Classical and Quantum Particles of the Galilei and Carroll groups

“Scenes from a kinematical Erlanger Programme”

José Miguel Figueroa-O’Farrill, 24 June 2026, Collège de France

Plan

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- Kinematical groups and kinematical Klein geometries

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- Kinematical Cartan geometries

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- Applications

Kinematical groups
and
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Klein's Erlanger Programme

"Geometry from Symmetry"

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- These geometries are "static" (time plays no rôle)

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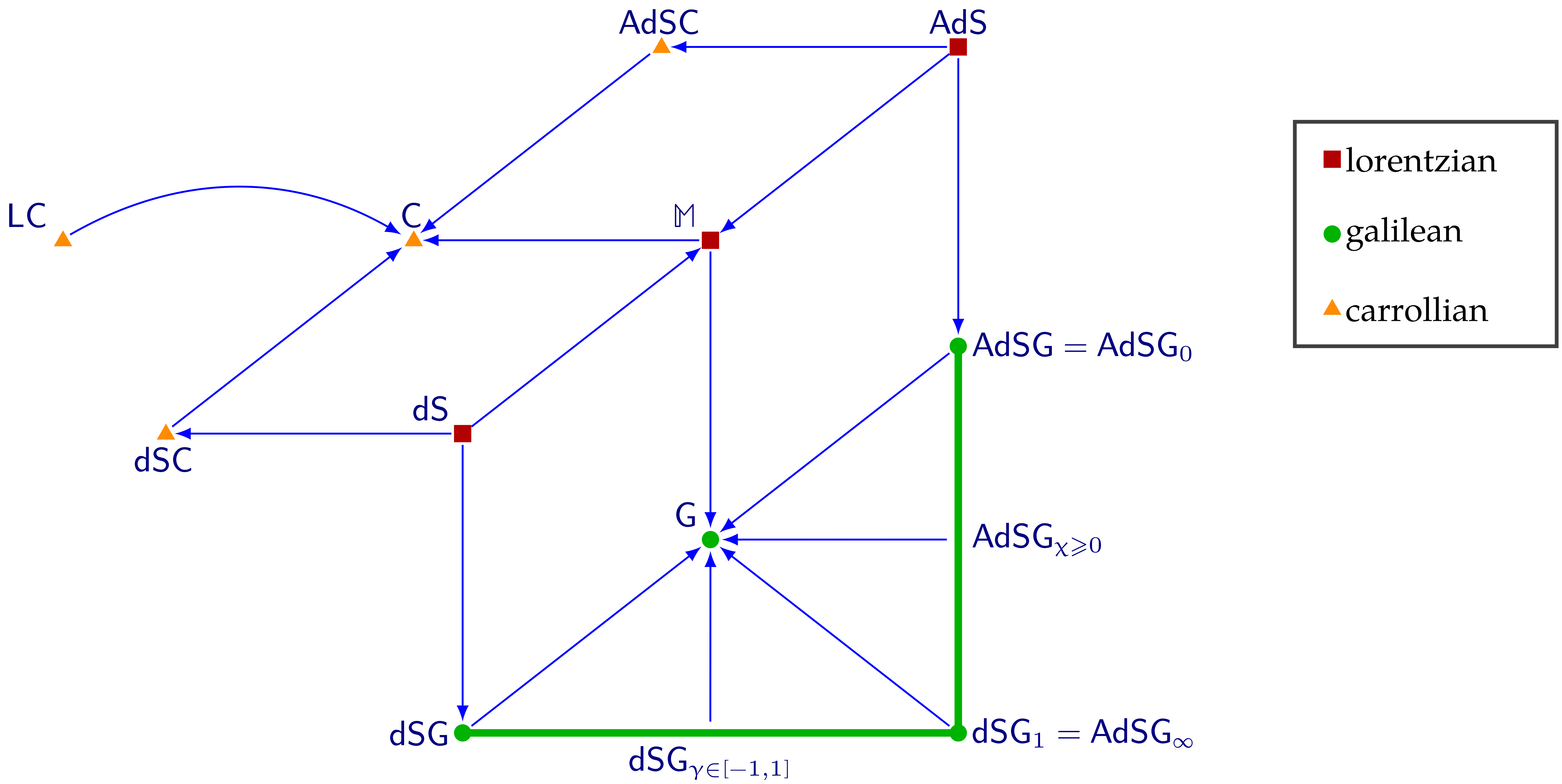
- The (spatially isotropic) kinematical Klein geometries were classified (in general dimension) by myself and **Stefan Prohazka** in 2018. They form a long list, but they fall into a few general classes: riemannian, lorentzian, galiliean and carrollian.


