Efficient Greedy Sampling for Model Order Reduction

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Image by Bart van Overbeeke Photography

Outline

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Configuration Optimization Problems

- Greedy Sampling Method

Randomized Polytope Division Method

- Connection to birth processes

Convergence results

Numerical Examples

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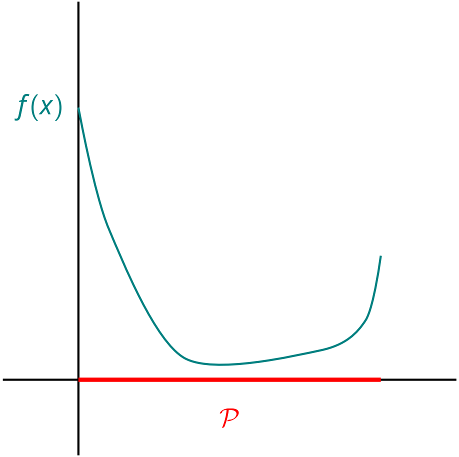
Example: Piecewise linear interpolation

- $\mathcal{P} \subset \mathbb{R}^d$
- $\eta = \{p_1, \ldots, p_n\} \in \Omega$
- Loss function: $\mathcal{L} : \Omega \to \mathbb{R}$



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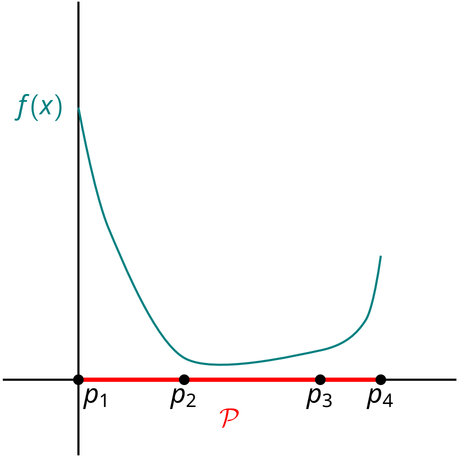

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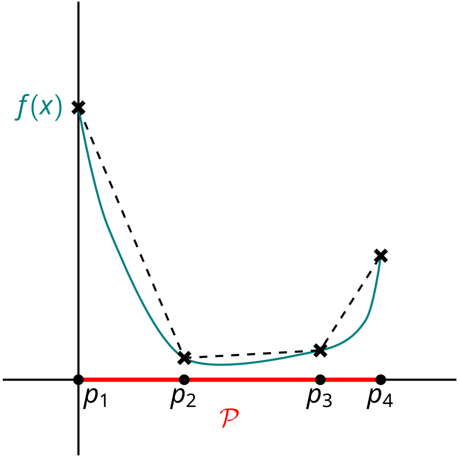

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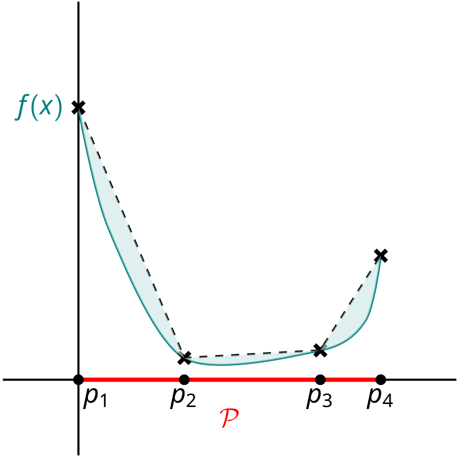

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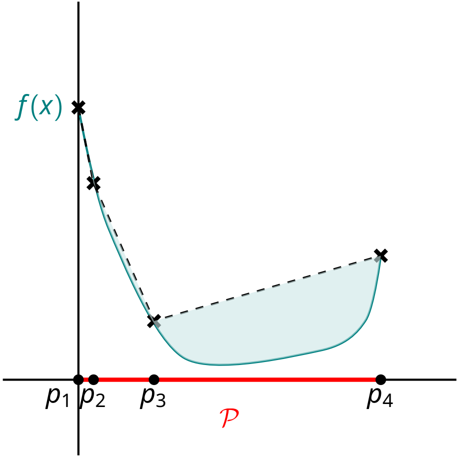

