_2) \tag{3.8}$$

### C. Band Structure

Let $m = A{,}B$ denote the sublattice index ('isospin'). The Fourier-transformed operators are:

$$c_m(\boldsymbol{r}) = \frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, c_{\boldsymbol{k}m}, \qquad N = \#\text{ of unit cells} \tag{3.9}$$

The Hamiltonian in momentum space is:

$$\hat{H}_0 = \sum_{\boldsymbol{k}}\sum_{mn} c^\dagger_{\boldsymbol{k}m}\, H_0(\boldsymbol{k})_{mn}\, c_{\boldsymbol{k}n} \tag{3.10}$$

In real space, the tight-binding Hamiltonian is (hopping between A and B sublattices):

$$\hat{H}_0 = -t\sum_{\boldsymbol{r}\in B}\sum_{\alpha=1}^{3} c^\dagger_A(\boldsymbol{r}+\boldsymbol{\delta}_\alpha)\, c_B(\boldsymbol{r}) + \text{h.c.} \tag{3.11}$$

The $\boldsymbol{k}$-space Hamiltonian (in the sublattice basis) is:

$$H_0(\boldsymbol{k}) = d_x(\boldsymbol{k})\,\sigma_x + d_y(\boldsymbol{k})\,\sigma_y \tag{3.12}$$

with:

$$d_x(\boldsymbol{k}) = -t\sum_{\alpha=1}^{3}\cos(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha), \qquad d_y(\boldsymbol{k}) = -t\sum_{\alpha=1}^{3}\sin(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha) \tag{3.13}$$

and the Pauli matrices:

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3.14}$$

### D. Eigenvalues and Eigenvectors of a two level system (Intermezzo)

For a general two-level Hamiltonian $d_0\sigma_0 + \boldsymbol{d}\cdot\boldsymbol{\sigma}$ with $\sigma_0 = \mathbf{1}$, the eigenvalues are:

$$\varepsilon_\pm = d_0 \pm \|\boldsymbol{d}\|, \qquad \|\boldsymbol{d}\| = \sqrt{d_x^2 + d_y^2 + d_z^2} \tag{3.15}$$

Parametrize by the unit vector:

$$\boldsymbol{n}(\boldsymbol{k}) = \frac{\boldsymbol{d}(\boldsymbol{k})}{\|\boldsymbol{d}(\boldsymbol{k})\|} = \begin{pmatrix} \cos\varphi_k \sin\theta_k \\ \sin\varphi_k \sin\theta_k \\ \cos\theta_k \end{pmatrix} \tag{3.16}$$

The eigenstates are:

$$|u_+(\boldsymbol{k})\rangle = \cos\frac{\theta_k}{2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta_k}{2}\, e^{i\varphi_k}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.17}$$

$$|u_-(\boldsymbol{k})\rangle = \sin\frac{\theta_k}{2}\, e^{-i\varphi_k}\begin{pmatrix}1\\0\end{pmatrix} - \cos\frac{\theta_k}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.18}$$

up to a choice of phase $\varphi_k$. Another gauge choice is:

$$|u_+(\boldsymbol{k})\rangle = e^{-i\varphi_k}\cos\frac{\theta_k}{2}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta_k}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.19}$$

$$|u_-(\boldsymbol{k})\rangle = \sin\frac{\theta_k}{2}\begin{pmatrix}1\\0\end{pmatrix} - e^{i\varphi_k}\cos\frac{\theta_k}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.20}$$

Using $\mathbf{d}\cdot\sigma\,|u_\pm\rangle = \pm\|\boldsymbol{d}\||u_\pm\rangle$, it is easily shown that:

$$\langle u_\pm|\boldsymbol{\sigma}|u_\pm\rangle = \pm\,\boldsymbol{n}$$

### E. Graphene bandstructure (continued)

In graphene, $\theta_k = \pi/2$ (since $d_z = 0$), so:

$$\|\boldsymbol{d}\| = t\left[\left(\sum_\alpha \cos(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha)\right)^2 + \left(\sum_\alpha \sin(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha)\right)^2\right]^{1/2} \tag{3.21}$$

$$\varepsilon_\pm(\boldsymbol{k}) = \pm\|\boldsymbol{d}(\boldsymbol{k})\|, \qquad \cos\varphi_k = \frac{\sum_\alpha \cos(\boldsymbol{k}\cdot\boldsymbol{\delta}_\alpha)}{\|\boldsymbol{d}\|} \tag{3.22}$$

The eigenstates reduce to:

$$|u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\varphi_k}\end{pmatrix}, \qquad |u_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-e^{i\varphi_k}\end{pmatrix} \tag{3.23}$$

### F. Dirac Points

Let's expand the hamiltonian around the $\boldsymbol{K}$-point, setting $\boldsymbol{k} = \boldsymbol{K} + \boldsymbol{q}$. A simple calculation leads to:

$$H_0 \simeq \hbar v_F \begin{pmatrix} 0 & q_x - i q_y \\ q_x + i q_y & 0 \end{pmatrix} = \hbar v_F (q_x\, \sigma_x + q_y\, \sigma_y) = \hbar v_F \boldsymbol{q} \cdot \boldsymbol{\sigma}, \qquad \hbar v_F = \frac{3at}{2} \tag{3.24}$$

with $v_F \simeq 10^6 m s^{-1} \simeq c/300$ is the Fermi velocity.

Around $-\boldsymbol{K}$, one obtains instead ($\hbar = 1$):

$$H_0 \simeq \hbar v_F \begin{pmatrix} 0 & -q_x - i q_y \\ -q_x + i q_y & 0 \end{pmatrix} = -\hbar v_F\ \boldsymbol{q} \cdot \boldsymbol{\sigma}^* \tag{3.25}$$

where $*$ denotes complex conjugation, consistent with $-\boldsymbol{K}$ being time-reversed of $\boldsymbol{K}$ so that:

$$H(-\boldsymbol{K} + \boldsymbol{q}) = H(\boldsymbol{K} - \boldsymbol{q})^* \tag{3.26}$$

The eigenvalues are valley independent: $\varepsilon_\pm = \pm \hbar v_F ||\boldsymbol{q}||$.

In the vicinity of each valley, we have a Dirac spectrum with opposite chirality

The low-energy theory of graphene can thus be described by a $4 \times 4$ hamiltonian, introducing the valley degree of freedom and corresponding Pauli matrix $\tau^z$:

$$\hat{H}_4 = -i\hbar v_F\ [\sigma_x \otimes \tau_z\, \partial_x + \sigma_y \otimes 1\, \partial_y] \tag{3.27}$$

### G. Symmetries

*a. Time reversal:* $t \to -t$, $\boldsymbol{r} \to \boldsymbol{r}$, $\boldsymbol{k} \to -\boldsymbol{k}$:

$$\psi(\boldsymbol{r},t) \to \psi^*(\boldsymbol{r},-t) \tag{3.28}$$

Hence under time-reversal:

$$H(\boldsymbol{k}) \to H^*(-\boldsymbol{k})$$

If one can show that these two hamiltonians are related by a $\boldsymbol{k}$-independent unitary: $H^*(-\boldsymbol{k}) = U^\dagger\, H(\boldsymbol{k})\, U$, then the system is T-invariant. However this is not a gauge-independent statement. In the simplest case with $U = 1$ the condition:

$$d_x(\boldsymbol{k}) = d_x(-\boldsymbol{k}), \qquad d_y(\boldsymbol{k}) = -d_y(-\boldsymbol{k}), \qquad d_z(\boldsymbol{k}) = d_z(-\boldsymbol{k}) \tag{3.29}$$

insures that the system is T-invariant.

*b. Inversion:* $\boldsymbol{r} \to -\boldsymbol{r}$, $\boldsymbol{k} \to -\boldsymbol{k}$ (and A site $\to$ B site):

$$\psi(\boldsymbol{r},t) \to \psi(-\boldsymbol{r},t) \tag{3.30}$$

Under inversion, $H_{AA} \to H_{BB}(-\boldsymbol{k})$ and $H_{AB}(\boldsymbol{k}) \to H_{BA}(-\boldsymbol{k})$, and hence inversion corresponds to:

$$H \to \sigma_x H(-\boldsymbol{k})\,\sigma_x$$

As before, if $H(-\boldsymbol{k}) = U^\dagger\, H(\boldsymbol{k})\, U$, with $U$ a $\boldsymbol{k}$-independent unitary, then the system is invariant under inversion (beware of the gauge, however). For example ($U = \sigma_x$), the condition

$$d_x(\boldsymbol{k}) = d_x(-\boldsymbol{k}), \qquad d_y(\boldsymbol{k}) = -d_y(-\boldsymbol{k}), \qquad d_z(\boldsymbol{k}) = -d_z(-\boldsymbol{k}) \tag{3.31}$$

insures I-invariance.

*c. Combined TI invariance* Combined time-reversal and inversion invariance is satisfied if $d_z = 0$, with $d_x$ even and $d_y$ odd. With TI invariance, the existence of Dirac points is *robust*: the two conditions

$$d_x(k_x,k_y) = 0, \qquad d_y(k_x,k_y) = 0 \tag{3.32}$$

generically can have solutions (two equations in two unknowns).

### H. Massive Dirac Spectrum (hBN)

When A and B sites are occupied by different atoms (as in hexagonal boron nitride), a staggered potential breaks the sublattice symmetry:

$$m\left(c_A^\dagger c_A - c_B^\dagger c_B\right) \tag{3.33}$$

The Hamiltonian becomes $H(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$ with $d_z(\boldsymbol{k}) = m$. Near $\pm K$ (with $\xi K + \boldsymbol{q} = \boldsymbol{k}$):

$$\varepsilon_\pm(\boldsymbol{k}) = \pm\sqrt{v_F^2 q^2 + m^2} \tag{3.34}$$

Hence a gapped spectrum (gap $2m$) with separate bands. The low-energy hamiltonian in the sublattice-valley $4 \times 4$ notation now reads:

$$\hat{H}_4 = -i\hbar v_F\,\left[\sigma_x \otimes \tau_z\,\partial_x + \sigma_y \otimes 1\,\partial_y\right] + m\,\sigma_z \otimes 1 \tag{3.35}$$

## IV. TOPOLOGY OF A TWO-LEVEL SYSTEM

### A. Berry curvature and Chern number

For $H(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$:

$$\partial_i H(\boldsymbol{k}) = \partial_i d_\alpha \, \sigma_\alpha \tag{4.1}$$

$$(\varepsilon_1 - \varepsilon_2)^2 = (\varepsilon_+ - \varepsilon_-)^2 = 4|\boldsymbol{d}|^2 \tag{4.2}$$

From the perturbation theory expression for the Berry curvature:

$$F_{xy}^{(n)} = \frac{i}{4d^2} \langle u_n | \partial_x H(\boldsymbol{k}) \, \partial_y H(\boldsymbol{k}) | u_n \rangle + \text{c.c.} \tag{4.3}$$

Using:

$$\partial_x \hat{H} \, \partial_y \hat{H} = (\partial_x d_\alpha)(\partial_y d_\beta) \, \sigma_\alpha \sigma_\beta \tag{4.4}$$

and the identity $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$:

$$\partial_x \hat{H} \, \partial_y \hat{H} = \partial_x d_\alpha \cdot \partial_y d_\alpha + i \, \varepsilon_{\alpha\beta\gamma} \, \partial_x d_\alpha \, \partial_y d_\beta \, \sigma_\gamma \tag{4.5}$$