AdSC
AdS
LC
C
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dS
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- the subgroup of the affine group preserving the clock and ruler is the **Galilei group**

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(schematically)
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“The views of space and time that I wish to lay before you have sprung from the **soil of experimental physics**, and therein lies their strength.”

Hermann Minkowski (1908)

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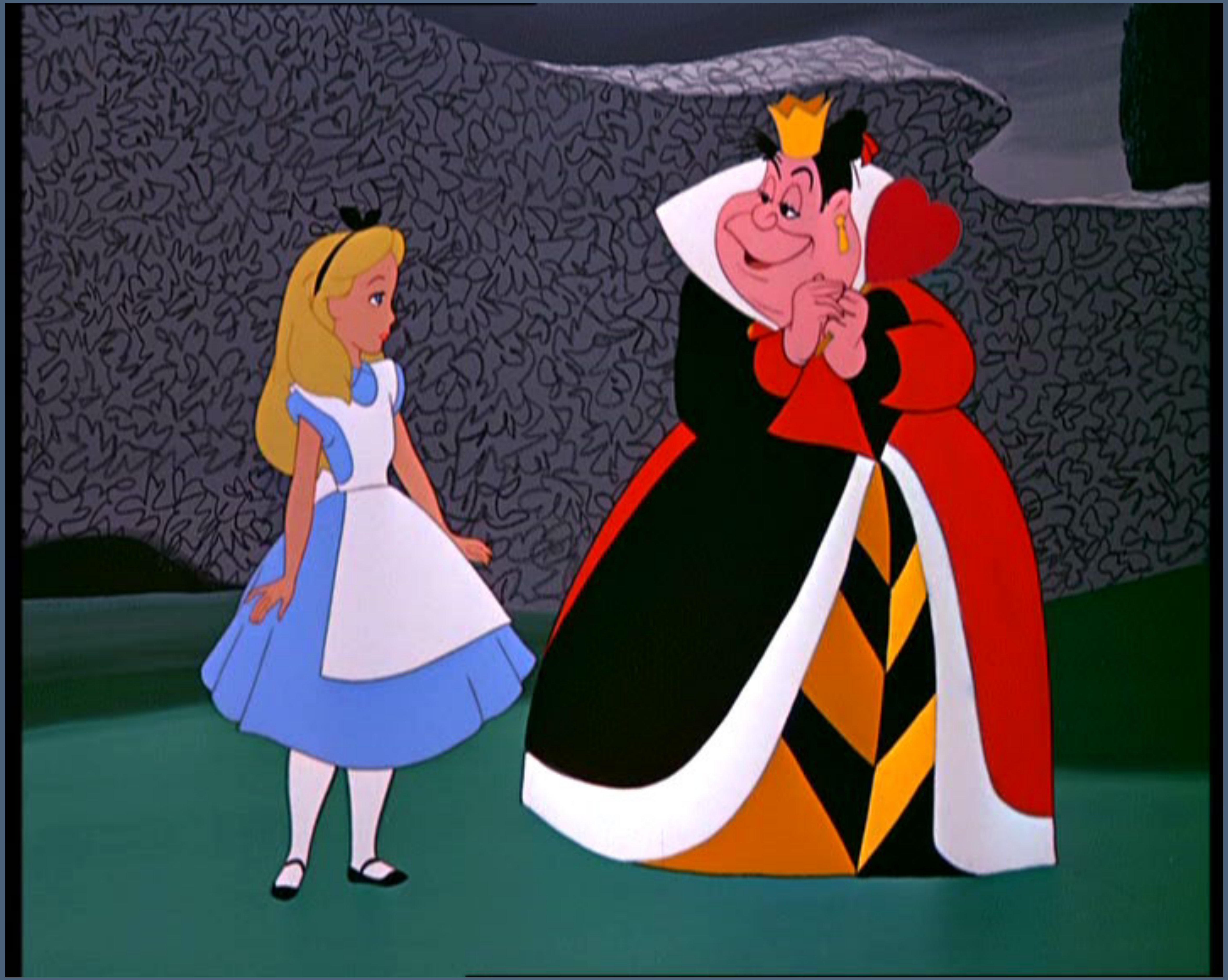
- Why Carroll?