Example: Reduced Basis Method

$$a(u(p), v; p) = f(v; p), \qquad \forall v \in \mathbb{V}, p \in \mathcal{P}.$$

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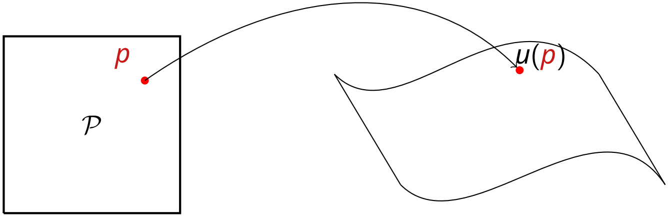

Parameter space

Solution manifold

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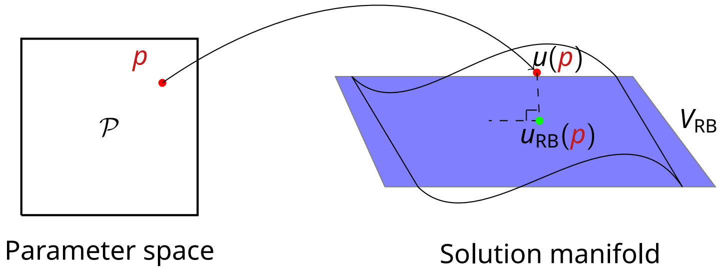

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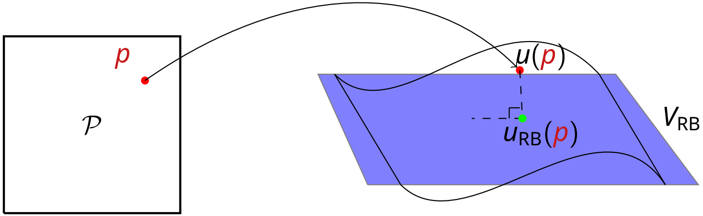

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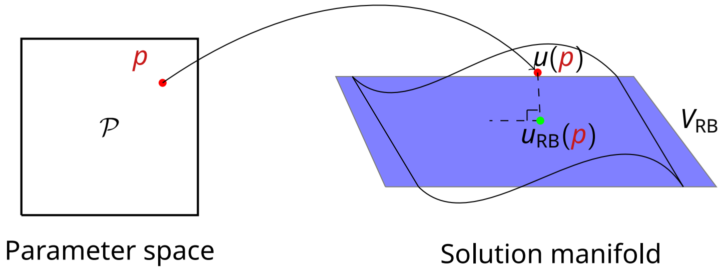

- $V_{RB}(\eta) := \mathrm{span}\{u(p_1), \ldots, u(p_n)\}$.
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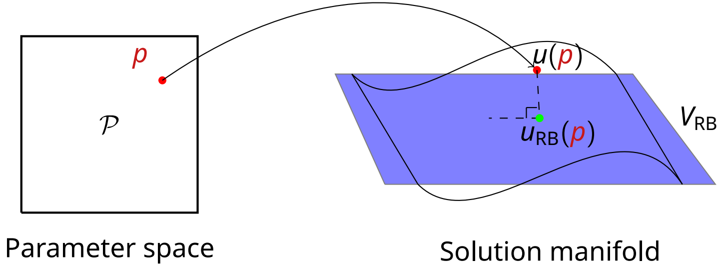

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- Loss function: $\mathcal{L}_{\text{RB}}(\eta) = \max_{q \in P} \|u(q) - u_{\text{RB}}(q)\| \approx \max_{q \in P} J(q, \eta)$

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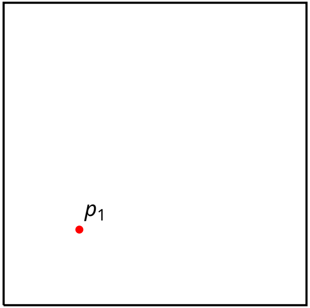

Greedy Sampling

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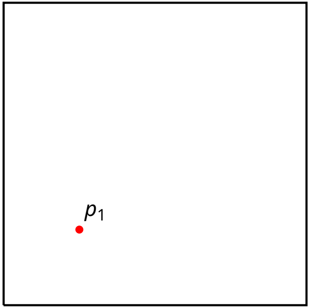
- Specify sample set $\mathcal{S} \subset \mathcal{P}$ (random or deterministic)