The first (symmetric) term together with its c.c. does not contribute. Hence:

$$F_{xy}^{(n)} = -\frac{1}{2d^2} \, \varepsilon_{\alpha\beta\gamma} \langle u_n | \sigma_\gamma | u_n \rangle \, \partial_x d_\alpha \, \partial_y d_\beta \tag{4.6}$$

Let us evaluate this for the lower band, using: $\langle u_- | \boldsymbol{\sigma} | u_- \rangle = -\boldsymbol{n}$:

$$F_{xy} = +\frac{1}{2d^2} \, \varepsilon_{\alpha\beta\gamma} \, n_\gamma \, \partial_x d_\alpha \, \partial_y d_\beta \tag{4.7}$$

Using $\partial_x d_\alpha = n_\alpha \, \partial_x \|\boldsymbol{d}\| + \|\boldsymbol{d}\| \, \partial_x n_\alpha$ (the first term vanishes upon contraction with $\varepsilon_{\alpha\beta\gamma}$):

$$F_{xy} = \frac{1}{2} \, \varepsilon_{\alpha\beta\gamma} \, n_\gamma \, \partial_x n_\alpha \, \partial_y n_\beta = \frac{1}{2} \, \boldsymbol{n} \cdot (\partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n}) = \frac{1}{2||\boldsymbol{d}||^3} \, \boldsymbol{d} \cdot (\partial_x \boldsymbol{d} \times \partial_y \boldsymbol{d}) \tag{4.8}$$

The Chern number of the lower band is thus:

$$C = \frac{1}{2\pi} \int F_{xy} \, dk_x \wedge dk_y = \frac{1}{4\pi} \int dk_x \wedge dk_y \, \boldsymbol{n} \cdot (\partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n}) \tag{4.9}$$

An explicit computation yields:

$$C = \frac{1}{4\pi} \int dk_x \, dk_y \, [\partial_x \theta_k \, \partial_y \varphi_k - \partial_y \theta_k \, \partial_x \varphi_k] \sin \theta_k \tag{4.10}$$

Noting that the term in parenthesis is the Jacobian of $k_x$,$k_y \to \theta_k$,$\varphi_k$:

$$C = \frac{1}{4\pi} \int_{Im[T_2]} sin\theta d\theta d\varphi \tag{4.11}$$

This is obviously the surface of the image of the BZ torus $T_2$ wrapped onto $S_2$, hence an integer describing the homotopy from $T_2$ to $S_2$.

When $(k_x,k_y)$ run over the full BZ ($T_2$), $\boldsymbol{n}$ covers an area of the sphere $S_2$. If this area is contractible, the Chern number vanishes. If it wraps the sphere once, $C = 1$ etc.

Application to hBN: slides (cf. Cayssol and Fuchs)

A simpler expression of the Chern number can be obtained, which only involves an evaluation at discrete points in the BZ instead of an integral. It amounts to counting (algebraically) the number of times a ray coming from the origin $||\boldsymbol{d}|| = 0$ intersects the manifold spanned by $\boldsymbol{d}$. Let us choose this ray as being the $Oz$ axis, corresponding to $\boldsymbol{d}$ vectors pointing to the North or South poles. Call $\mathcal{D}$ the corresponding set of $\boldsymbol{k}$ vectors in the BZ, namely the points such that $d_x = d_y = 0$ (corresponding to the location of the Dirac points when the gap is closed). Then one can show that:

$$\boxed{C = \frac{1}{2}\sum_{\boldsymbol{k}\in\mathcal{D}} \operatorname{sign}[d_z(\boldsymbol{k})]\,\operatorname{sign}\left[(\partial_x\boldsymbol{d}\times\partial_y\boldsymbol{d})_z\right]} \tag{4.12}$$

### B. Berry connection and effective magnetic monopole

#### 1. Berry Connection in Two Gauges

We work with the lower eigenstate $|u_-(\boldsymbol{k})\rangle$ of the two-level Hamiltonian $H = \boldsymbol{d}\cdot\boldsymbol{\sigma}$, parametrised by spherical angles $(\theta_k,\varphi_k)$ on the Bloch sphere.

##### a. *Gauge I (singular at the south pole)*

$$|u_-(\boldsymbol{k})\rangle = \sin\frac{\theta_k}{2}\,e^{-i\varphi_k}\begin{pmatrix}1\\0\end{pmatrix} - \cos\frac{\theta_k}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{4.13}$$

Clearly, this spinor is well defined everywhere except at the south pole $\theta = \pi$, since $\phi$ is ill-defined there (multivalued).

We compute the derivatives of $|u_-(\theta,\varphi)\rangle$:

$$|\partial_\theta u\rangle = \frac{1}{2}\cos\frac{\theta}{2}\,e^{-i\varphi}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\sin\frac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{4.14}$$

$$\langle u|\partial_\theta u\rangle = \frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0 \tag{4.15}$$

$$|\partial_\varphi u\rangle = -i\sin\frac{\theta}{2}\,e^{-i\varphi}\begin{pmatrix}1\\0\end{pmatrix} \tag{4.16}$$

$$\langle u|\partial_\varphi u\rangle = -i\sin^2\frac{\theta}{2} \tag{4.17}$$

The Berry connection components are $A_\mu = i\langle u|\partial_\mu u\rangle$:

$$A_\theta^{\mathrm{I}} = 0, \qquad \boxed{A_\varphi^{\mathrm{I}} = \sin^2\frac{\theta}{2}} \tag{4.18}$$

*b. Gauge II (singular at the north pole)*

$$|u_-\rangle = \sin\frac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} - \cos\frac{\theta}{2}\,e^{+i\varphi}\begin{pmatrix}0\\1\end{pmatrix} \tag{4.19}$$

This spinor is, in contrast, singular at the north pole.

$$|\partial_\varphi u\rangle = -i\cos\frac{\theta}{2}\,e^{i\varphi}\begin{pmatrix}0\\1\end{pmatrix} \tag{4.20}$$

$$\boxed{A_\varphi^{\mathrm{II}} = -\cos^2\frac{\theta}{2}} \tag{4.21}$$

The two Berry connections are related by:

$$A_\varphi^{\mathrm{I}} - A_\varphi^{\mathrm{II}} = 1 = \partial_\varphi\,\varphi \tag{4.22}$$

which is precisely a gauge transformation $A \to A - \partial_\varphi\varphi$ (with gauge function $\varphi$).

### 2. Berry Phase Along a Closed Loop

Consider the closed loop $\mathcal{C}_\theta$ defined by the azimuthal angle $\varphi$ at fixed polar angle $\theta$ (a latitude circle on the Bloch sphere). The Berry phases in the two gauges are:

$$\Phi_{\mathrm{I}} = \oint_{\mathcal{C}_\theta} d\varphi\, A_\varphi^{(\mathrm{I})} = 2\pi\sin^2\frac{\theta}{2} \tag{4.23}$$

$$\Phi_{\mathrm{II}} = \oint_{\mathcal{C}_\theta} d\varphi\, A_\varphi^{(\mathrm{II})} = -2\pi\cos^2\frac{\theta}{2} \tag{4.24}$$

The difference is:

$$\Phi_{\mathrm{I}} - \Phi_{\mathrm{II}} = 2\pi \tag{4.25}$$

hence the two gauges describe the same physical phase (they differ by $2\pi$, i.e. by a full winding).

3. Choice of gauge: trivial vs. topological insulators

In a trivial two-band insulator, the map $T^2 \to S^2$ does not cover the whole sphere. Hence, one can always work in a single non-singular gauge, in which the phase of the wave-function is well defined everywhere. If $n_{\boldsymbol{k}}$ does not reach the south pole ($\theta = \pi$), one can pick gauge I (and conversely). For a topological insulator in contrast, in which $n_k$ covers the sphere, there is no unique choice of gauge which is non-singular everywhere. One can use the two gauges, switching from one to the other on different parts of the Bloch sphere.

4. Berry Curvature

The Berry curvature in spherical coordinates on the Bloch sphere is:

$$F_{\theta\varphi} = \partial_\theta A_\varphi - \partial_\varphi A_\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta \tag{4.26}$$

This result holds in both gauges (the curvature is gauge invariant).

The Chern number is:

$$C = \int \frac{d\theta\, d\varphi}{2\pi} F_{\theta\varphi} = \frac{1}{4\pi}\int d\theta\, d\varphi\, \sin\theta \tag{4.27}$$

as seen before (wrapping number of $\boldsymbol{n} : T^2 \to S^2$).

5. Berry Connection in Cartesian Coordinates

Instead of $\theta$,$\phi$, we consider the vector $\boldsymbol{d} = d\boldsymbol{n}$ and the corresponding Berry phase in Cartesian (parameter-space) coordinates:

$$\tilde{\boldsymbol{A}} = i\langle u|\nabla_d u\rangle = i\left\langle u \left| \left( \boldsymbol{e}_\theta \frac{1}{d}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\varphi \frac{1}{d\sin\theta}\frac{\partial}{\partial\varphi} \right) \right| u \right\rangle \tag{4.28}$$

In the two gauges this gives:

$$\boxed{\tilde{\boldsymbol{A}}^{\mathrm{I}} = \frac{1-\cos\theta}{2d\sin\theta}\,\boldsymbol{e}_\varphi} \tag{4.29}$$

$$\boxed{\tilde{\boldsymbol{A}}^{\mathrm{II}} = -\frac{1+\cos\theta}{2d\sin\theta}\,\boldsymbol{e}_\varphi} \tag{4.30}$$

As expected, the first one is singular at the south pole, and the second one at the north pole.

The Berry curvature is:

$$\begin{aligned} \boldsymbol{F} = \nabla_d \times \tilde{\boldsymbol{A}} &= \frac{1}{d\sin\theta}\frac{\partial}{\partial\theta}(A_\varphi d\sin\theta)\,\boldsymbol{e}_r \\ &= \frac{1}{2d^2}\,\boldsymbol{e}_r \end{aligned} \tag{4.31}$$

This is the "magnetic field" in parameter space of a **magnetic monopole** located at $d = 0$. The two choices of vector potential above correspond to two ways of placing the 'Dirac string' (since there cannot be a unique non-singular choice of vector potential corresponding to a monopole). The point $d = 0$ corresponds to the degeneracy of the energy levels.

## V. CHERN INSULATORS: THE HALDANE MODEL

The foundational reference is F.D.M. Haldane *Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the 'Parity Anomaly'* Phys. Rev. Letters 61, 2015 (1988). Here, I start by introducing a simpler version defined on the square lattice.

### A. A simpler square-lattice version of the Haldane Model

We consider a model defined on a square lattice with:
– $t_1 e^{\pm i\pi/4}$ for a hop along/opposite the black arrows
– Semenoff mass (crystal field) $m$ (optional)
– $+t_2$ along the two diagonals of even plaquettes, $-t_2$ in odd plaquettes (next-nearest-neighbour hopping)

The off-diagonal element of the Hamiltonian is (note that there are several possible gauge choices):

$$H_{12} = d_x - i d_y = -2t_1 \left[ e^{-i\pi/4} \cos k_x + e^{i\pi/4} \cos k_y \right] \tag{5.1}$$

with:

$$d_x = -\sqrt{2}\, t_1 \left(\cos k_x + \cos k_y\right) \tag{5.2}$$

$$d_y = \sqrt{2}\, t_1 \left(\cos k_x - \cos k_y\right) \tag{5.3}$$

$$d_x^2 + d_y^2 = 4t_1^2 \left(\cos^2 k_x + \cos^2 k_y\right) \tag{5.4}$$

$$\begin{aligned} d_z &= m - 2t_2 \left[\cos(k_x + k_y) - \cos(k_x - k_y)\right] \\ &= m + 4t_2 \sin k_x \sin k_y \end{aligned} \tag{5.5}$$

The energies of the two bands are:

$$\varepsilon_\pm(\boldsymbol{k}) = \pm||\boldsymbol{d}|| = \pm \left[ 4t_1^2 \left(\cos^2 k_x + \cos^2 k_y\right) + \left(m + 4t_2 \sin k_x \sin k_y\right)^2 \right]^{1/2} \tag{5.6}$$

– For $m = t_2 = 0$ the model displays four Dirac points at $\epsilon_x \pi/2, \epsilon_y \pi/2$ with $\epsilon_{x,y} = \pm 1$.
– As for hBN, a staggered mass $m$ opens a gap between the two bands $\Delta_g = 2m$
– Interestingly, $t_2 \neq 0$ also opens a gap even for $m = 0$ $\Delta_g = 8|t_2|$
– More generally, the gap at the $(\pi/2, \pi/2)$ point is $\Delta_g = 2|m + 4t_2|$, while at $(\pi/2, -\pi/2)$ it is $\Delta_g = 2|m - 4t_2|$. One of the two gaps vanish at $m = \pm 4t_2$.

