A doubly reduced approximation for the solution to PDE's based on a domain truncation and a reduced basis method: Application to Navier-Stokes equations

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Symposium *Mathematical and applied aspects of complexity reduction methods*

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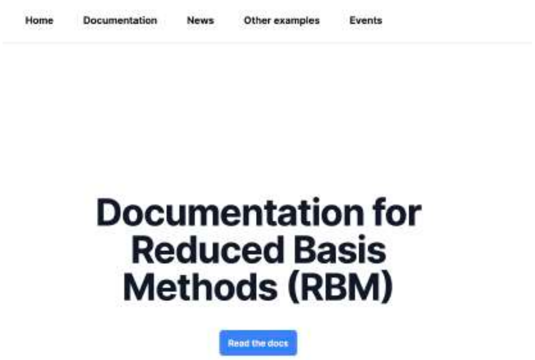
1 A small digression

2 Introduction

3 Rectification

4 New rectification
- Navier-Stokes equations
- NIRB method with parabolic equations

Reduced basis methods: https://grosjean1.github.io



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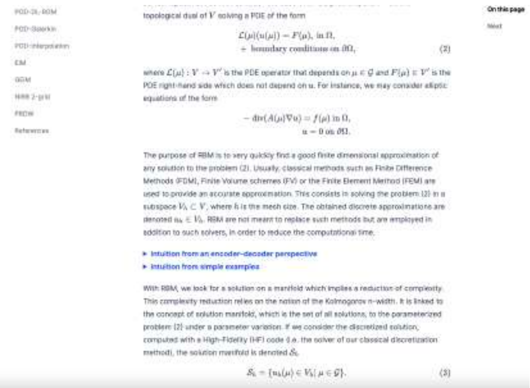

POD-DL-ROM
POD-Galerkin
POD-interpolation
EIM
GGM
NIRB 2-grid
PBDW
References

topological dual of V solving a PDE of the form

$$\begin{aligned} \mathcal{L}(\mu)(u(\mu)) &= F(\mu), \text{ in } \Omega, \\ &+ \text{ boundary conditions on } \partial\Omega, \end{aligned} \tag{2}$$

where $\mathcal{L}(\mu) : V \to V'$ is the PDE operator that depends on $\mu \in \mathcal{G}$ and $F(\mu) \in V'$ is the PDE right-hand side which does not depend on u. For instance, we may consider elliptic equations of the form

$$\begin{aligned} -\operatorname{div}(A(\mu)\nabla u) &= f(\mu) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The purpose of RBM is to very quickly find a good finite dimensional approximation of any solution to the problem (2). Usually, classical methods such as Finite Difference Methods (FDM), Finite Volume schemes (FV) or the Finite Element Method (FEM) are used to provide an accurate approximation. This consists in solving the problem (2) in a subspace $V_h \subset V$, where h is the mesh size. The obtained discrete approximations are denoted $u_h \in V_h$. RBM are not meant to replace such methods but are employed in addition to such solvers, in order to reduce the computational time.

- Intuition from an encoder-decoder perspective
- Intuition from simple examples

With RBM, we look for a solution on a manifold which implies a reduction of complexity. This complexity reduction relies on the notion of the Kolmogorov n-width. It is linked to the concept of solution manifold, which is the set of all solutions, to the parameterized problem (2) under a parameter variation. If we consider the discretized solution, computed with a High-Fidelity (HF) code (i.e. the solver of our classical discretization method), the solution manifold is denoted $\mathcal{S}_h$.

$$\mathcal{S}_h = \{u_h(\mu) \in V_h | \mu \in \mathcal{G}\}. \tag{3}$$

On this page

Next

POD-DL-ROM
POD-Galerkin
POD-Interpolation
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▶ See here for the complex Hilbert space

Proof [eigenvalue problem in real Hilbert space]

We start with the Lagrangian of (1):

$$J[\Phi_i] = \mathbb{E}[|(u,\Phi_i)|^2] - \lambda(\|\Phi_i\|^2 - 1).$$

We consider its Gâteaux derivative at Φ_i in a direction Ψ:

$J[\Phi_i + \delta\Psi] = \mathbb{E}[|(u,\Phi_i+\delta\Psi)|^2] - \lambda(\|\Phi_i+\delta\Psi\|^2 - 1).$

Since we want to maximise J, we have $\frac{d}{d\delta}J[\Phi_i+\delta\Psi]|_{\delta=0} = 0.$

- $(u,\Phi_i+\delta\Psi) = (u,\Phi_i) + \delta(u,\Psi)$. So,
  $(u,\Phi_i+\delta\Psi)^2 = (u,\Phi_i)^2 + 2\delta(u,\Phi_i)(u,\Psi) + O(\delta^2)$
  and with average:
  $\mathbb{E}[(u,\Phi_i+\delta\Psi)^2] = \mathbb{E}[(u,\Phi_i)^2] + 2\delta\mathbb{E}[(u,\Phi_i)(u,\Psi)] + O(\delta^2)$
  Thus, $\boxed{\frac{d}{d\delta}\mathbb{E}[(u,\Phi_i+\delta\Psi)^2]\Big|_{\delta=0} = 2\mathbb{E}[(u,\Phi_i)(u,\Psi)]}$
- $(\Phi_i+\delta\Psi,\Phi_i+\delta\Psi) = (\Phi_i,\Phi_i) + 2\delta(\Phi_i,\Psi) + O(\delta^2)$, so
  $\boxed{\frac{d}{d\delta}(\Phi_i+\delta\Psi,\Phi_i+\delta\Psi)\Big|_{\delta=0} = 2(\Phi_i,\Psi)}$