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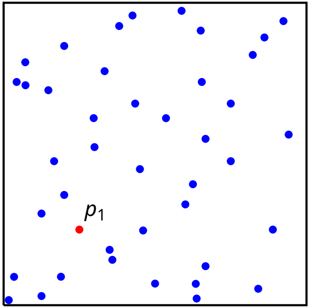

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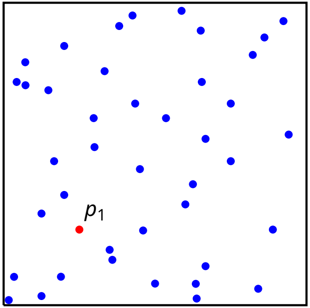

Compute

$\xrightarrow{J(q,\eta)}$

Greedy Sampling

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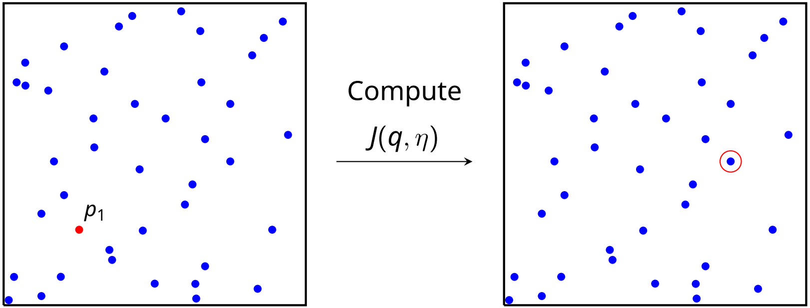

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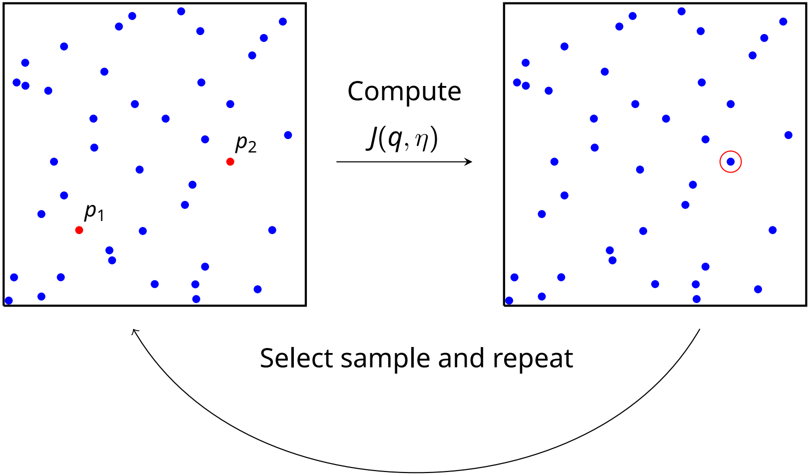
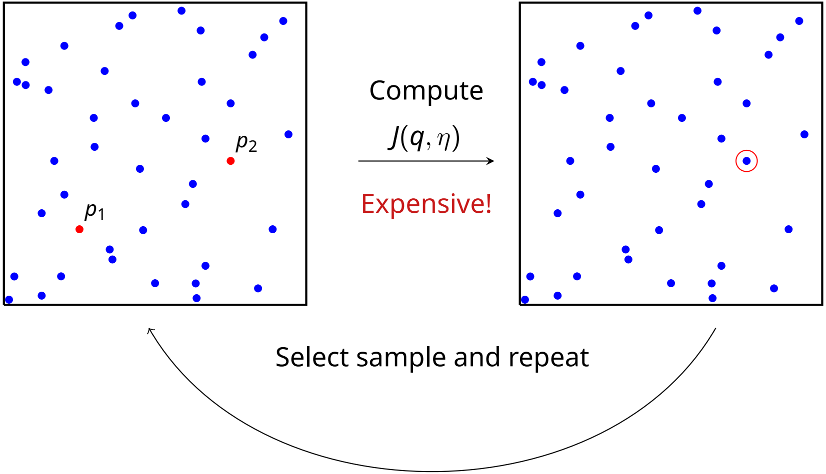


- Specify sample set $\mathcal{S} \subset \mathcal{P}$ (random or deterministic) Difficult in high-dimensions!
- Choose p_1 randomly and set $\eta = \{p_1\}$.