Hence, we have two gapped insulators (e.g. $m \neq 0, t_2 = 0$ and $m = 0, t_2 \neq 0$) with very similar spectra. The properties of these two insulators are very different however, as detailed below.

*Symmetries* In this gauge, symmetries are a bit subtle. Physically, we expect the model with $t_1$ only to be T-reversal invariant. Indeed, under T, a flux $\pi$ is changed into $-\pi$ but $-\pi + 2\pi = \pi$ and hence $\pm\pi$ are gauge equivalent. However, we see that $H(-k)^* \neq H(k)$ ($d_x$ and $d_y$ are both even) when $t_2 = 0$, so what's going on? The answer is that, in this gauge, time-reversal takes the following form $H(k_x,k_y) \to H(k_y, -k_x)^*$. We also see that $m$ does not break $T$, but $t_2$ does, as expected. There are of course other gauges in which symmetries take a simpler form. I am grateful to Jean-Noël Fuchs for a discussion about this point.

1. Dirac Points and Topological Condition

At the Dirac points:

$$d_z\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right) = m + 4t_2, \qquad d_z\left(\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right) = m - 4t_2 \tag{5.7}$$

Since $d_x = d_y = 0$, the $\boldsymbol{d}$-vector points at either the North or South poles.

Condition for non-trivial topology: opposite signs of these two values. This is a necessary condition for the $T_2 \to S_2$ map to cover the sphere (both poles are reached). It turns out to be also a sufficient condition.

$$m^2 - (4t_2)^2 < 0 \implies \boxed{|t_2| > |m|/4} \tag{5.8}$$

2. Parametric Equation for $\boldsymbol{d}(k_x,k_y)$

Define rescaled variables:

$$\tilde{d}_x = \frac{d_x}{2\sqrt{2}\,t_1}, \qquad \tilde{d}_y = \frac{d_y}{2\sqrt{2}\,t_1}, \qquad \tilde{d}_{x,y} \in [-1,1] \tag{5.9}$$

Then:

$$\cos k_x = -\left(\tilde{d}_x - \tilde{d}_y\right) \tag{5.10}$$

$$\cos k_y = -\left(\tilde{d}_x + \tilde{d}_y\right) \tag{5.11}$$

$$\left(\frac{d_z - m}{4t_2}\right)^2 = \left\{\left[1 - \left(\tilde{d}_x + \tilde{d}_y\right)^2\right]\left[1 - \left(\tilde{d}_x - \tilde{d}_y\right)^2\right]\right\}^{1/2} \tag{5.12}$$

The surface contains $\boldsymbol{d} = 0$ if and only if $|t_2| > |m|/4$.

In particular, for $\tilde{d}_y = 0$:

$$\left(\frac{d_z - m}{4t_2}\right)^2 + \tilde{d}_x^2 = 1 \tag{5.13}$$

where $\tilde{d}_z \equiv d_z/(4t_2)$. This describes a **circle of radius unity centered at** $m$ in the $(\tilde{d}_x,\tilde{d}_z)$ plane. It contains the origin iff $|m| < 4|t_2|$.

The Chern number reads, using the expression in Sec.IV:

$$C = \mathrm{sign}(t_2)\,\frac{1}{2}\left[\mathrm{sign}\left(\frac{m}{t_2}+4\right) - \mathrm{sign}\left(\frac{m}{t_2}-4\right)\right] \tag{5.14}$$

Hence $C = \mathrm{sign}(t_2)$ for $|m| < 4|t_2|$ and $C = 0$ otherwise.

#### 3. Topological phase transition.

*Connecting a trivial and topological insulator:* Imagine starting from the 'atomic' limit in which $t_1 = t_2 = 0$ but with a non-zero $m$ - i.e. two different atoms on sites A and B. Then turn on $t_1$: we obtain two bands continuously connected to this atomic insulator. Turn on $t_2$: at small $t_2/m$, we stay in the same phase. But at $t_2 = \pm m/4$ the gap closes at one of the two Dirac points and one enters a distinct phase - a Chern insulator. This phase transition is not characterized by a local order parameter à la Landau - and the gap must close and reopen through the transition: the two phases are not adiabatically connected.

### B. Haldane Model on the Honeycomb Lattice

#### 1. Hamiltonian

Add next-nearest-neighbour (nnn) hopping $t_2 e^{i\phi}$ with sign such that the flux is:
– $-3\phi$ in A-sites up-pointing triangle
– $+3\phi$ in A-sites down-pointing triangle
Vectors Connecting nnn Sites:

$$\boldsymbol{b}_1 = \boldsymbol{\delta}_2 - \boldsymbol{\delta}_3 = \frac{a}{2}\left(-\sqrt{3}\,\boldsymbol{e}_x + 3\boldsymbol{e}_y\right) \tag{5.15}$$

$$\boldsymbol{b}_2 = \boldsymbol{\delta}_3 - \boldsymbol{\delta}_1 = -\frac{a}{2}\left(\sqrt{3}\,\boldsymbol{e}_x + 3\boldsymbol{e}_y\right) \tag{5.16}$$

$$\boldsymbol{b}_3 = \boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 = \sqrt{3}\,a\,\boldsymbol{e}_x \tag{5.17}$$

The Hamiltonian reads:

$$H(\boldsymbol{k}) = d_0(\boldsymbol{k}) + \boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma} \tag{5.18}$$

with $d_x$, $d_y$ unchanged from graphene and:

$$d_0(\boldsymbol{k}) = 2t_2\cos\phi\sum_{i=1}^{3}\cos(\boldsymbol{k}\cdot\boldsymbol{b}_i) \tag{5.19}$$

$$d_z(\boldsymbol{k}) = m + 2t_2\sin\phi\sum_{i=1}^{3}\sin(\boldsymbol{k}\cdot\boldsymbol{b}_i) \tag{5.20}$$

$$d_x(\boldsymbol{k}) = d_x(-\boldsymbol{k}), \qquad d_y(\boldsymbol{k}) = -d_y(-\boldsymbol{k}) \tag{5.21}$$

and: $d_z(-\boldsymbol{k}) = -d_z(\boldsymbol{k})$ for $m = 0$,$t_2 \neq 0$ while $d_z(-\boldsymbol{k}) = d_z(\boldsymbol{k})$ for $t_2 = 0$,$m \neq 0$. Hence, the different terms obey the following symmetries

– $m = t_2 = 0$: T and I symmetry
– $m \neq 0$,$t_2 = 0$: T-invariant, I broken
– $m = 0$,$t_2 \neq 0$, with $\phi \neq 0$,$\pi$: T broken, I symmetry
– $m \neq 0$,$t_2 \neq 0$: T and I broken

3. Topological Properties

We need to find out if:

– The map $\boldsymbol{k} \mapsto \boldsymbol{n}(\boldsymbol{k})$ covers the sphere entirely.
– Equivalently, whether the surface spanned by $\boldsymbol{d}(\boldsymbol{k})$ contains the origin $\boldsymbol{d} = 0$ or not.

At $\boldsymbol{k} = \pm K$, $d_x = d_y = 0$, hence $\boldsymbol{d}$ points towards the north or south pole. A *necessary condition* for non-trivial topology is that $\boldsymbol{d}(K)$ and $\boldsymbol{d}(-K)$ point at distinct poles.

From:

$$d_z(\pm K) = m \mp 3\sqrt{3}\, t_2 \sin\phi \tag{5.22}$$

we see that this necessary condition reads:

$$d_z(K)\, d_z(-K) = m^2 - \left(3\sqrt{3}\, t_2 \sin\phi\right)^2 < 0 \tag{5.23}$$

Hence:

$$\boxed{\left|\frac{m}{t_2}\right| < 3\sqrt{3}\,|\sin\phi|} \tag{5.24}$$

This condition turns out to be also sufficient.

The Chern number is given by: The Chern number reads, using the expression in Sec.IV:

$$C = \frac{1}{2}\left[\mathrm{sign}\left(\frac{m}{t_2} + 3\sqrt{3}\sin\phi\right) - \mathrm{sign}\left(\frac{m}{t_2} - 3\sqrt{3}\sin\phi\right)\right] \tag{5.25}$$

The Hall conductance is given by:

$$\sigma_{xy} = -\frac{e^2}{h}\, C \tag{5.26}$$

This is a **quantized Hall effect with zero net magnetic field** (but T-breaking): a **Chern insulator**.

### C. Edge states

The transition between a trivial (atomic) and topological insulator takes place when the 'mass' $m \pm 3\sqrt{3}t_2 \sin\phi$ changes sign. Then, the gap closes at either $K$ or $K'$.

Consider an interface located at $y = 0$ between the topological phase and the trivial phase, on which $m(y)$ changes sign and $m(y = 0) = 0$ (with $m(y > 0) > 0$ and $m(y < 0) < 0$). In the regime where the mass is small, we look for solutions of the Dirac equation at the relevant Dirac point, e.g.:

$$[m(y)\sigma_z - i\sigma_x\partial_x - i\sigma_y\partial_y]\,\psi(x,y) = E\psi(x,y) \tag{5.27}$$

with $\psi$ a two-component spinor and $\hbar v_F = 1$. Explicitly:

$$\begin{cases} m(y)\psi_1 - i\partial_x\psi_2 - \partial_y\psi_2 = E\psi_1 \\ -m(y)\psi_2 - i\partial_x\psi_1 + \partial_y\psi_1 = E\psi_2 \end{cases} \tag{5.28}$$

Define:

$$\psi_\pm = \frac{1}{\sqrt{2}}\left[\psi_1 \pm \psi_2\right] \tag{5.29}$$

The system of equations becomes:

$$\begin{cases} -i\partial_x\psi_+ + \partial_y\psi_- + m(y)\psi_- = E\psi_+ \\ \quad i\partial_x\psi_- - \partial_y\psi_+ + m(y)\psi_+ = E\psi_- \end{cases} \tag{5.30}$$

Using translation invariance along $x$, we can look for solutions of the form:

$$\psi_\pm(x,y) = e^{iqx}\,u_\pm(y) \tag{5.31}$$

with the dispersion relation (restoring $\hbar v_F$):

$$E = \hbar v_F\, q \tag{5.32}$$

The equations reduce to:

$$\begin{cases} \partial_y u_- + m(y)\,u_- = 0 \\ \partial_y u_+ - m(y)\,u_+ = 0 \end{cases} \tag{5.33}$$

Given that $m(y > 0) > 0$ and $m(y < 0) < 0$, we must choose $u_+ = 0$ to obtain a normalizable solution. We finally obtain:

$$\boxed{\psi(x,y) \propto e^{iqx}\, e^{-\int_0^y m(y')\,dy'}\, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \tag{5.34}$$

*Physical Interpretation:* this solution represents a freely propagating **right-moving** plane wave along $x$, localised around the interface: a **chiral edge state**.