Altogether, we obtain:

$$\boxed{\mathbb{E}[(u,\Phi_i)(u,\Psi)] - \lambda(\Phi_i,\Psi) = 0}$$

Let C be such that $(C\Phi_i,\Psi) = \mathbb{E}[(u,\Phi_i)(u,\Psi)]$. Then

$$(C\Phi_i,\Psi) = \lambda(\Phi_i,\Psi), \forall\Psi \in V_N.$$

On this page
Offline
Online
Codes:
Details:
POD algorithm
How to choose ?
References
Backlinks
POD-Interpolation
EIM
Documentation

Reduced basis methods: https://grosjean1.github.io

THE GALERKIN-POD METHOD

Script by Elise Grosjean (ENSTA-Paris, elise.grosjean@ensta-paris.fr)

[70]:
```
import sys
!{sys.executable} -m pip install numpy
!{sys.executable} -m pip install matplotlib
!{sys.executable} -m pip install scikit-fem
```

Let us consider a parameterized problem. Having (previously computed) solutions for different parameter values, we aim at approaching much faster the solution associated to a new parameter value. The Galerkin-Proper Orthogonal Decomposition (POD) algorithm is decomposed in two parts:

- The "offline part" : First we compute a reduced basis (denoted Φ_i) from several snapshots of the solution (solutions for different parameter values in a parameter set $\mathcal{G}$),
- The "online part" : then we project onto it the solution associated to the new parameter value that we are interested in, denoted by $\mu \in \mathcal{G}$. To avoid computing the solution associated to this new parameter, we are going to solve instead a "reduced problem", that is the projection of the model onto the reduced space. This step is the key ingredient of the method: Instead of solving a problem of dimension the number of degrees of freedom, we are going to solve the reduced problem of dimension the number of functions in the reduced basis (also called modes). Thus, we assume here that this number can be much smaller than the number of degrees of freedom (see the paragraph on Kolmogorov n-width for details on this assumption) and that makes the algorithm runs much faster than the execution of usual high fidelity solver (e.g. FEM).

[2]:
```
# import packages
import skfem  # for Finite Element Method
import numpy as np
import matplotlib.pyplot as plt
```

1) The 2D-driven cavity problem:

We are going to use in this example the famous 2D-driven cavity problem with the Finite Element Method (FEM), which consists in solving on a unit square (denoted Ω) the following equations:

$$-\nu\Delta u + (u \cdot \nabla)u + \nabla p = 0, \text{ in } \Omega,$$
$$\nabla . u = 0, \text{ in } \Omega,$$

Reduced basis methods: https://grosjean1.github.io

Home Documentation News Other examples Events

Events

Welcome to the RBM Events Calendar!

Conferences/Workshops/Courses

- Upcoming

2026: RBM courses at College de France with Yvon Maday **link**

- Previous

Summer School on Reduced Order Methods in Computational Fluid Dynamics (2022,Trieste, Italy)

Reduced Order Models: Bridging PDE Models and Simulation Data (2024, Palaiseau, France)

MORTech 2025

ICOSAHOM 2025 (2025,Montréal,Canada)

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Motivation

Motivation

Further reduce the online cost of the NIRB 2-grid algorithm

NIRB 2-grid method

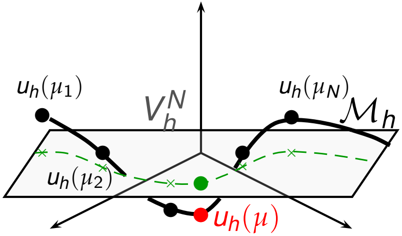
$PDE : \mu \to u(\mu)$

$\mu \in \mathcal{G}$: Parameter of interest

$u(\mu) \in V$: Solution

$(\cdot,\cdot) = (\cdot,\cdot)_V$

$h = \max_{K \in \mathcal{M}_h} h_K$, where $h_K = \sup_{x,y \in K} |x - y|$.



Solution manifold

Offline Part

$\mathcal{M}_h = \{u_h(\mu) | \ \mu \in \mathcal{G}\}$
V_h^N L^2-orthonormalized Reduced space
Parameters $\mu_1, \ldots, \mu_N \in \mathcal{G}$

Online Part

Approximation onto V_h^N : $u_h^N(\mu)$

NIRB 2-grid method

$PDE : \mu \to u(\mu)$

$\mu \in \mathcal{G}$: Parameter of interest

$u(\mu) \in V$: Solution

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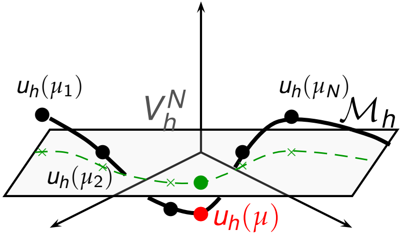

Solution manifold

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$\mathcal{M}_h = \{u_h(\mu) | \ \mu \in \mathcal{G}\}$

V_h^N L^2-orthonormalized Reduced space

Parameters $\mu_1, \ldots, \mu_N \in \mathcal{G}$

Online Part

Approximation onto V_h^N : $u_h^N(\mu)$

NIRB two-grid method

$u(\mu)$: Exact solution

μ $\qquad$ $u(\mu)$

[1] Chakir, R. & Maday, Y. (2009). *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE*.