Better scaling properties

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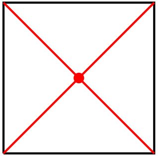
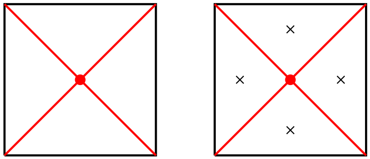
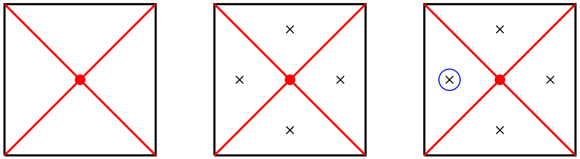
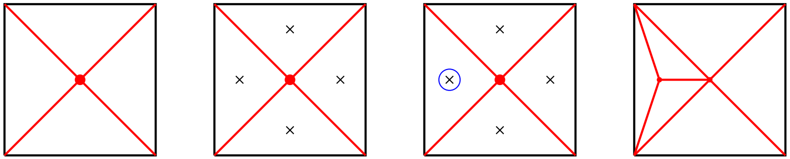




Polytope Division Method





Polytope Division Method

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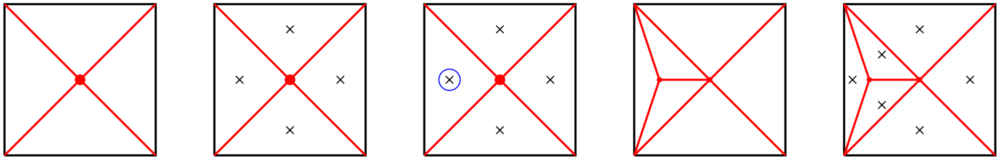
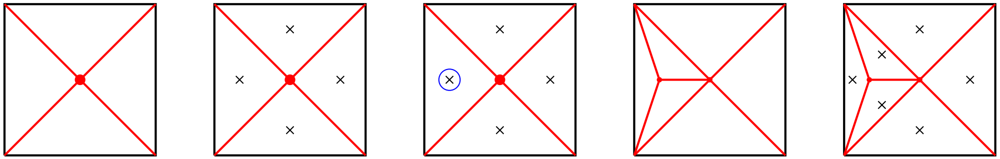
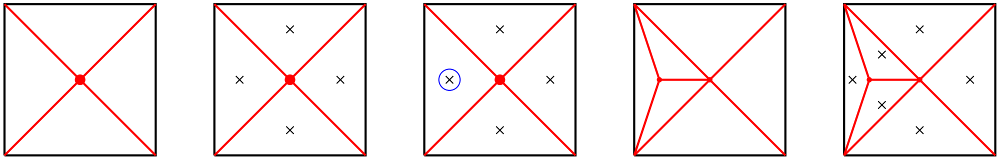




Benefits of randomization:



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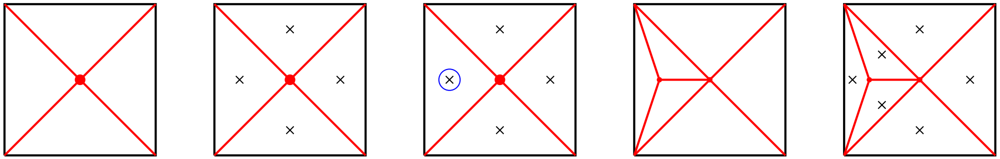
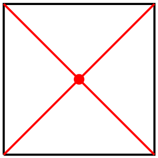
- Explorability



Benefits of randomization:

- Explorability
- Convergence proof



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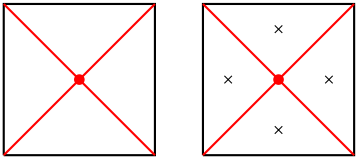