- For an interface between a trivial insulator and the $C = -1$ Chern insulator, we would have a **left-moving** chiral edge state.

– If $m(y)$ vanishes at $y = 0$ but does not change sign, there is no such normalizable continuous solution: **no edge state**.

The diagonalization of $H(k_x{,}y)$ in the mixed momentum–real space representation (i.e. Fourier transforming along $x$ only, keeping $y$ in real space) gives the edge state spectrum directly: *see slides.*

## VI. THE FRACTIONAL QUANTUM HALL EFFECT

*Useful references*

– S. Girvin, *Introduction to the Fractional Quantum Hall Effect Séminaire Poincaré* **2** (2004) 53
– M. O. Goerbig, *Quantum Hall effects*, arXiv:0909.1998 — Les Houches Lectures
– David Tong, *The Quantum Hall effect*, TIFR Infosys lectures, arXiv:1606.06687
– I am grateful to Nicolas Regnault for inspiring discussion on this topic.

*Remarkable discovery: Tsui, Störmer and Gossard (1982)* The $\nu = \frac{1}{3}$ state: at a magnetic field three times higher than for the $\nu = 1$ IQHE plateau, fractional quantization of $\sigma_{xy}$ is observed:

$$\sigma_{xy} = \nu \frac{e^2}{h}, \qquad \sigma_{xx} \to 0 \tag{6.1}$$

with the filling fraction:

$$\nu = \frac{N}{N_\phi} = \frac{1}{3} \tag{6.2}$$

The system is in a **partially filled Landau level**. There is no kinetic energy and there is macroscopic degeneracy. Consequently, <u>interactions are a singular perturbation</u>.

### A. The $\nu = 1$ Many-Body Wave Function

Recall the single-particle wave functions in the symmetric gauge:

$$z^m \, e^{-|z|^2/4} \tag{6.3}$$

where $z = (x - iy)/\ell$ is normalised to the magnetic length $\ell$.

Fill all levels $m = 0{,}1,\ldots,N-1$ with one electron each and form the Slater determinant:

$$f_N(z) = \begin{vmatrix} z_1^0 & \cdots & z_N^0 \\ \vdots & & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = -\prod_{i<j} (z_i - z_j) \tag{6.4}$$

This is the **Vandermonde determinant**. For $N = 2$:

$$\begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} = z_2 - z_1 \tag{6.5}$$

Including the Gaussian factors, the full $\nu = 1$ many-body wave function is:

$$\boxed{\Psi_N(z_1 \cdots z_N) = \prod_{i<j} (z_i - z_j) \, e^{-\sum_i |z_i|^2/4}} \tag{6.6}$$

#### B. The Laughlin Wave Function

To take care of the repulsive interaction between electrons, let's multiply the Slater determinant above by a Jastrow factor:

$$\psi_L(z_1 \cdots z_N) = \prod_{i<j}(z_i - z_j)^{2s} \prod_{i<j}(z_i - z_j)\, e^{-\sum_i |z_i|^2/4} \tag{6.7}$$

where $2s$ is an even integer, preserving antisymmetry.

For which filling fraction could this be a valid variational wave-function?

The corresponding average density is not immediately apparent from the wave-function. It can be obtained from the plasma analogy below, or guessed from the following argument.

Consider a given electron $k$. There are $N-1$ factors $(z_k - z_\ell)^{2s+1}$ with $\ell \neq k$. Hence the highest power of $z_k$ is $z_k^{(2s+1)(N-1)}$. This must equal to the available number of states in the Landau level:

$$(2s+1)(N-1) \sim N_\phi \tag{6.8}$$

In the large-$N$ limit:

$$\nu \equiv \frac{N}{N_\phi} = \frac{1}{2s+1} \tag{6.9}$$

The Laughlin wave-function for filling factor $\nu = 1/p$ ($p$ odd) is:

$$\boxed{\psi_L^{[p]}(z_1 \cdots z_N) = \prod_{i<j}(z_i - z_j)^p\, e^{-\sum_i |z_i|^2/4}} \tag{6.10}$$

It turns out to be an excellent variational wave-function (see slides and below). It is quite interesting to read Bob Laughlin's account of how he was led to propose this wavefunction - exact diagonalisations on small systems played an important role in building up his intuition.

Note: We have introduced a Jastrow factor, but *no backflow* - i.e. the orbitals in the Slater determinant are unmodified from the free electron ones, with no dependence on the coordinates of the other electrons. Indeed, the backflow correction - introduced by Feynman for Helium 4, is unimportant here - in contrast to Helium 4. The reason is that the backflow correction is mainly about the kinetic energy which is absent in the present case!

Viewing the Laughlin wavefunction as the product of a Jastrow factor by a Slater determinant is also the first step towards the *composite fermion* picture of Jainendra Jain (see below).

#### C. Plasma Analogy (Laughlin)

The squared modulus of the Laughlin wave function can be written as a Boltzmann weight:

$$|\Psi(z)|^2 = \prod_{i<j} |z_i - z_j|^{2p} \, e^{-\frac{1}{2}\sum_i |z_i|^2} = e^{-\beta \, U(z_1 \cdots z_N)} \tag{6.11}$$

##### Normalisation

$$\mathcal{Z} = \int dz_1 \cdots dz_N \, |\Psi(z)|^2 \tag{6.12}$$

The effective potential is:

$$U(z_1, \ldots, z_N) = p^2 \sum_{i<j} (-\ln |z_i - z_j|) + \frac{p}{4} \sum_i |z_i|^2 \tag{6.13}$$

with $\beta = 2/p$. Here $p = 1$.

– **First term:** particles of "charge" $p$ interacting with the 2D Coulomb potential.
– **Second term** (Poisson equation):

$$-\nabla^2 \frac{1}{4} |z|^2 = -\frac{1}{\ell^2} \equiv 2\pi \varrho_B \tag{6.14}$$

$\varrho_B$: uniform negative charge background.

$$\varrho_B = -\frac{1}{2\pi\ell^2} \tag{6.15}$$

**Note:** $2\pi\ell^2$ contains 1 flux quantum, so $\varrho_B = -B/\Phi_0$.

#### D. Neutrality Condition and Filling Fraction

The neutrality condition $np + \varrho_B = 0$ gives:

$$n = \frac{1}{p} \frac{1}{2\pi\ell^2} \tag{6.16}$$

$$\boxed{\nu \equiv 2\pi\ell^2 \, n = \frac{1}{p}} \tag{6.17}$$

### E. Haldane Pseudopotentials

#### 1. The Two-Body Problem

Let $m$ denote the relative angular momentum (odd) and $M$ the centre-of-mass angular momentum. The two-body eigenstates are:

$$\psi_{mM}(z_1,z_2) = (z_1 - z_2)^m (z_1 + z_2)^M \, e^{-\frac{1}{4}(|z_1|^2+|z_2|^2)} \tag{6.18}$$

These are exact solutions of the two-body problem for any interaction $V(r)$, with $r \equiv |z_1 - z_2|$.

The **Haldane pseudopotential** is:

$$V_m = \frac{\langle mM|\hat{V}|mM\rangle}{\langle mM|mM\rangle} \qquad \text{(independent of } M\text{)} \tag{6.19}$$

$\Rightarrow$ Plot $V_m$ vs. $m$ for the Coulomb interaction.

Note that with a magnetic field, a repulsive interaction leads to bound states!

**N.B.** Because the average distance of 2 particles with relative angular momentum $m$ is $\ell\sqrt{2m}$:

$$V_m \simeq V\Big(|z| = \ell\sqrt{2m}\Big) \tag{6.20}$$

#### 2. The Many-Body Problem

The relative angular momenta $L_{12}, L_{13}, L_{23}, \ldots$ do not commute. Hence there is no exact solution in general.

The interaction can be written as:

$$\hat{V} = \sum_{m=0}^{\infty} \sum_{i<j} V_m \, \hat{P}_m(ij) \tag{6.21}$$

where $\hat{P}_m$ is the projection operator onto states where $(ij)$ have relative angular momentum $m$.

Consider the truncated interaction:

$$\hat{W}_m : \quad V_{m'} = 0 \text{ for } m' \geq m \tag{6.22}$$

Then the Laughlin state for $\nu = 1/m$ is an **exact zero-energy eigenstate**:

$$\hat{W}_m \, \psi_L^{[m]} = 0 \tag{6.23}$$

Every pair in $\psi_L^{[m]}$ has relative angular momentum at least $m$. Hence:

$$\hat{P}_{m'}(ij) \, \psi_L^{[m]} = 0 \qquad m' < m \tag{6.24}$$

It should be noted [see Tong], however, that there is a an infinite family of exact zero-energy eigenstates of the hamiltonian with a truncated interaction. The Laughlin wave-function multiplied by *any* symmetric polynomial $P(\{z_i\})$ is a zero-energy eigenstate. However the Laughlin wave-function is the one which is the most compact in real-space. Hence, in the presence of a confining potential which lifts the degeneracy, the Laughlin wave function with $P = 1$ will have the lowest energy. A term proportional to the total angular momentum $J$ can for example be added to the hamiltonian to lift the degeneracy.

### F. Flux attachment and composite particles

#### 1. Intermezzo (reminder): the Aharonov-Bohm effect

The phase accumulated by the wave function of a quantum particle along a path $\mathcal{P}$ in the presence of a vector potential $\boldsymbol{A}$ is:

$$\theta = \frac{e}{\hbar}\int_{\mathcal{P}} \boldsymbol{A}\cdot d\boldsymbol{\ell} \tag{6.25}$$

Along a closed loop:

$$\Delta\theta = \frac{e}{\hbar}\oint_C \boldsymbol{A}\cdot d\boldsymbol{\ell} = 2\pi\,\frac{\phi}{\phi_0} \tag{6.26}$$

with $\phi$ the flux through $C$ and $\phi_0 = h/e$ the flux quantum.

In the Laughlin wave function for $\nu = 1/m$, the antisymmetric factor:

$$\prod_{i<j}(z_i - z_j)^m \tag{6.27}$$

implies that if particle 2 makes a complete rotation $(2\pi)$ around particle 1, the wave function changes by a phase:

$$\Delta\theta = 2\pi m \tag{6.28}$$

corresponding to $m$ flux quanta. The Laughlin wave function can be rewritten as:

$$\psi_L = \prod_{i<j}\left(\frac{z_i - z_j}{|z_i - z_j|}\right)^m \prod_{i<j}|z_i - z_j|^m\, e^{-\sum_i |z_i|^2/4} \tag{6.29}$$

The first product is $\prod_{i<j} e^{im\theta_{ij}}$, which is the singular gauge transformation that attaches $m$ flux quanta to each electron, while the second one is a symmetric wave-function - hence describing a *bosonic* system. Indeed, each electron with an off number $(m)$ of flux attachment has acquired bosonic statistics. Note that $Nm\phi_0 = N\phi_0/\nu = \phi_B$, the total flux of the applied field, hence all the applied field has been used in the flux attachment procedure with no residual field acting on these bosonic particles, and there is no residual field. The gas of bosons then forms a condensate. This *composte boson* viewpoint is used as a starting point of some Chern-Simons effective field theories of the FQHE.