“My dear, here we must run as fast as we can, just to stay in place.”

Dodgson [1865]

Lévy-Leblond [1965]

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- On Carroll spacetime with coordinates (t, x^a) we have $L_{ab} \mapsto x_a\partial_b - x_b\partial_a$, $B_a \mapsto x^a\partial_t$, $P_a \mapsto \partial_a$ and $H \mapsto \partial_t$

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- The triple $(\mathbb{A}^4, \kappa, h)$ is **Carroll spacetime**

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- Paradigmatic example: coadjoint orbits of K , relative to the **Kirillov—Kostant—Souriau** symplectic form

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- Classical elementary particles of the Galilei, Poincaré and Carroll are (up to coverings) coadjoint orbits of the Bargmann, Poincaré and Carroll groups, respectively.

Classical trajectories from coadjoint orbits

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Particle trajectories: $t \mapsto \varpi_o(\gamma(t)) = \gamma(t) \cdot o$

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  JMF+Pérez+Prohazka [2305.06730]
- Both Galilei and Carroll cases are intimately related to “planons” and “fractons”, novel hypothetical particles with restricted mobility.

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$$\mathbf{M}(m, E, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{j}) \in \mathfrak{g}^*$$

Coadjoint orbits of the Bargmann group

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$$m \qquad \|\boldsymbol{p}\|^2 - 2Em \qquad \|m\boldsymbol{j} + \boldsymbol{p} \times \boldsymbol{k}\|^2$$

Coadjoint orbits of the Bargmann group

(in 4 dimensions)

#	Orbit representative $\alpha = \mathsf{M}(m, E, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{j})$	Stabiliser G_α	$\dim \mathcal{O}_\alpha$	Equations for orbits
1	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\{\mathsf{g}(a_+, a_-, \mathbf{0}, \mathbf{0}, R)\}$	6	$m = m_0 \neq 0, \frac{1}{2m}(\\|\boldsymbol{p}\\|^2 - 2Em) = E_0, m\boldsymbol{j} - \boldsymbol{p} \times \boldsymbol{k} = \mathbf{0}$
2	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, s\boldsymbol{e}_3)$	$\{\mathsf{g}(a_+, a_-, \mathbf{0}, \mathbf{0}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	8	$m = m_0 \neq 0, \frac{1}{2m}(\\|\boldsymbol{p}\\|^2 - 2Em) = E_0, \\|m\boldsymbol{j} - \boldsymbol{p} \times \boldsymbol{k}\\| = s > 0$
3	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	G	0	$m = 0, E = E_0, \boldsymbol{p} = \boldsymbol{k} = \boldsymbol{j} = \mathbf{0}$
4	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, s\boldsymbol{e}_3)$	$\{\mathsf{g}(a_+, a_-, \boldsymbol{a}, \boldsymbol{v}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	2	$m = 0, E = E_0, \boldsymbol{p} = \boldsymbol{k} = \mathbf{0}, \\|\boldsymbol{j}\\| = s > 0$
5	$\mathsf{M}(0, E_0, \mathbf{0}, k_0\boldsymbol{e}_3, h\boldsymbol{e}_3)$	$\{\mathsf{g}(a_+, a_-, \boldsymbol{a}, v\boldsymbol{e}_3, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	4	$m = 0, E = E_0, \boldsymbol{p} = \mathbf{0}, \\|\boldsymbol{k}\\| = k_0 > 0, \boldsymbol{j} \cdot \boldsymbol{k} = hk_0$
6	$\mathsf{M}(0, 0, p_0\boldsymbol{e}_3, \mathbf{0}, \mathbf{0})$	$\{\mathsf{g}(a_+, 0, a\boldsymbol{e}_3, \boldsymbol{v}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3, \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0\}$	6	$m = 0, \\|\boldsymbol{p}\\| = p_0 > 0, \boldsymbol{p} \times \boldsymbol{k} = \mathbf{0}$
7	$\mathsf{M}(0, 0, p_0\boldsymbol{e}_3, k_0\boldsymbol{e}_2, \mathbf{0})$	$\{\mathsf{g}(a_+, 0, a\boldsymbol{e}_3, v\boldsymbol{e}_2, I)\}$	8	$m = 0, \\|\boldsymbol{p}\\| = p_0 > 0, \\|\boldsymbol{p} \times \boldsymbol{k}\\| = p_0k_0 > 0$