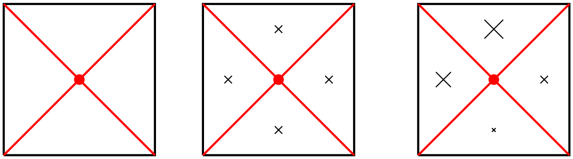

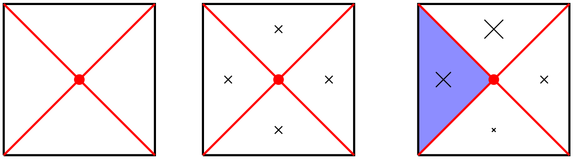

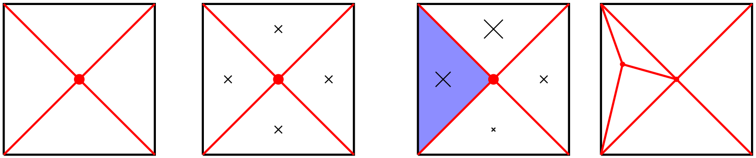

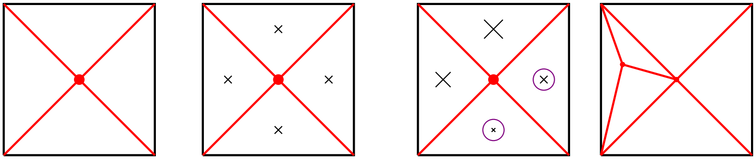

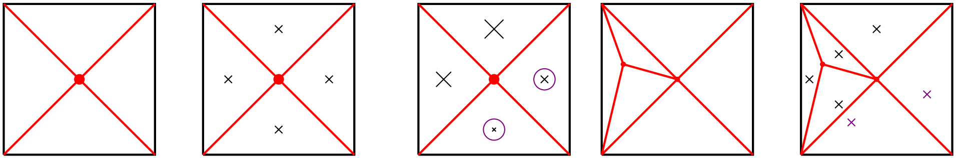

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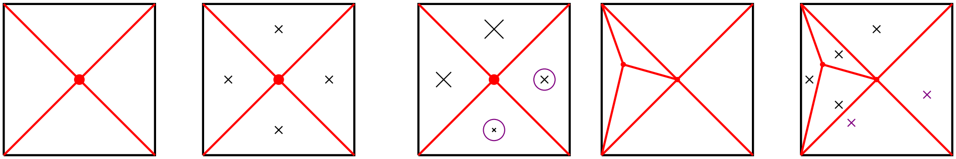
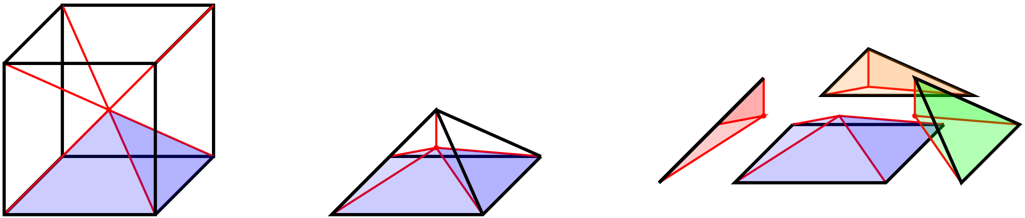
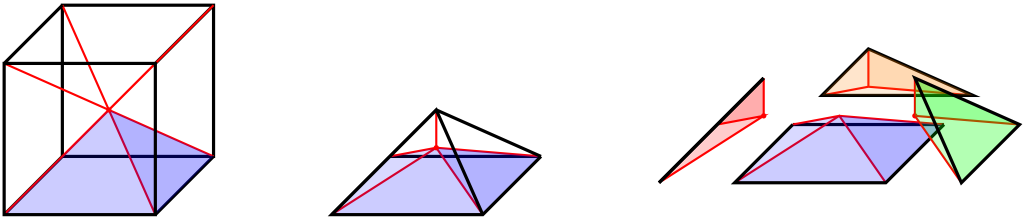
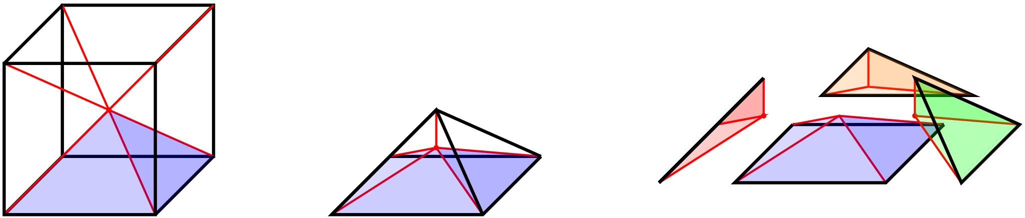


More methods on division of parameter domains exist (Burkardt, Gunzburger, and Lee 2006; Eftang and Stamm 2012; Guignard and Mula 2024)





- Can we still find the facets in the higher-dimensional case?