NIRB two-grid method

$u(\mu)$: Exact solution

Fine mesh h
with HF code $u_h(\mu)$

μ $\rightarrow$ $u(\mu)$

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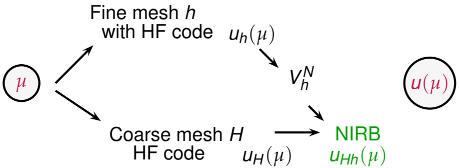
NIRB two-grid method

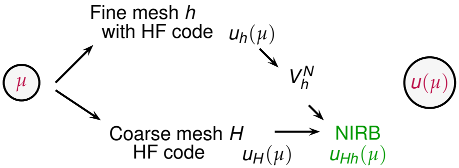
$\boxed{u(\mu) : \text{Exact solution} \;-\; u^N_{Hh}(\mu) : \text{NIRB Solution}}$[1]



[1] Chakir, R. & Maday, Y. (2009). *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE*.

NIRB two-grid method

$u(\mu)$: Exact solution – $u_{Hh}^N(\mu)$: NIRB Solution[1]



Decomposition

$$u_h^N(\mathbf{x},\mu) = \sum_{j=1}^{N} a_j^h(\mu)\Phi_j^h(\mathbf{x})$$

$(\Phi_j^h)_{j=1,\ldots,N} \in V_h^N$: fine L^2-orthonormalized modes

Optimal coefficients: $a_j^h(\mu) = (u_h(\mu), \Phi_j^h(\mathbf{x}))_{L^2(\Omega)}$,

Our choice: $b_j^H(\mu) = (u_H(\mu), \Phi_j^h(\mathbf{x}))_{L^2(\Omega)}$

[1] Chakir, R. & Maday, Y. (2009). *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE*.

NIRB two-grid method

$u(\mu)$: Exact solution – $u^N_{Hh}(\mu)$: NIRB Solution[1]

μ

Fine mesh h
with HF code $u_h(\mu)$
$\|u - u_h\|_{H^1} \leq C\,h$

V_h^N

Coarse mesh H
HF code $u_H(\mu)$
$\|u - u_H\|_{H^1} \leq C\,H$

NIRB
$u_{Hh}(\mu)$

$u(\mu)$

Convergence (FEM / FV ...)[1,2]

$$\|u(\mu) - u^N_{Hh}(\mu)\|_V \leq \underbrace{\varepsilon(N)}_{T_1} + \underbrace{C_1 h}_{T_2} + \underbrace{C_2(N) H^2}_{T_3}$$

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[1] Chakir, R. & Maday, Y. (2009). *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE.*

[2] G., E. & Maday, Y. (2021). *Error estimate of the non-intrusive reduced basis method with finite volume schemes*

NIRB two-grid method

$u(\mu)$: Exact solution – $u_{Hh}^N(\mu)$: NIRB Solution[1]

μ

Fine mesh h
with HF code $u_h(\mu)$
$\|u - u_h\|_{H^1} \leq C\,h$

V_h^N

Coarse mesh H
HF code $u_H(\mu)$
$\|u - u_H\|_{H^1} \leq C\,H$

NIRB
$u_{Hh}(\mu)$

$u(\mu)$

Post-treatment: rectification[1]

$$B_{ij} = b_j^H(\mu^i) \longrightarrow A_{ij} = a_j^h(\mu^i) \in \mathbb{R}^{N\times N}$$

Least-square approach:

$$\underset{R^i}{\arg\min}\, \|BR^i - A^i\|_2^2 + \lambda\|R^i\|_2^2.$$

$$\boxed{R_i = (B^T B + \lambda I_N)^{-1} B^T A_i, \quad i = 1, \dots, N.}$$

$$u_{Hh}^N(\mu) = \sum_{i,j=1}^{N} R_{ij}\, b_j^H(\mu)\, \Phi_i^h.$$

[1] Chakir, R. & Maday, Y. (2009). *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE*.

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Rectification

[a] R. Chakir *Contribution à l'analyse numérique de quelques problèmes en chimie quantique et mécanique, PhD*. 2009

[b] R. Chakir, Y. Maday, P. Parnaudeau, *A non-intrusive reduced basis approach for parametrized heat transfer problems*. 2019

[c] R. Chakir, B. Streichenberger, P. Chatellier, *A Non-intrusive reduced basis method for urban flows simulation.* 2021

[d] E. Grosjean, *Variations and further developments on the Non-Intrusive Reduced Basis two-grid method, PhD.* 2022

[e] O. Mula, *Quelques contributions vers la simulation parallèle de la cinétique neutronique et la prise en compte de données observées en temps réel, PhD.* 2014

Rectification setting

- Fine reduced space: $V_h^N = \mathrm{span}\{\Phi_1^h, \dots, \Phi_N^h\} = \mathrm{span}\{u_h(\mu_1), \dots, u_h(\mu_N)\} \subset V_h$.
- Fine reduced projection:

$$P_N^h : V_h \to V_h^N, \qquad P_N^h u_h(\mu) = \sum_{j=1}^{N} a_j^h(\mu) \Phi_j^h.$$