An alternative view of the $\nu = 1/m$ state is to attach $m-1$ (even) number of flux quanta to each electrons. These composite particles are now *fermionic*. This is the viewpoint developed by Jainendra Jain. The Laughlin wave function is written as:

$$\psi_L = \prod_{i<j} |z_i - z_j|^{m-1} \prod_{i<j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right)^{m-1} \prod_{i<j} (z_i - z_j) e^{-\sum_i |z_i|^2/4} \tag{6.30}$$

The middle term attaches $m-1$ flux quanta, the last term is the Slater determinant which is the ground-state of an *integer* quantum Hall problem at an effective filling fraction $\nu^* = 1$ and the first term is a Jastrow factor.

In that picture we have distributed the total number $N_\phi = Nm$ as:

$$N_\phi = N(m-1) + N \Rightarrow \nu^* = 1$$

So that exactly we are left with N composte fermions seeing exactly N residual flux quanta, i.e the composite fermions are in an integer quantum Hall state.

More generally, the *Jain series* ($p$ odd):

$$\nu = \frac{p}{2s\,p \pm 1}$$

can be reached by attaching $2s$ flux quanta to each electron. For $s = 1$ (2 attached fluxed quanta, as above) this yields the series:

$$\nu = \frac{p}{2p+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \cdots \quad , \; \nu = \frac{p}{2p-1} = 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \cdots$$

which converges to the $\nu = 1/2$ state as $p \to \infty$. In this construction the total number of flux quanta has been distributed as:

$$\frac{N}{\nu} = 2s\,N + \frac{N}{\nu^*} \Rightarrow \frac{1}{\nu^*} = \frac{1}{\nu} - 2s$$

So that:

$$\nu^* = p$$

The composite fermions fill $p$ Landau levels and see a residual field (for $\nu = p/(2sp+1)$)

$$\frac{B^*}{B} = \frac{\nu}{\nu^*} = \frac{1}{2sp+1}$$

with the $\nu = p/(2sp-1)$ series obtained by reversing $B^*$.

One can also construct a Chern-Simons field theory description based on composite fermions, *distinct* from the one based on composite bosons.

### G. Excitations: quasi-particles

In the following the filling fraction is $\nu = 1/m$ ($m$ odd).
Key take-home points:

- There is an **energy gap**.
- Quasiparticle (charged) excitations have:
  - Fractional charge $\pm e/m$
  - Fractional statistics (anyon)
- There is also a collective neutral excitation: the *magneto-roton* which will be described in the next section.

#### 1. The quasi-hole wave-function and fractional charge

Consider the wave function obtained by inserting a zero at a fixed position $\eta$:

$$\psi_h(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)\, \psi_L(z_1, \ldots, z_N) \tag{6.31}$$

The electron density now vanishes at $z = \eta$: this is a hole in the electron fluid.

*Heuristic Explanation of Fractional Charge.* Consider now $m$ quasi-holes, with wave-function:

$$\prod_{i=1}^{N}(z_i - \eta)^m\, \psi_L$$

We have created a deficit of one full electron at position $\eta$. Hence $\psi_h$ must carry charge $e^* = e/m$.

This can also be seen from the plasma analogy. The effective plasma potential for one quasi-hole is:

$$U_h[\{z_i\}] = -m^2 \sum_{i<j} \ln|z_i - z_j| + \frac{m}{4}\sum_i |z_i|^2 - m\sum_i \ln|z_i - \eta| \tag{6.32}$$

The last term corresponds to an impurity in the plasma. It can be rewritten as:

$$-m^2\, \frac{1}{m} \sum_i \ln|z_i - \eta| \tag{6.33}$$

which makes clear that each electron interacts with a charged impurity of charge $-e/m$. Hence the charge deficit is $+e/m$.

#### 2. Fractional charge deduced from charge pumping

Consider the Laughlin argument for the IQHE in Corbino geometry. Pierce the centre of the sample with a magnetic flux $\phi(t)$ which is increased from $\phi(0) = 0$

to $\phi(t_f) = \phi_0 = h/e$, the flux quantum. Do this adiabatically, i.e. over time scales much slower than $\hbar/\Delta$, with $\Delta$ the excitation gap.

The vector potential due to the inserted flux is:

$$\delta \boldsymbol{A}(t) = \frac{\phi(t)}{2\pi r}\, \boldsymbol{e}_\theta \tag{6.34}$$

At the end of the process:

$$\delta \boldsymbol{A} = \frac{\phi_0}{2\pi r}\, \boldsymbol{e}_\theta \tag{6.35}$$

corresponding to an Aharonov–Bohm phase:

$$\frac{e}{\hbar} \oint \delta \boldsymbol{A} \cdot d\boldsymbol{\ell} = \frac{e}{\hbar}\, \phi_0 = 2\pi \tag{6.36}$$

Hence $H(\phi_0)$ and $H(0)$ are unitarily related. Note that $\delta\boldsymbol{A}$ is a singular gauge transformation. $H(0)$ and $H(\phi_0)$ have identical spectra, but that does *not* imply that the system remains in the same state.

What happens is the following:

- The Hamiltonian $H[\phi(t)]$ is time-dependent.
- $H[\phi_0]$ at the final time is unitarily equivalent to $H(t=0)$ since $\phi_0$ corresponds to a $2\pi$ Aharonov–Bohm phase change.
- The system remains in the ground state of $H(t)$ at all times.
- For the IQHE, it returns to the original (unique) ground state at the final time.
- For the FQHE state, this is *not* the case. $H(0)$ and $H(\phi_0)$ are related by a <u>singular</u> gauge transformation.

Let's analyse this in the Corbino geometry with an outer and inner edge. At a plateau (protected by the gap, hence dissipationless):

$$\sigma_{xx} = \sigma_{yy} = \rho_{xx} = \rho_{yy} = 0 \tag{6.37}$$

The varying flux induces a time-dependent electric field. From Faraday's law along a contour at radius $R$:

$$\oint_r \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{\partial \phi}{\partial t}, \qquad 2\pi R\, \boldsymbol{E} = -\dot{\phi}\, \boldsymbol{e}_\theta \tag{6.38}$$

Since the current is purely transverse:

$$\boldsymbol{j} = \sigma_{xy}\, \boldsymbol{E} \times \hat{z} \tag{6.39}$$

$$\boldsymbol{j} = -\sigma_{xy} E\, \boldsymbol{e}_r = -\sigma_{xy}\, \frac{\dot{\phi}}{2\pi R}\, \boldsymbol{e}_r \tag{6.40}$$

Integrating over the volume delimited by the contour $\Gamma$ (one can extend to a cylinder) and using the continuity equation:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0 \tag{6.41}$$

we obtain the charge flowing into the contour:

$$\frac{dQ}{dt} = \sigma_{xy} \frac{d\phi}{dt} \tag{6.42}$$

Hence after the full cycle, the charge that has flowed in is:

$$Q = \sigma_{xy}\,\phi_0 = \frac{1}{m}\frac{e^2}{h} \cdot \frac{h}{e} = \frac{e}{m} \tag{6.43}$$

In the open Corbino geometry, one must conclude that the system is now in an eigenstate of $H(0)$ which corresponds to exciting one quasiparticle in the inner region and one quasi-hole in the outer one. **Fractional quantization of $\sigma_{xy}$ implies excitations with fractional charge!**

Fractionally charged excitations have been revealed experimentally using shot-noise spectroscopy (see slides).

*Torus geometry* In the *torus geometry* (i.e. identifying the inner and outer ring, so no charge can flow), the quasi-hole and quasi-electron annihilate. Hence the system must be in a ground state of $H(0)$. We have to conclude that $H(0)$ has **degenerate ground states on the torus** and that the system ends up into a ground-state of $H(0)$ belonging to a different sector (corresponding to a twist of boundary conditions).

There are $m$ distinct eigenstates on the torus, with phase differences $2\pi/m$ between them. The flux insertion procedures shifts between these different ground-states and one returns to the original one only after $m$ insertions of $\phi_0$. The fact that the ground-state degeneracy depends on the genus of the manifold on which we consider the system is a hallmark of *topological order* (Wen).

*Heuristic argument.* The degeneracy of the ground-state on the torus can be understood from a heuristic argument as follows. Call $T_x$ and $T_y$ the two translations along each of the cycles of the torus. Consider a quasi-hole and quasi-particle created by Laughlin pumping. Fix the position of one and apply to the other the operation $T_x T_y T_X^{-1} T_y^{-1}$, before recombining them: this corresponds to a full cycle of the moved particle around the other (figure), hence the wave-function must undergo a phase change $e^{i2\pi/m}$, implying:

$$T_x T_y \;=\; e^{i2\pi/m}\, T_y T_x$$

A single ground-state is not consistent with this algebra. Instead, the smallest representation has dimension $m$, with $m$ degenerate ground-states $|n\rangle$ ($n = 0, \cdots, m-1$)

and (in the Landau gauge):

$$T_x\,|n\rangle = \omega^n\ |n\rangle \quad T_y\,|n\rangle = |n+1\rangle$$

with $\omega \equiv e^{i2\pi/m}$ the $m$-th root of unity.

*Thin torus limit.* A much more precise resoning explicitly showing the ground-state degeneracy on the torus uses the limit in which one of the cycles is very small $L_y \ll L_x$. In this limit, the Landau orbitals which are labeled by the guiding center positions: $x_k = kL_x/N_\phi$ $(k = 0, \cdots, N_\phi - 1)$ are well separated in the x-direction and have a very narrow extension in the y-direction. A given many-body state can be labeled by the Fock occupancy $n_k$ of these orbitals ($n_k = 0,1$) with $\sum_k n_k = N$. The interaction matrix elements also simplify, and it can be shown that the nearest neighbor and next nearest neighbor repulsive interactions between the Landau orbitals are dominant. This means that if $k$ is occupied, $k+1$ and $k+2$ must be empty (for $m = 3$). Hence, the three ground-states correspond to:

$$|1\rangle = |1{,}0{,}0{,}1{,}0{,}0{,}1{,}0{,}0, \cdots\rangle \ \ , \ \ |2\rangle = |0{,}1{,}0{,}0{,}1{,}0{,}0{,}1, \cdots\rangle \ \ , \ \ |3\rangle = |0{,}0{,}1{,}0{,}0{,}1{,}0{,}0{,}1, \cdots\rangle \ \ ,$$

and the magnetic translation operator $\hat{t}_{L_x/N_\phi}$ connects these three ground-states. The three states have distinct center of mass momenta $K = \sum_k kn_k$.

See slides for an explicit expression of the Laughlin wave-function on the torus.

Importantly, the ground-state degeneracy also clarifies how the FQHE evades the Thouless et al. integer quantization argument based on the many-body Chern number: a key assumption there was that the ground-state is non-degenerate.

#### H. Fractional statistics

We will use a heuristic argument based on the composite fermion picture of the Laughlin state. Consider a quasi-hole:

$$\psi_h(z;\eta) = \prod_i (z_i - \eta)\,\psi_L(\{z_i\}) \tag{6.44}$$

The Laughlin charge pumping argument implies that it has fractional charge $+e/m$, corresponding to the insertion of one flux quantum $\phi_0$. Hence we can view the quasi-hole as a particle of charge $e/m$ to which one unit of flux has been attached.

Now consider two quasi-holes and exchange them. Each one makes half a circle around the other, hence acquires an Aharonov–Bohm phase:

$$\Delta\theta = \frac{1}{2}\,\frac{1}{\hbar}\frac{e}{m}\phi_0 = \frac{\pi}{m} \tag{6.45}$$

Hence quasi-holes are **anyons** with statistical angle $\theta = \pi/m$.

**Note:** Exchange of the two particles correspond to *half* of a braid for each. An explicit calculation of the Berry phase during exchange confirms the heuristic argument above (Arovas et al.).