Galilean particle dynamics

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#	$\alpha = \mathsf{M}(m, E, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{j}) \in \mathfrak{g}^*$	$\mathfrak{g}_\alpha$	$\mathfrak{g}_\alpha \cap \mathfrak{g}_o$	$\mathfrak{m}$
1	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\langle M, H, L_a \rangle$	$\langle M, L_a \rangle$	$\langle H \rangle$
2	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, s\boldsymbol{e}_3)$	$\langle M, H, L_3 \rangle$	$\langle M, L_3 \rangle$	$\langle H \rangle$
3	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\mathfrak{g}$	$\langle M, B_a, L_a \rangle$	$\langle H, P_a \rangle$
4	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, s\boldsymbol{e}_3)$	$\langle M, H, P_a, B_a, L_3 \rangle$	$\langle M, B_a, L_3 \rangle$	$\langle H, P_a \rangle$
5	$\mathsf{M}(0, E_0, \mathbf{0}, k_0\boldsymbol{e}_3, h\boldsymbol{e}_3)$	$\langle M, H, P_a, B_3, L_3 \rangle$	$\langle M, B_3, L_3 \rangle$	$\langle H, P_a \rangle$
6	$\mathsf{M}(0, 0, p_0\boldsymbol{e}_3, \mathbf{0}, \mathbf{0})$	$\langle M, P_3, B_1, B_2, L_3 \rangle$	$\langle M, B_1, B_2, L_3 \rangle$	$\langle P_3 \rangle$
7	$\mathsf{M}(0, 0, p_0\boldsymbol{e}_3, k_0\boldsymbol{e}_2, \mathbf{0})$	$\langle M, P_3, B_2 \rangle$	$\langle M, B_2 \rangle$	$\langle P_3 \rangle$

Coadjoint orbits of the Carroll group

(in 4 dimensions)

#	Particle description	Orbit representative $\alpha = (\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, E)$	$\dim \mathcal{O}_\alpha$	Equations for orbits
1	Massive spinless	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0)$	6	$E = E_0 \neq 0, E_0\boldsymbol{j} + \boldsymbol{p} \times \boldsymbol{k} = \mathbf{0}$
2	Massive with spin S	$(S\boldsymbol{u}, \mathbf{0}, \mathbf{0}, E_0)$	8	$E = E_0 \neq 0, \lVert\boldsymbol{j} + E_0^{-1}\boldsymbol{p} \times \boldsymbol{k}\rVert = S > 0$
3	Vacuum	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	0	$E = 0, \boldsymbol{p} = \mathbf{0}, \boldsymbol{k} = \mathbf{0}, \boldsymbol{j} = \mathbf{0}$
4	Spinning vacuum	$(j\boldsymbol{u}, \mathbf{0}, \mathbf{0}, 0)$	2	$E = 0, \boldsymbol{p} = \mathbf{0}, \boldsymbol{k} = \mathbf{0}, \lVert\boldsymbol{j}\rVert = j > 0$
5	Centrons	$(h\boldsymbol{u}, k\boldsymbol{u}, \mathbf{0}, 0)$	4	$E = 0, \boldsymbol{p} = \mathbf{0}, \lVert\boldsymbol{k}\rVert = k > 0, \boldsymbol{j} \cdot \boldsymbol{k} = h\lVert\boldsymbol{k}\rVert \in \mathbb{R}$
6	Massless with helicity h	$(h\boldsymbol{u}, \mathbf{0}, p\boldsymbol{u}, 0)$	4	$E = 0, \boldsymbol{k} = \mathbf{0}, \lVert\boldsymbol{p}\rVert = p > 0, \boldsymbol{j} \cdot \boldsymbol{p} = h\lVert\boldsymbol{p}\rVert \in \mathbb{R}$
7_+	Parallel massless	$(h\boldsymbol{u}, k\boldsymbol{u}, p\boldsymbol{u}, 0)$	4	$E = 0, \lVert\boldsymbol{p}\rVert = p > 0, \lVert\boldsymbol{k}\rVert = k > 0, \boldsymbol{p} \cdot \boldsymbol{k} = pk, \boldsymbol{j} \cdot \boldsymbol{p} = h\lVert\boldsymbol{p}\rVert \in \mathbb{R}$
7_-	Antiparallel massless	$(h\boldsymbol{u}, -k\boldsymbol{u}, p\boldsymbol{u}, 0)$	4	$E = 0, \lVert\boldsymbol{p}\rVert = p > 0, \lVert\boldsymbol{k}\rVert = k > 0, \boldsymbol{p} \cdot \boldsymbol{k} = -pk, \boldsymbol{j} \cdot \boldsymbol{p} = h\lVert\boldsymbol{p}\rVert \in \mathbb{R}$
8	Generic massless	$(\mathbf{0}, k\cos\theta\boldsymbol{u} + k\sin\theta\boldsymbol{u}_\perp, p\boldsymbol{u}, 0)$	6	$E = 0, \lVert\boldsymbol{p}\rVert = p > 0, \lVert\boldsymbol{k}\rVert = k > 0, \boldsymbol{p} \cdot \boldsymbol{k} = pk\cos\theta, \theta \in (0, \pi)$