- Cheap/coarse approximation expressed in the fine basis:

$$J_N^H : V_h \to V_h^N, \qquad J_N^H u_h(\mu) = \sum_{j=1}^{N} b_j^H(\mu) \Phi_j^h \in V_h^N.$$

- Rectification matrix:

$$R : \mathbb{R}^N \to \mathbb{R}^N, \text{ and } R_i = (B^T B + \lambda I_N)^{-1} B^T A_i, \quad i = 1, \dots, N.$$

- Rectified operator:

$$\boxed{P_N^R u_H(\mu) = \sum_{i,j=1}^{N} R_{ij} b_j^H(\mu) \Phi_i^h.}$$

A model problem

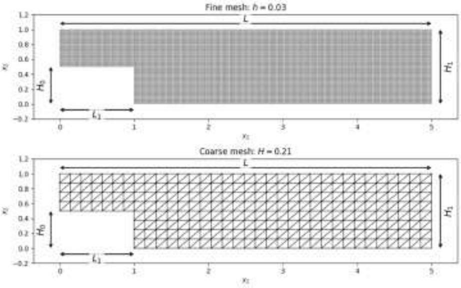
We solve the stationary Navier–Stokes problem: find $\mathbf{u} = (u_1, u_2) \in H^1(\Omega)^2$ and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0 \quad & \text{in } \Omega,\\
\nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega,\\
\mathbf{u} = \begin{pmatrix} g_{\text{in}}(x_2) \\ 0 \end{pmatrix} \quad & \text{on } \Gamma_{\text{in}},\\
\nu\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} = 0 \quad & \text{on } \Gamma_{\text{out}},\\
\mathbf{u} = 0 \quad & \text{on } \partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}).
\end{aligned}
$$

where the inflow profile is

$$g_{\text{in}}(x_2) = \frac{4}{H_0^2}(H_1 - x_2)(x_2 - H_1 + H_0).$$

$$\nu = \frac{1}{Re} \text{ and } \mu := Re \in \mathcal{G} = [30, 300].$$



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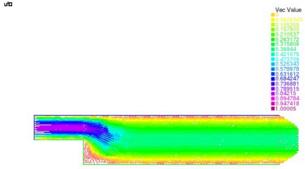
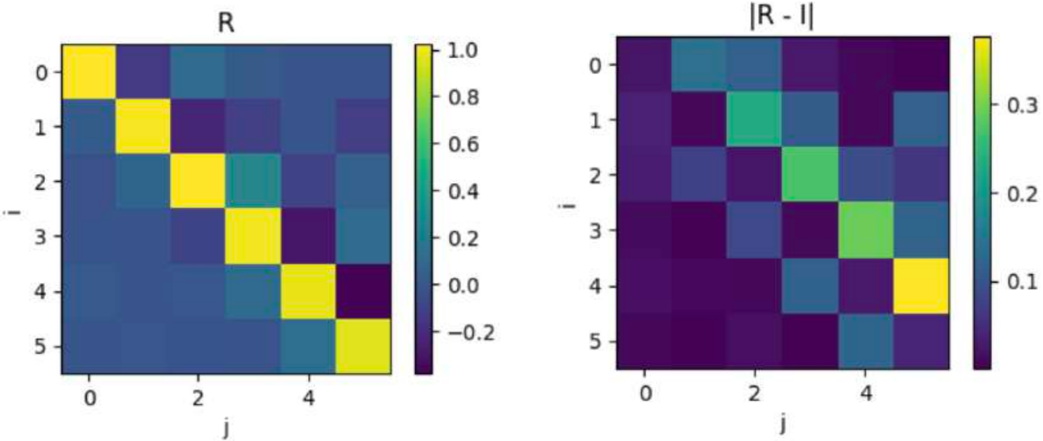
$\nu = 0.01$



Conditioning of R

Rectification matrix:

$$R : \mathbb{R}^N \to \mathbb{R}^N, \text{ and } R_i = (B^T B + \lambda I_N)^{-1} B^T A_i, \quad i = 1, \ldots, N.$$



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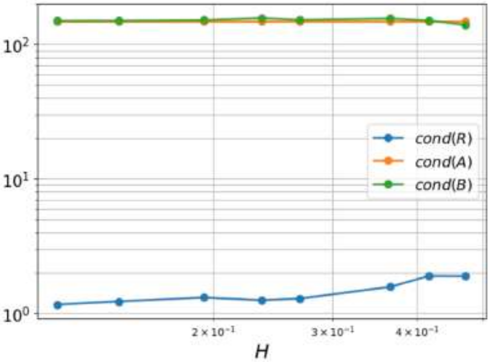
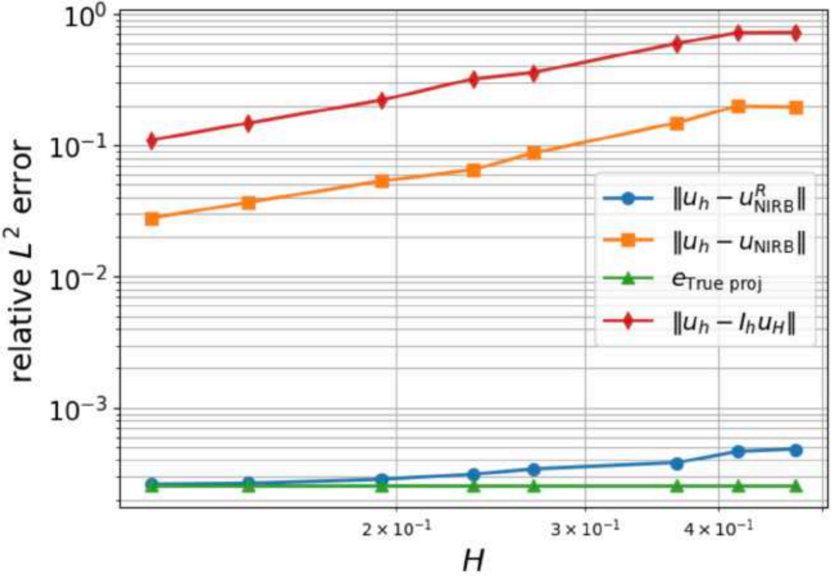