Experimental demonstration of fractional statistics: see seminar by Gwendal Fève on June 3.

### I. The magneto-roton

See slides

# VII. WANNIER FUNCTIONS AND TOPOLOGICAL OBSTRUCTIONS

## A. Conventions

In this section, I will use the conventions of Marzari and Vanderbilt, namely:
– Normalisation of Bloch functions (integral over all space):

$$\langle \psi_{nk} | \psi_{n'k'} \rangle = \frac{(2\pi)^d}{V_c} \, \delta_{nn'} \, \delta^d(\boldsymbol{k} - \boldsymbol{k}')$$

with $V_c$ the volume of the unit cell.
– Bloch functions $u_{nk}$ are normalized within the unit cell
– Note that $V_c \, V_{BZ} = (2\pi)^d$
– Fourier transforms: Lattice vectors $\boldsymbol{R} = \sum n_i \boldsymbol{a}_i$, $n_i \in \mathbb{Z}$.

$$\hat{f}(\boldsymbol{k}) = \sum_R e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \, f(\boldsymbol{R}), \qquad \boldsymbol{k} \in \mathrm{BZ} \tag{7.1}$$

$$f(\boldsymbol{R}) = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{R}} \, \hat{f}(\boldsymbol{k}) \tag{7.2}$$

$$\int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \, e^{i\boldsymbol{k}\cdot\boldsymbol{R}} = \frac{1}{V_c} \, \delta_{R,0} \tag{7.3}$$

$$\sum_R e^{i\boldsymbol{k}\cdot\boldsymbol{R}} = \frac{(2\pi)^d}{V_c} \sum_G \delta^d(\boldsymbol{k} - \boldsymbol{G}) \tag{7.4}$$

## B. Wannier functions for a single band

### 1. Definition

$$|W_R^n\rangle = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{R}} \, |\psi_{nk}\rangle \tag{7.5}$$

Unitary operation. Inverse expression:

$$|\psi_{nk}\rangle = \sum_R e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \, |W_R^n\rangle \tag{7.6}$$

Dropping the band index, $W_R(\boldsymbol{r}) = \langle \boldsymbol{r} | W_R \rangle = W_0(\boldsymbol{r} - \boldsymbol{R})$. Indeed:

$$W_R(\boldsymbol{r}) = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \, e^{ik(\boldsymbol{r}-\boldsymbol{R})} \, u_{nk}(\boldsymbol{r}) \tag{7.7}$$

with $u_{nk}(\boldsymbol{r}+\boldsymbol{R}) = u_{nk}(\boldsymbol{r})$ periodic. The $|W_{Rn}\rangle$ basis is equivalent to the Bloch state basis:

$$P_n = V_c \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \, |\psi_{nk}\rangle\langle\psi_{nk}| = \sum_R |W_R^n\rangle\langle W_R^n| \tag{7.8}$$

Wannier functions provide an *exact tight-binding representation*:

$$\langle W_0^n|\hat{H}|W_R^n\rangle = \varepsilon_n(\boldsymbol{R}) \tag{7.9}$$

with $\varepsilon_n(R)$ the Fourier transform of $\varepsilon_n(\boldsymbol{k})$ with the conventions above. There is an infinite family of Wannier functions (choice of phase).

*Example: one dimension.* $\text{BZ} = \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$, $V_c = a$, $V_{\text{BZ}} = \frac{2\pi}{a}$. Dropping the band index:

$$w(x - na) = a \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} \, e^{ik(x-na)} \, u_k(x) \tag{7.10}$$

Gauge freedom (cf. Kohn): $u_k$ is only defined up to a phase $\varphi_k$. Family of Wannier functions:

$$w(x - na) = a \int \frac{dk}{2\pi} \, e^{ik(x-na)} \, e^{i\varphi_k} \, u_k(x) \tag{7.11}$$

This phase freedom can be used to control the degree of localization of the Wannier function.

2. Where is $W_R$ centered?

$$\boxed{\langle W_n^R|\boldsymbol{r}|W_n^R\rangle = \boldsymbol{R} + V_c \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \, \boldsymbol{A}_n(\boldsymbol{k})} \tag{7.12}$$

with $A_n$ the Berry connection associated with band n:

$$\boldsymbol{A}_n(\boldsymbol{k}) \equiv i\langle u_{nk}|\nabla_k u_{nk}\rangle \tag{7.13}$$

More generally:

$$\langle W_n^0|\boldsymbol{r}|W_n^R\rangle = \boldsymbol{A}_n(R) \tag{7.14}$$

with $A_n(R)$ the Fourier transform of $A_n(\boldsymbol{k})$ according to the conventions above.

In $d = 1$ (or along a 1D cut of the BZ, corresponding to a closed path):

$$\bar{x}_n \equiv \langle W_n^0|x|W_n^0\rangle = a \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} \, \langle u_{nk}|i\partial_k u_{nk}\rangle = a \, \frac{\phi_n}{2\pi} \tag{7.15}$$

Since $\left[-\frac{\pi}{a}, \frac{\pi}{a}\right] \sim S^1$, this is a closed contour, hence this is the **Berry phase**. The centre $\bar{x}_n$ is therefore *gauge-independent*. An instructive example is the SSH model.

Drop band index; $W$ centered on $\mathbf{0}$:

$$w(\boldsymbol{r}) = \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, u_k(\boldsymbol{r}) \tag{7.16}$$

If $u_k(x)$ admits an analytic continuation onto a strip $|\mathrm{Im}\, k| < \alpha$ in the complex $k$-plane, then $|w(\boldsymbol{r})| \lesssim e^{-\alpha|r|}$ for large enough $|r| \to$ **exponential decay** (Paley-Wiener theorem on Fourier transforms).

In a topological band, there is no gauge in which $u_{\boldsymbol{k}}$ is smooth everywhere in the BZ: the phase has necessarily a singularity in the BZ $\to$ **obstruction to exponential localization**. In the Haldane model for example, considering the lower band isolated separated by a gap from the other, this phase singularity is a vortex located at $K$,$K'$ (and symmetry equivalent points). This leads to a Wannier function $w(\boldsymbol{r}-\boldsymbol{R})$ which decays as $1/|\boldsymbol{r}-\boldsymbol{R}|^2$ (a rather general result).

For an isolated band, exponential localization is possible if and only if the $U(1)$ bundle is trivial, i.e. if the Chern number $C_n = 0$.

### C. Wannierization of a group of bands

Consider now a set $B$ of bands (isolated, otherwise a disentangling procedure is needed). The unitary transformation from Bloch to Wannier functions can be generalized as:

$$|W_n^R\rangle = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{R}} \sum_{m\in B} U_{mn}(\boldsymbol{k})\, |\psi_{mk}\rangle \tag{7.17}$$

The unitary matrix $U_{mn}(\boldsymbol{k})$ can be used to optimise the localisation of the $W_n^R$ (their spatial decay). In the Marzari-Vanderbilt maximal localization procedure, the following spread function is minimized:

$$\Omega[U] = \sum_{n\in B} \left[ \langle W_n|\boldsymbol{r}^{\mathbf{2}}|W_n\rangle - \langle W_n|\boldsymbol{r}|W_n\rangle^2 \right]$$

with $W_n$ centered on, say, $R = 0$. However the condition under which exponential localization is possible is less simple than in the isolated band case:

– A *necessary* condition is that the *total* Chern number vanishes:

$$C_{tot} = \sum_{n\in B} C_n = 0$$

– This condition is not sufficient however. Situations exist in which $C_{tot} = 0$ but exponential localization is only posisble if some additional *topologically trivial* bands are introduced in the set (*fragile topology*, see Vishwanath et al.). This situation occurs in TBLG for example.

The spread functional can be written in terms of the projectors:

$$\hat{P} = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \sum_{n\in B} |\psi_{nk}\rangle\langle\psi_{nk}| = \sum_{R,n\in B} |W_n^R\rangle\langle W_n^R|, \qquad \hat{Q} \equiv \mathbf{1} - \hat{P} \tag{7.18}$$

One can show:

$$\Omega = \Omega_I + \tilde{\Omega} \tag{7.19}$$

where $\Omega_I$ is gauge invariant, $\tilde{\Omega}$ is gauge dependent, and both $\Omega_I$,$\tilde{\Omega} \geq 0$.

$$\Omega_I = \mathrm{Tr}\Big[\hat{P}\,\hat{r}\,\hat{Q}\,\hat{r}\,\hat{P}\Big] \qquad \text{Tr: all bands} \tag{7.20}$$

$$\tilde{\Omega} = \sum_n \sum_{R\neq 0} |\langle W_R^m|\hat{r}|W_0^n\rangle|^2 \tag{7.21}$$

The MaxLoc procedure (optimising $U_{nm}(\boldsymbol{k})$) acts on $\tilde{\Omega}$ only. The standard code implementing this procedure is Wannier90.

### D. Localization of Wannier functions and the quantum metric

The following expressions can be derived:

$$\langle W_n^R|\boldsymbol{r}|W_m^0\rangle = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\, \langle u_{nk}|i\nabla_k u_{mk}\rangle \tag{7.22}$$

$$\langle W_n^R|r^2|W_m^0\rangle = -V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\, \langle u_{nk}|\nabla_k^2 u_{mk}\rangle \tag{7.23}$$

Let us focus for simplicity on a single isolated band in two dimensions. Then:

$$\begin{aligned}\Omega_I &= \langle w_0|\hat{r}\hat{Q}\hat{r}|w_0\rangle \\ &= \langle w_0|\hat{r}^2|w_0\rangle - \sum_R |\langle w_R|\hat{r}|w_0\rangle|^2 \end{aligned}\tag{7.24}$$

$$\tilde{\Omega} = \sum_{R\neq 0} |\langle w_R|\hat{r}|w_0\rangle|^2 \tag{7.25}$$

Recalling the quantum geometric tensor for this band:

$$T_{ab}(\boldsymbol{k}) = \langle \partial_a u_k|1 - P_k|\partial_b u_k\rangle, \qquad 1 - P_k = 1 - |u_k\rangle\langle u_k| = Q_k \tag{7.26}$$

In terms of the quantum metric and Berry curvature:

$$T = g - \frac{i}{2}F \tag{7.27}$$

One can show:

$$\Omega_I = V_c \int_{\mathrm{BZ}} \frac{d^d k}{(2\pi)^d} \operatorname{tr} g(\boldsymbol{k}) \tag{7.28}$$

One can further show $\operatorname{tr} g(\boldsymbol{k}) \geq |F_{xy}(\boldsymbol{k})|$, hence:

$$\begin{aligned} \Omega_I &\geq V_c \int_{\mathrm{BZ}} \frac{d^2 k}{(2\pi)^2} \, |F_{xy}(\boldsymbol{k})| \\ &\geq \frac{V_c}{2\pi} \left| \int \frac{d^2 k}{2\pi} \, F_{xy}(\boldsymbol{k}) \right| \end{aligned} \tag{7.29}$$

$$\boxed{\Omega_I \geq \frac{V_c}{2\pi} |C|} \tag{7.30}$$

*Ideal Flat Bands* “Ideal” flat bands saturate this bound, with:

$$g_{xx} = g_{yy}, \qquad g_{xy} = 0 \tag{7.31}$$

$$\det g = g_{xx} \, g_{yy} = \frac{1}{4} F_{xy}^2 \tag{7.32}$$

$$g_{ab}(\boldsymbol{k}) = \frac{1}{2} |F_{xy}(\boldsymbol{k})| \, \delta_{ab} \tag{7.33}$$

### VIII. INTRODUCTION TO TWISTED BILAYER GRAPHENE

See slides of lecture 6

#### Annexe A: The Berry Phase

The Berry phase is a geometric phase acquired by a quantum system when its Hamiltonian depends adiabatically on external parameters that are varied along a closed loop. Unlike the dynamical phase, which depends on the energy and the time spent in evolution, the Berry phase depends only on the geometry of the path in parameter space.