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1	$\mathsf{M}(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$\langle H, L_a \rangle$	$\langle L_a \rangle$	$\langle H \rangle$
2	$\mathsf{M}(s\boldsymbol{e}_3, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$\langle H, L_3 \rangle$	$\langle L_3 \rangle$	$\langle H \rangle$
3	$\mathsf{M}(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$\mathfrak{c}$	$\mathfrak{c}_o$	$\langle P_a, H \rangle$
4	$\mathsf{M}(j\boldsymbol{e}_3, \mathbf{0}, \mathbf{0}, 0)$	$\langle L_3, B_a, P_a, H \rangle$	$\langle L_3, B_a \rangle$	$\langle P_a, H \rangle$
5	$\mathsf{M}(h\boldsymbol{e}_3, k\boldsymbol{e}_3, \mathbf{0}, 0)$	$\langle L_3, B_3, P_a, H \rangle$	$\langle L_3, B_3 \rangle$	$\langle P_a, H \rangle$
6	$\mathsf{M}(h\boldsymbol{e}_3, \mathbf{0}, p\boldsymbol{e}_3, 0)$	$\langle L_3, B_a, P_3, H \rangle$	$\langle L_3, B_a \rangle$	$\langle P_3, H \rangle$
$7_\pm$	$\mathsf{M}(h\boldsymbol{e}_3, \pm k\boldsymbol{e}_3, p\boldsymbol{e}_3, 0)$	$\left\langle L_3, B_3, P_a - \frac{p}{k} B_a, H \right\rangle$	$\langle L_3, B_3 \rangle$	$\left\langle P_a - \frac{p}{k} B_a, H \right\rangle$
8	$\mathsf{M}(\mathbf{0}, k\cos\theta\boldsymbol{e}_3 + k\sin\theta\boldsymbol{e}_2, p\boldsymbol{e}_3, 0)$	$\left\langle B_2 - \frac{k}{p}\cos\theta P_2, B_3 + \frac{k}{p}\sin\theta P_2, P_3, H \right\rangle$	$\langle \sin\theta B_2 + \cos\theta B_3 \rangle$	$\left\langle P_2 - \frac{p}{k}(\cos\theta B_2 - \sin\theta B_3), P_3, H \right\rangle$

Quantum particles and fields

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- For kinematical groups $K \cong G \ltimes T$ with T abelian, these are precisely the (projective) UIRs constructed using the **Wigner—Mackey** method of induced representations.

Rawnsley [1975]

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- For other kinematical groups it is largely unexplored.

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  - applying a group-theoretical Fourier transform to pass from the momentum orbit to the spacetime

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- Of course, there is also a growing body of work constructing Galilei/Carroll invariant field theories by taking nonrelativistic/ultrarelativistic limits of Poincaré-invariant field theories.

Applications

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- **Planons** are particles with conserved ***electric charge*** Q, ***dipole moment*** $\boldsymbol{D}$ and ***trace of quadrupole moment*** $Z = \int d^3x\rho(x)||\boldsymbol{x}||^2$

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- The absence of boosts means the fracton spacetime is aristotelian

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Summary

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- Each UIR can be viewed as a classical field on K/G subject to its free field equations (and perhaps some gauge invariance)
- The Carroll and Bargmann groups are intimately related to fractons and planons