NIRB Error



10^0
10^{-1}
10^{-2}
10^{-3}
relative L^2 error
2×10^{-1}
3×10^{-1}
4×10^{-1}
H
$\|u_h - u^R_{\mathrm{NIRB}}\|$
$\|u_h - u_{\mathrm{NIRB}}\|$
$e_{\mathrm{True\ proj}}$
$\|u_h - I_h u_H\|$

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3 Rectification

4 New rectification
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New method

How can we further reduce the online costs?

New method

Decomposition

$$u_h^N(\mathbf{x},\mu) = \sum_{j=1}^{N} a_j^h(\mu)\Phi_j^h(\mathbf{x})$$

$(\Phi_j^h)_{j=1,\dots,N} \in V_h^N$: fine L^2-orthonormalized modes

Optimal coefficients: $a_j^h(\mu) = (u_h(\mu), \Phi_j^h(\mathbf{x}))_{L^2(\Omega)}$,

Our new choice: $b_j^H(\mu) = (u_H(\mu), \Phi_j^H(\mathbf{x}))_{L^2(\Omega)}$

Our new choice: $b_j^H(\mu) = (u_H(\mu), \Phi_j^H(\mathbf{x}))_{L^2(\omega)}$, where $\omega \subset \Omega$

New method

Decomposition

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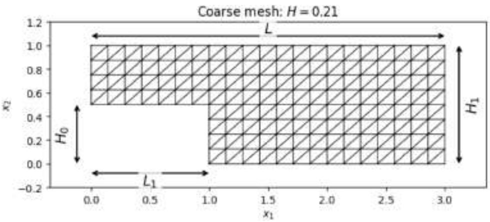
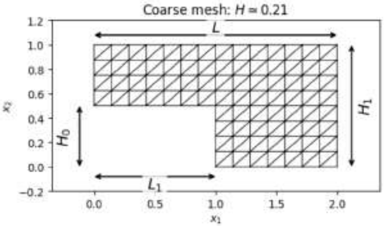
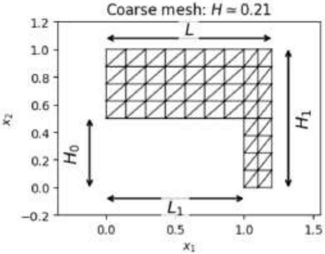
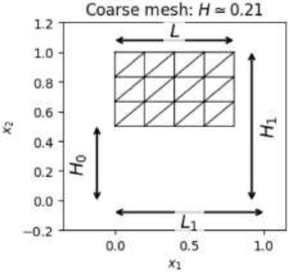
Natural truncations $\omega \subset \Omega$









Error contributions, $\omega = \Omega$

The fine and coarse coefficients are

$$a_i^h = (u_h, \Phi_i^h)_{L^2(\Omega)}, \qquad b_i^H = (u_H, \Phi_i^h)_{L^2(\Omega)}.$$

The coarse reduced coefficients are

$$\gamma_i^H = (u_H, \Phi_i^H)_{L^2(\Omega)}.$$

Define the cross-Gram matrix $(C_H)_{ij} = (\Phi_i^H, \Phi_j^h)_{L^2(\Omega)}$.
Then

$$b^H \approx C_H^T \gamma^H.$$

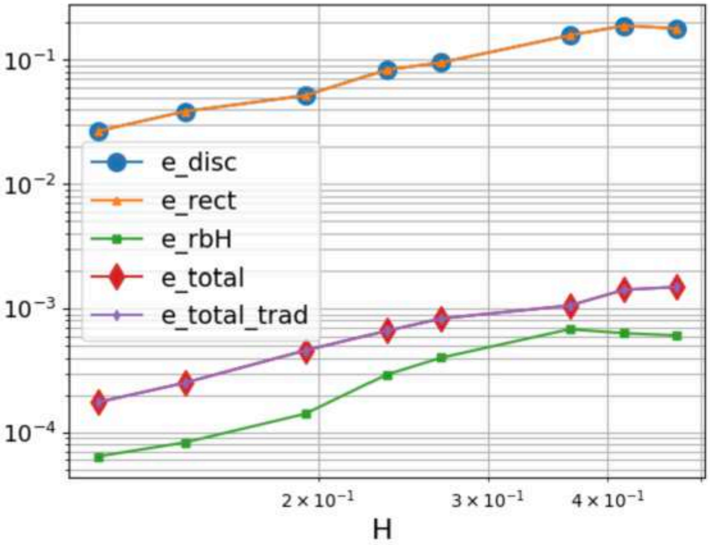
The online coefficient error satisfies

$$\boxed{a^h - R\gamma^H = (a^h - b^H) + (b^H - C_H^T\gamma^H) + (C_H^T - R)\gamma^H.}$$