Beyond its conceptual elegance, the Berry phase plays a central role in modern condensed matter physics, molecular physics, and quantum field theory. Its differential-geometric formulation naturally introduces the notions of Berry connection and Berry curvature, in close analogy to gauge potentials and field strengths.

Let the Hamiltonian depend on a set of parameters living on a manifold $\mathcal{M}$

$$\lambda = \{\lambda^a\}.$$

We consider an **adiabatic evolution** $\lambda(t)$, where the system follows a given instantaneous eigenstate. Assume:

– No level crossing,

– The eigenstate is non-degenerate.

The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}[\lambda(t)]\left|\psi(t)\right\rangle.$$

#### Instantaneous Eigenstates

At each time $t$, consider a representative of the instantaneous eigenstate:

$$\hat{H}[\lambda]\left|n(\lambda)\right\rangle = E_n(\lambda)\left|n(\lambda)\right\rangle.$$

The eigenstate $\left|n(\lambda)\right\rangle$ is defined up to an arbitrary phase.

#### Adiabatic Ansatz

We write the state as

$$\left|\psi(t)\right\rangle = e^{i\phi(t)}\left|n(\lambda(t))\right\rangle.$$

Compute the time derivative:

$$\frac{\partial}{\partial t}\left|\psi\right\rangle = i\dot{\phi}e^{i\phi}\left|n\right\rangle + e^{i\phi}\dot{\lambda}\,\partial_\lambda\left|n\right\rangle.$$

Plugging into Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = -\hbar\dot{\phi}e^{i\phi}|n\rangle + i\hbar\dot{\lambda}e^{i\phi}\partial_\lambda|n\rangle = E_n[\lambda(t)]e^{i\phi}|n\rangle\,.$$

Projecting onto $\langle n|$ gives

$$\dot{\phi} = i\dot{\lambda}\,\langle n|\partial_\lambda n\rangle - \frac{1}{\hbar}E_n[\lambda(t)].$$

Integrating in time:

$$\phi(t) = -\frac{1}{\hbar}\int_0^t dt'\,E_n[\lambda(t')] + i\int_{\lambda(0)}^{\lambda(t)} d\lambda\,\langle n|\partial_\lambda n\rangle\,.$$

We identify:

– Dynamical phase:

$$\phi_{\mathrm{dyn}}(t) = -\frac{1}{\hbar}\int_0^t dt'\,E_n[\lambda(t')].$$

– Berry (geometric) phase:

$$\boxed{\phi_B^{(n)} = i\int_{\lambda(0)}^{\lambda(t)} d\lambda\,\langle n|\partial_\lambda n\rangle}$$

For a closed loop $C$ in parameter space:

$$\boxed{\phi_B^{(n)}(C) = i\oint_C d\lambda\,\langle n|\partial_\lambda n\rangle}$$

**Berry connection**

$$\boxed{A_a(\lambda) \equiv i\,\langle n_\lambda\,|\frac{\partial}{\partial\lambda_a}n_\lambda\rangle}$$

Because of the normalisation $\langle n_\lambda|n_\lambda\rangle = 1$, the Berry connection is real and can be written as:

$$A_a(\lambda) \equiv -\,\mathrm{Im}\langle n_\lambda\,|\partial_a n_\lambda\rangle$$

The expression of the Berry phase above in terms of the Berry connection is clearly analogous to the Aharonov-Bohm phase if we interpret the Berry connection as a *vector potential in parameter space* (i.e defined on the manifold $\mathcal{M}$ of parameters. It is a one-form on this manifold.

**Gauge dependence.**

The eigenstates $|n_\lambda\rangle$ are only defined up to an arbitrary phase. Let us consider a change of the phase:

$$|\tilde{n}_\lambda\rangle = e^{i\Omega(\lambda)}\,|n_\lambda\rangle$$

$$\tilde{A}_a = i\,\langle\tilde{n}|\partial_a\tilde{n}\rangle = A_a - \partial_a\Omega.$$

Hence $A_a$ transforms like a gauge field.
The Berry phase over a closed loop is gauge invariant:

$$\oint A_a d\lambda^a = \oint \tilde{A}_a d\lambda^a.$$

**Berry Curvature**

Define the Berry curvature:

$$F_{ab} = \partial_a A_b - \partial_b A_a \;\;,\;\; F = \nabla_\lambda \times A.$$

Compute explicitly:

$$\boxed{F_{ab} = i\partial_a\,\langle n|\partial_b n\rangle - i\partial_b\,\langle n|\partial_a n\rangle = -2\,\mathrm{Im}\,\langle\partial_a n|\partial_b n\rangle}$$

The Berry curvature is a 2-form on the manifold $\mathcal{M}$ on which the eigenstates $|n\rangle$ define a complex vector bundle.
From Stokes theorem, the Berry phase on a closed contour is the flux of the Berry connection through the surface of $\mathcal{M}$ enclosed by the contour C:

$$\phi_B = \oint_C \mathbf{A}.d\lambda = \int \mathbf{F}\cdot d\mathbf{S}$$

**Expression in Terms of Hamiltonian Matrix Elements**

Start from

$$\hat{H}\,|n\rangle = E_n\,|n\rangle\,.$$

Differentiate:

$$\partial_a \hat{H} |n\rangle + \hat{H}\partial_a |n\rangle = \partial_a E_n |n\rangle + E_n \partial_a |n\rangle .$$

Project onto $n$:

$$\boxed{\partial_a E_n = \langle n| \partial_a \hat{H} |n\rangle}$$

Project onto $m \neq n$:

$$(E_n - E_m) \langle m|\partial_a n\rangle = \langle m|\partial_a \hat{H}|n\rangle , \qquad m \neq n.$$

Thus

$$\boxed{\langle m|\partial_a n\rangle = \frac{\langle m|\partial_a \hat{H}|n\rangle}{E_n - E_m} \ , \ m \neq n}$$

Important remark: first order perturbation theory does not fix the phase of the component of the perturbed wave-function along $|n\rangle$. The full expression of the new wave-function is:

$$\boxed{|n + \delta n\rangle \ = \ e^{i\delta\phi} |n\rangle + \sum_{m\neq n} \frac{\langle m|\delta_a \hat{H}|n\rangle}{E_n - E_m} |m\rangle}$$

with $i\delta\phi = \langle n|\delta n\rangle$ given by the Berry phase.

Using completeness,

$$\langle \partial_a n|\partial_b n\rangle = \sum_{m\neq n} \frac{\langle n|\partial_a \hat{H}|m\rangle \langle m|\partial_b \hat{H}|n\rangle}{(E_n - E_m)^2}.$$

Therefore,

$$\boxed{F_{ab}^{(n)} = i \sum_{m\neq n} \frac{\langle n|\partial_a \hat{H}|m\rangle \langle m|\partial_b \hat{H}|n\rangle - (a \leftrightarrow b)}{(E_n - E_m)^2}}$$

This expression makes explicit that Berry curvature becomes large near degeneracies.

Note: the $m = n$ term vanishes due to normalization. Indeed:

$$\langle n|n\rangle = 1 \Rightarrow \langle \partial_a n|n\rangle + \langle n|\partial_a n\rangle = 0$$

Hence:

$$\langle \partial_a n|n\rangle \langle n|\partial_b n\rangle = - \langle n|\partial_a n\rangle \langle n|\partial_b n\rangle$$

is a symmetric expression upon exchanging $a$,$b$.

#### Annexe B: Berry phase and fiber bundles

The underlying mathematical structure is that of a *fiber bundle*. To each point of the manifold (base space) $\mathcal{M}$ one associates a vector space - here, the one-dimensional space generated by the vector $|n\rangle$ - the *fiber*. Note that the fiber does not depend on the arbitrary choice of phase of $|n\rangle$ - it has $U(1)$ invariance.

An infinitesimal change of $|n\rangle$ can be decomposed as:

$$d\,|n\rangle = \langle n|dn\rangle\,|n\rangle + (1 - |n\rangle\,\langle n|)\,d\,|n\rangle$$

the first term is the change along the fiber ('vertical' change in the language of line bundles), while the second is the 'horizontal' change. A change of phase of $|n\rangle$ corresponds to a purely vertical change (along the fiber).

*Parallel transport* of a vector means that the change is purely 'horizontal' (i.e. *perpendicular* to the fiber - despite the terminology...). Hence the condition for parallel transport is:

$$\langle n|dn\rangle = 0$$

Define the covariant derivative as:

$$D \equiv d - \langle n|dn\rangle = d + iA \;\;,\;\; A \equiv i\,\langle n|dn\rangle$$

with A the connection as defined above. The covariant derivative selects the *horizontal* component of the change:

$$D\,|n\rangle = (1 - |n\rangle\,\langle n|)\,|dn\rangle$$

If we change $|n\rangle$ by a pure phase, then $D\,|n\rangle = 0$. $D\,|n\rangle$ is the change of the vector that does not include a pure phase change ('torsion').

Caution: do not confuse the $U(1)$ *line bundle* defined for each point $\lambda$ on the base space by the one-dimensional vector space generated by the eigenstate $|n_\lambda\rangle$ that we consider, and the *principal bundle* which would attach to each $\lambda$ the *full* Hilbert space $\mathcal{H}$. In all cases considered in these lectures, the principal bundle is topologically trivial, but the line bundle (or its generalisation to a restricted set of eigenstates) may not be.

#### Annexe C: Chern number and Berry connection

If the parameter space is closed (e.g. a two-dimensional torus), the integral of the Berry curvature over the manifold defines the first Chern number:

$$C_1^{(n)} = \frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{F}^{(n)}$$

This integer is topologically protected and underlies phenomena such as the quantization of the Hall conductance.

The Berry phase reveals that quantum mechanics naturally incorporates geometric and topological structures. The Berry connection behaves as a gauge potential over parameter space, while the Berry curvature acts as a field strength. Near degeneracies, curvature behaves as if sourced by monopoles, and global integrals of curvature give rise to topological invariants.

These ideas provide a unifying language across many areas of modern physics, from molecular dynamics to topological phases of matter.

## Annexe D: Kubo formula

### 1. Setup: uniform electric field via a time-dependent vector potential

We consider a many-body system with an unperturbed Hamiltonian $H_0$ and equilibrium density matrix

$$\rho_0 = \frac{e^{-\beta H_0}}{Z}, \qquad Z = \mathrm{Tr}\big(e^{-\beta H_0}\big). \tag{D1}$$

To probe the electrical response, we apply a *spatially uniform* time-dependent vector potential

$$\boldsymbol{A}(t) = \frac{\boldsymbol{E}}{i\omega} e^{-i\omega t}, \tag{D2}$$

so that in the gauge $\phi = 0$,

$$\boldsymbol{E}(t) = -\partial_t \boldsymbol{A}(t) = \boldsymbol{E}\, e^{-i\omega t}. \tag{D3}$$

We will ultimately take the physical (retarded) prescription $\omega \to \omega + i0^+$.

#### a. Coupling to the current

For a uniform vector potential, the perturbation can be written as

$$H'(t) = -\int d^d r\, \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(t) \equiv -\boldsymbol{J} \cdot \boldsymbol{A}(t), \tag{D4}$$

where $\boldsymbol{j}(\boldsymbol{r})$ is the current density and $\boldsymbol{J} = \int d^d r\, \boldsymbol{j}(\boldsymbol{r})$ is the total current operator.