Hence

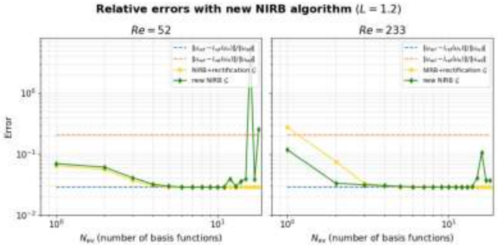
$$\boxed{\|a^h - R\gamma^H\| \leq \underbrace{\|a^h - b^H\|}_{e_{disc}} + \underbrace{\|b^H - C_H^T\gamma^H\|}_{e_{rbH}} + \underbrace{\|(C_H^T - R)\gamma^H\|}_{e_{rect}}.}$$

NIRB Error

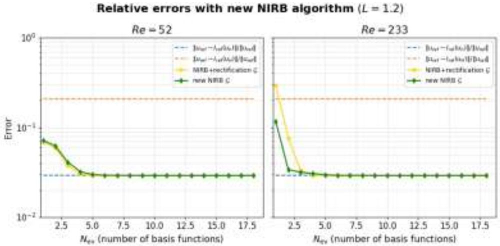


10^{-1}
10^{-2}
10^{-3}
10^{-4}
2×10^{-1}
3×10^{-1}
4×10^{-1}
H
e_disc
e_rect
e_rbH
e_total
e_total_trad

NIRB Error with $L = 1.2$



$\lambda = 0$



$\lambda = 10^{-10}$

Conditioning with $H = 0.15$

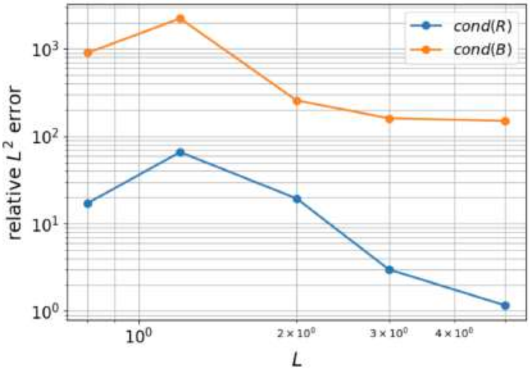


Figure: conditioning

Runtimes

Table: FEM runtimes min:sec

	FEM high fidelity solver	**FEM coarse solution**
	00:43	00:01
	NIRB Offline ($N = 18$)	**classical NIRB online**
	14:30	00:10
	new NIRB offline ($N = 18$)	**new NIRB online**
$\omega = \Omega$	14:52	00:10
$L = 3$	14:52	00:09
$L = 1.2$	14:50	00:09

3D results

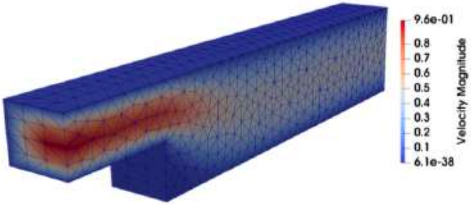


Figure: 3D domain (clip)

Results

Table: Relative L^2 errors, $N \in \{2, 3, 5\}$ with $\lambda = 10^{-10}$

	FEM results		NIRB results		
N	fine mesh	coarse mesh	$L = 5\ (\omega = \Omega)$	$L = 3\ (\omega \subset \Omega)$	$L = 1.2\ (\omega \subset \Omega)$
$N = 2$	**2.50E^{-2}**	**9.80E^{-2}**	**1.14E^{-1}**	**1.15E^{-1}**	**1.46E^{-1}**
$N = 3$	**2.50E^{-2}**	**9.80E^{-2}**	**2.85E^{-2}**	**3.55E^{-2}**	**4.51E^{-2}**
$N = 5$	**2.50E^{-2}**	**9.80E^{-2}**	**2.53E^{-2}**	**2.57E^{-2}**	**2.65E^{-2}**

NIRB convergence results

$\mathbf{u}(\mu) \in H^s(\Omega)$ with $s \simeq 1.54448$. Thus, a standard FEM yields reduced convergence rates and the following estimates hold for any $\mu \in \mathcal{G}$ with a uniform mesh of size h :

$$\|\mathbf{u}(\mu) - \mathbf{u}_h(\mu)\|_{H^1(\Omega)} \leq Ch^{s-1}\|u\|_{H^s(\Omega)},$$
$$\|\mathbf{u}(\mu) - \mathbf{u}_h(\mu)\|_{L^2(\Omega)} \leq Ch^{2(s-1)}\|u\|_{H^s(\Omega)},$$

[1] C. Bernardi and G. Raugel, *Méthodes d'éléments finis mixtes pour les équations de stokes et de navier-stokes dans un polyone non convexe*, 1981.

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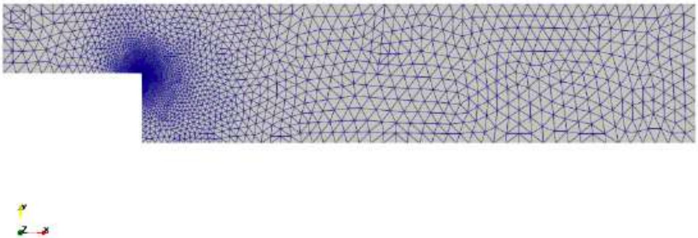
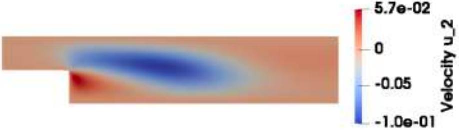
Convergence (FEM / FV ...)

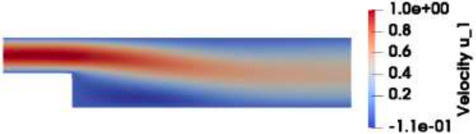
$$\|u(\mu) - u_{Hh}^N(\mu)\|_V \leq \underbrace{\varepsilon(N)}_{T_1} + \underbrace{C_1 h}_{T_2} + \underbrace{C_2(N)H^2}_{T_3}$$

[1] C. Bernardi and G. Raugel, *Méthodes d'éléments finis mixtes pour les équations de stokes et de navier-stokes dans un polyone non convexe*, 1981.

Results







$$h = \min_{K \in \mathcal{T}_h} h_K, \qquad h_K = \sup_{x,y \in K} |x - y|.$$

Results

Reference mesh size $= 2.54 \times 10^{-5}$.

Table: Relative errors with the semi H^1-norm, refined meshes $N = 6$ with $\lambda = 0$

		NIRB with $H = 0.320156$			NIRB with $H = 0.707107$		
h	FEM with the fine refined mesh	NIRB + Rectification	New NIRB $\omega = \Omega$	New NIRB with $L = 3$ $(\omega \subset \Omega)$	NIRB + Rectification	New NIRB $\omega = \Omega$	New NIRB with $L = 3$ $(\omega \subset \Omega)$
1.99E^{-4}	**1.93E^{-3}**	**1.93E^{-3}**	**1.93E^{-3}**	**1.93E^{-3}**	**1.93E^{-3}**	**1.93E^{-3}**	**1.95E^{-3}**
1.79E^{-4}	**1.70E^{-3}**	**1.71E^{-3}**	**1.71E^{-3}**	**1.71E^{-3}**	**1.71E^{-3}**	**1.71E^{-4}**	**1.73E^{-3}**
1.02E^{-4}	**9.72E^{-4}**	**9.73E^{-4}**	**9.73E^{-4}**	**9.74E^{-4}**	**9.74E^{-4}**	**9.74E^{-4}**	**9.81E^{-4}**
5.09E^{-5}	**5.39E^{-4}**	**5.42E^{-4}**	**5.42E^{-4}**	**5.42E^{-4}**	**5.43E^{-4}**	**5.42E^{-4}**	**5.42E^{-4}**

FEM Error estimates within the parabolic context

NIRB two-grid method applied to parabolic equations

FEM Error estimates within the parabolic context

$$
\begin{aligned}
&\partial_t u - \nabla \cdot (A(p)\nabla u) = f, \text{ dans } \Omega \times]0, T], \\
&u(\mathbf{x}, 0) = u^0(\mathbf{x}), \text{ in } \Omega, \\
&\partial_n u(\cdot, t) = 0, \text{ on } \partial\Omega \times]0, T],
\end{aligned}
$$

$\mu = (t, p) \in \mathcal{G} \subset \mathbb{R}^+ \times \mathbb{R}^P$: parameter

$u(t, \mathbf{x}; p)$: unknowns

FEM error estimates

Theorem [E G.-Y M.] (H^1 semi-norm error estimate)[2]

Pod-greedy[3] or greedy-greedy algorithm for the BR construction

$$\forall n,\ |u(t^n)(p) - u_{Hh}^{N,n}(p)|_{H^1(\Omega)} \leq \overbrace{\varepsilon(N)}^{T_1} + \underbrace{C_1(p)(h + \Delta t_F)}_{T_2} + \overbrace{C_2(N)(H^2 + \Delta t_G^2)}^{T_3}$$

$C_1,\ C_2$: constants independent of h et H

$u_h^n(p^k) = u_h(p^k, t^n) \in V_h$, for $n = 1, \dots, N_T$: Fine snapshots

$u_H^m(p) = u_H(p, \tilde{t}^m) \in V_H$, for $m = 1, \dots, M_T$: Coarse solution, $H \gg h$.

[1] V. Thomée, *Galerkin finite element methods for parabolic problems*, 1984.

[2] E. G., Y. Maday, *Error estimate of the non-intrusive reduced basis two-grid method with parabolic equations.* 2023.

[3] B. Haasdonk, *Convergence rates of the pod-greedy method*, 2013.