*a. Remark (diamagnetic term).* If the microscopic Hamiltonian is obtained by minimal coupling $\boldsymbol{p} \to \boldsymbol{p} - q\boldsymbol{A}$ (for particles of charge $q$), the physical current contains a part linear in $\boldsymbol{A}$:

$$J_\alpha(t) = J^p_\alpha(t) + \sum_\beta D_{\alpha\beta} A_\beta(t), \tag{D5}$$

where $J^p$ is the *paramagnetic* current (independent of $\boldsymbol{A}$ at $\boldsymbol{A} = 0$), and $D_{\alpha\beta}$ is the *diamagnetic* (or "contact") term. For a continuum electron gas $D_{\alpha\beta} = \frac{nq^2}{m}\delta_{\alpha\beta}$, and on a lattice $D_{\alpha\beta}$ is related to the kinetic-energy (stress) tensor. We keep $D_{\alpha\beta}$ general.

**2. Linear response formula for the current**

We work in the interaction picture with respect to $H_0$:

$$O_I(t) = e^{iH_0 t} O\, e^{-iH_0 t}, \qquad H_I'(t) = -\boldsymbol{J}_I(t) \cdot \boldsymbol{A}(t). \tag{D6}$$

The interaction-picture density matrix obeys

$$\frac{d\rho_I(t)}{dt} = -i\, [H_I'(t), \rho_I(t)]. \tag{D7}$$

To first order in the perturbation,

$$\rho_I(t) = \rho_0 - i \int_{-\infty}^{t} dt'\, [H_I'(t'), \rho_0] + O(A^2), \tag{D8}$$

where the lower limit $-\infty$ (with an implicit adiabatic factor $e^{\eta t'}$, $\eta \to 0^+$) enforces causality.

The induced expectation value of an operator $O$ is

$$\begin{aligned} \delta \langle O(t) \rangle &= \mathrm{Tr}(\rho_I(t) O_I(t)) - \mathrm{Tr}(\rho_0 O_I(t)) \\ &= -i \int_{-\infty}^{t} dt'\ \mathrm{Tr}([H_I'(t'), \rho_0]\, O_I(t)) \\ &= -i \int_{-\infty}^{t} dt'\ \langle [O_I(t), H_I'(t')] \rangle_0 \,, \end{aligned} \tag{D9}$$

where $\langle \cdots \rangle_0 \equiv \mathrm{Tr}(\rho_0 \cdots)$.

Specializing to the paramagnetic current $O = J_\alpha^p$, we obtain

$$\delta \langle J_\alpha^p(t) \rangle = -i \int_{-\infty}^{t} dt'\ \langle [J_\alpha^p(t), H_I'(t')] \rangle_0 = i \sum_\beta \int_{-\infty}^{t} dt'\ \left\langle [J_\alpha^p(t), J_\beta^p(t')] \right\rangle_0 A_\beta(t'). \tag{D10}$$

By time-translation invariance of equilibrium correlation functions, define the *retarded current-current correlator*

$$\chi_{\alpha\beta}^R(t - t') \equiv -i\, \theta(t - t')\ \left\langle [J_\alpha^p(t), J_\beta^p(t')] \right\rangle_0 . \tag{D11}$$

Then the response becomes

$$\delta \langle J_\alpha^p(t) \rangle = \sum_\beta \int_{-\infty}^{\infty} dt'\, \chi_{\alpha\beta}^R(t - t')\, A_\beta(t'). \tag{D12}$$

Including the explicit diamagnetic contribution to the physical current,

$$\delta \langle J_\alpha(t) \rangle = \delta \langle J_\alpha^p(t) \rangle + \sum_\beta D_{\alpha\beta} A_\beta(t). \tag{D13}$$

#### 3. Kubo formula for the conductivity tensor

Fourier transform with convention

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} f(\omega), \qquad f(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t). \tag{D14}$$

Then

$$\delta \langle J_\alpha(\omega) \rangle = \sum_\beta \left[ \chi^R_{\alpha\beta}(\omega) + D_{\alpha\beta} \right] A_\beta(\omega). \tag{D15}$$

Using $\boldsymbol{E}(\omega) = i\omega \boldsymbol{A}(\omega)$ (since $\boldsymbol{E} = -\partial_t \boldsymbol{A}$ and we take $e^{-i\omega t}$ time dependence), we define the conductivity tensor by

$$\delta \langle J_\alpha(\omega) \rangle = \sum_\beta \sigma_{\alpha\beta}(\omega)\, E_\beta(\omega). \tag{D16}$$

Therefore,

$$\sigma_{\alpha\beta}(\omega) = \frac{1}{i\omega} \left[ \chi^R_{\alpha\beta}(\omega) + D_{\alpha\beta} \right]. \tag{D17}$$

Equivalently, writing the retarded correlator directly as a commutator integral,

$$\begin{aligned} \chi^R_{\alpha\beta}(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \chi^R_{\alpha\beta}(t) \\ &= -i \int_0^{\infty} dt\, e^{i\omega t} \left\langle [J^p_\alpha(t), J^p_\beta(0)] \right\rangle_0, \end{aligned} \tag{D18}$$

so the Kubo formula reads

$$\boxed{\sigma_{\alpha\beta}(\omega) = \frac{1}{i\omega} \left[ -i \int_0^{\infty} dt\, e^{i(\omega + i0^+)t} \left\langle [J^p_\alpha(t), J^p_\beta(0)] \right\rangle_0 + D_{\alpha\beta} \right]} \tag{D19}$$

The $+i0^+$ ensures convergence and enforces the retarded (causal) response.

#### 4. Spectral (Lehmann) representation

Let $\{|n\rangle\}$ be a complete set of eigenstates of $H_0$:

$$H_0 |n\rangle = E_n |n\rangle. \tag{D20}$$

At finite temperature,

$$\langle X \rangle_0 = \frac{1}{Z} \sum_n e^{-\beta E_n} \langle n | X | n \rangle. \tag{D21}$$

Use

$$J^p_\alpha(t) = e^{iH_0 t} J^p_\alpha e^{-iH_0 t}, \tag{D22}$$

and insert resolutions of identity to evaluate the equilibrium commutator:

$$\begin{aligned}\Big\langle [J^p_\alpha(t), J^p_\beta(0)] \Big\rangle_0 &= \frac{1}{Z}\sum_n e^{-\beta E_n} \langle n|\, J^p_\alpha(t) J^p_\beta - J^p_\beta J^p_\alpha(t)\, |n\rangle \\ &= \frac{1}{Z}\sum_{n,m} e^{-\beta E_n} \Big( e^{i(E_n - E_m)t} \langle n|\, J^p_\alpha \,|m\rangle \langle m|\, J^p_\beta \,|n\rangle - e^{-i(E_n - E_m)t} \langle n|\, J^p_\beta \,|m\rangle \langle m|\, J\end{aligned} \tag{D23}$$

Define matrix elements $J^\alpha_{nm} \equiv \langle n|\, J^p_\alpha \,|m\rangle$. Then

$$\Big\langle [J^p_\alpha(t), J^p_\beta(0)] \Big\rangle_0 = \frac{1}{Z}\sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) e^{i(E_n - E_m)t}\, J^\alpha_{nm} J^\beta_{mn}. \tag{D24}$$

Insert this into the retarded correlator:

$$\begin{aligned}\chi^R_{\alpha\beta}(\omega) &= -i \int_0^\infty dt\, e^{i(\omega + i0^+)t}\, \frac{1}{Z}\sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) e^{i(E_n - E_m)t}\, J^\alpha_{nm} J^\beta_{mn} \\ &= \frac{1}{Z}\sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) \frac{J^\alpha_{nm} J^\beta_{mn}}{\omega + i0^+ + E_n - E_m}.\end{aligned} \tag{D25}$$

Hence the conductivity tensor becomes

$$\sigma_{\alpha\beta}(\omega) = \frac{1}{i\omega}\left[\frac{1}{Z}\sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) \frac{J^\alpha_{nm} J^\beta_{mn}}{\omega + i0^+ + E_n - E_m} + D_{\alpha\beta}\right]. \tag{D26}$$

#### a. Real part and absorption (optional standard form)

Using

$$\frac{1}{x + i0^+} = \mathcal{P}\frac{1}{x} - i\pi\delta(x), \tag{D27}$$

one finds for $\omega \neq 0$ the dissipative part

$$\mathrm{Re}\ \sigma_{\alpha\beta}(\omega) = \frac{\pi}{\omega}\frac{1}{Z}\sum_{n,m} \left(e^{-\beta E_n} - e^{-\beta E_m}\right) J^\alpha_{nm} J^\beta_{mn}\, \delta(\omega + E_n - E_m) \quad (\omega \neq 0), \tag{D28}$$

which makes positivity transparent for $\alpha = \beta$ and $\omega > 0$.

#### b. Comments on $\omega \to 0$ and the Drude weight

Equation (D17) also contains singular contributions at $\omega \to 0$, often written as

$$\mathrm{Re}\ \sigma_{\alpha\beta}(\omega) = \pi D^{\mathrm{Drude}}_{\alpha\beta}\delta(\omega) + \sigma^{\mathrm{reg}}_{\alpha\beta}(\omega), \tag{D29}$$

where the weight of the $\delta(\omega)$ piece depends on the interplay between the diamagnetic term $D_{\alpha\beta}$ and the low-frequency behavior of $\chi^R_{\alpha\beta}(\omega)$. This is system-dependent (metal, insulator, superfluid/superconductor, etc.) and can be analyzed starting from (D17).

### 5. Limit of zero temperature, Hall conductivity

In the $T \to 0$ ($\beta \to \infty$) limit, either $n$ or $m$ must be the ground-state (assumed here to be non-degenerate). The exponential factor then cancels the partition function and we obtain:

$$\sigma_{\alpha\beta}(\omega) = \frac{1}{i\omega}\sum_{n\neq 0}\left[\frac{J^\alpha_{n0}J^\beta_{0n}}{\omega + i0^+ + E_n - E_0} - \frac{J^\alpha_{0n}J^\beta_{n0}}{\omega + i0^+ + E_0 - E_n}\right] + \frac{D_{\alpha\beta}}{i\omega} \tag{D30}$$

The ac Hall conductivity thus reads:

$$\boxed{\sigma_{xy}(\omega) = \frac{1}{i\omega}\sum_{n\neq 0}\left[\frac{J^y_{0n}J^x_{n0}}{\omega + i0^+ + E_n - E_0} - \frac{J^x_{0n}J^y_{n0}}{\omega + i0^+ + E_0 - E_n}\right]} \tag{D31}$$

To obtain the dc limit $\omega \to 0$, we expand:

$$\frac{1}{\omega + \Delta E} = \frac{1}{\Delta E} - \frac{\omega}{\Delta E^2} + \cdots$$

The first term would give a contribution:

$$\frac{1}{i\omega}\sum_{n\neq 0}\frac{J^y_{0n}J^x_{n0} + (x \leftrightarrow y)}{E_n - E_0}$$

which appears divergent, but actually vanishes. This can be shown from current conservation or gauge invariance and can be guessed from the fact that it would be symmetric in $(x,y)$. We thus finally obtain the dc Hall conductivity as (restoring $\hbar$):

$$\boxed{\sigma_{xy} = -i\hbar\sum_{n\neq 0}\frac{J^x_{0n}J^y_{n0} - J^y_{0n}J^x_{n0}}{(E_n - E_0)^2}} \tag{D32}$$

Note that the quantum states involved in this expression are the full many-body states and that this expression is general: it applies to both interacting and non-interacting systems.

## RÉFÉRENCES