FEM error estimates

Theorem [E G.-Y M.] (H^1 semi-norm error estimate)[2]

$$\forall n,\ |u(t^n)(p) - u_{Hh}^{N,n}(p)|_{H^1(\Omega)} \leq \overbrace{\varepsilon(N)}^{T_1} + \underbrace{C_1(p)(h + \Delta t_F)}_{T_2} + \overbrace{C\sqrt{\lambda_N}(H^2 + \Delta t_G^2)}^{T_3}$$

λ_N: $\forall v \in X_h^N, \int_\Omega \nabla\Phi_h \cdot \nabla v = \lambda \int_\Omega \Phi_h \cdot v$,
and C_1, C: constants independent of h, H and N.

$$\|u(\tilde{t}^m) - u_H^m\|_{L^2(\Omega)} \leq C\, H^2 \Big[\|u^0\|_{H^2(\Omega)} + \int_0^{\tilde{t}^m} \|u_t\|_{H^2(\Omega)}\, ds\Big] + C\,\Delta t_G^2 \big[\Big(\int_0^{\tilde{t}^m} \|u_{ttt}\, ds\|_{L^2(\Omega)}^2\Big)^{1/2} + \Big(\int_0^{\tilde{t}^m} \|\Delta u_{tt}\|_{L^2(\Omega)}^2\, ds\Big)^{1/2}\big].$$

[1] V. Thomée, *Galerkin finite element methods for parabolic problems*, 1984.
[2] E. G., Y. Maday, *Error estimate of the non-intrusive reduced basis two-grid method with parabolic equations.* 2023.

Decomposition

Projection coefficients

$$u_h(\mathbf{x}, t^n; p) = \sum_j a_j^h(p, t^n)\, \Phi_j^h(\mathbf{x}), \quad n = 1, \ldots, N_T$$

$(\Phi_j^h)_{j=1,\ldots,N} \in X_h^N$: L^2-orthonormalized basis functions (modes)

Coefficients $a_j^h(\mu) = a_j^h(p, t^n)$

Optimal coefficients: $(u_h(p, t^n), \Phi_j^h(\mathbf{x})), n = 1, \ldots, N_T$

Our choice with classical rectification: $(I_n(u_H(p, \tilde{t}^m)), \Phi_j^h(\mathbf{x}))_{L^2},\ m = 1, \ldots, M_T$

Our choice with new rectification: $(I_n(u_H(p, \tilde{t}^m)), \Phi_j^H(\mathbf{x}))_{L^2(\omega)},\ m = 1, \ldots, M_T$

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Results

We have taken the parameter set to be $\mathcal{G} = [0.5, 9.5]^2$.
For $\boldsymbol{\mu} = (1, 1)$,

$$u(\mathbf{x}, t; 1) = (t + x^2/2 - x^3/3)(t - y^2/2 + y^3/3),$$

for a right-hand side function

$$f(\mathbf{x}, t) = x^2(1 - x)^2 + t[-2 + 12x - 11x^2 - 2x^3 + x^4],$$

where $\mathbf{x} = (x, y)$.

Table: Relative $l^{\infty}(0,\ldots,N_T)$ errors, $N \in \{5, 10, 15\}$ with $\lambda = 10^{-10}$

Number of modes	mesh sizes	FEM results		NIRB results		
N	h	fine mesh	coarse mesh	NIRB + Rect. $(\omega = \Omega)$	$\omega = [0.2, 0.8]^2$	$\omega = [0.4, 0.6]^2$
$N = 5$	0.01	**6.20E^{-3}**	**5.88E^{-2}**	**6.25E^{-3}**	**1.08E^{-2}**	**2.01E^{-2}**
	0.02	**1.08E^{-2}**	**8.88E^{-2}**	**1.11E^{-2}**	**1.41E^{-2}**	**2.02E^{-2}**
	0.05	**2.91E^{-2}**	**1.50E^{-1}**	**2.97E^{-2}**	**3.17E^{-2}**	**3.73E^{-2}**
	0.1	**5.88E^{-2}**	**1.90E^{-1}**	**5.99E^{-2}**	**6.18E^{-2}**	**6.37E^{-2}**
$N = 10$	0.01	**6.20E^{-3}**	**5.88E^{-2}**	**6.23E^{-3}**	**6.38E^{-3}**	**1.99E^{-2}**
	0.02	**1.08E^{-2}**	**8.88E^{-2}**	**1.11E^{-2}**	**1.14E^{-2}**	**2.02E^{-2}**
	0.05	**2.91E^{-2}**	**1.50E^{-1}**	**2.97E^{-2}**	**2.97E^{-2}**	**3.74E^{-2}**
	0.1	**5.88E^{-2}**	**1.90E^{-1}**	**5.99E^{-2}**	**6.00E^{-2}**	**6.37E^{-2}**
$N = 15$	0.01	**6.20E^{-3}**	**5.88E^{-2}**	**6.23E^{-3}**	**6.38E^{-3}**	**1.99E^{-2}**
	0.02	**1.08E^{-2}**	**8.88E^{-2}**	**1.11E^{-2}**	**1.14E^{-2}**	**2.02E^{-2}**
	0.05	**2.91E^{-2}**	**1.50E^{-1}**	**2.97E^{-2}**	**2.97E^{-2}**	**3.74E^{-2}**
	0.1	**5.88E^{-2}**	**1.90E^{-1}**	**5.99E^{-2}**	**6.00E^{-2}**	**6.37E^{-2}**

Conclusions and perspectives

Perspectives

- ◇ More insights on the theory
- ◇ More realistic problem
- ◇ Application of this strategy after removing the leading singular term
- ◇ Put the NIRB 2-grid CG-T on the RB website!

Thank you!