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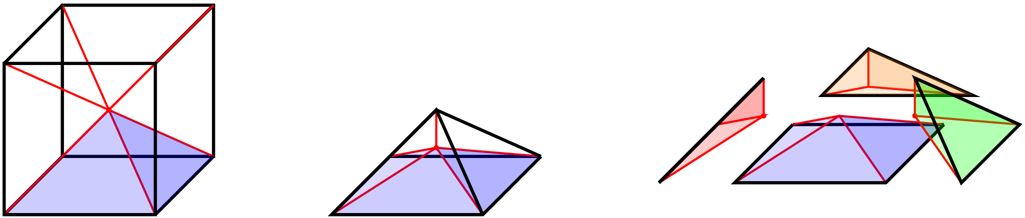
- Do we keep the satisfactory scaling properties?



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Yes!

(Nielen, Tse, and Veroy 2025)

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Connection to birth processes

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with transition kernel $\lambda(\eta_t, dy)$

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- y is called the newborn particle
- The transition kernel $\lambda(\eta_t, dy)$ expresses the likelihood of a particle being born at position y given configuration η_t.

- Generator:

$$LF(\eta) := \int_{\mathcal{P}} [F(\eta \cup y) - F(\eta)]\, \lambda(\eta, dy) \quad \text{for all } F \in C_b(\Omega)$$

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- Here, $\langle F, P_t \rangle := \int_\Omega F(\eta)\, P_t(d\eta)$

Assumption on global error function

TU/e

Assumption

Let $G : \Omega \to \mathbb{R}^+$, then G satisfies:

1. *(Boundedness). There exists a $c_0 \in \mathbb{R}^+$ such that $G(\{q\}) \leq c_0$ for all $q \in \mathcal{P}$.*

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$$G(\eta \cup y) \leq G(\eta).$$

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3. *(Saturation property) If $G(\eta) = G(\eta \cup y)$ for $\lambda(\eta, dy)$-almost every y, and P_∞-almost every η, then $G(\eta) = 0$.*

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$$G(\eta \cup y) \leq G(\eta).$$
3. *(Saturation property) If $G(\eta) = G(\eta \cup y)$ for $\lambda(\eta, dy)$-almost every y, and P_∞-almost every η, then $G(\eta) = 0$.*
4. *(Improvement factor) There exists a $\gamma \in (0,1), \delta > 0$, $\beta \in [0,1]$, and for every η there exists a set $B(\eta) \subset \mathcal{P}$ with $\lambda(\eta, B(\eta)) > \delta$, such that*
$$\fint_{B(\eta)} \left(G(\eta) - G(\eta \cup y)\right) \lambda(\eta, dy) \geq \frac{\gamma}{|\eta|^\beta} G(\eta)$$

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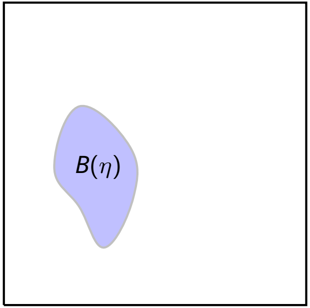

Parameter space

Improvement factor

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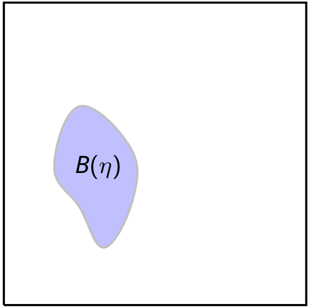
$$\int_{B(\eta)} \left(G(\eta) - G(\eta \cup y)\right) \lambda(\eta, dy) \geq \gamma \, G(\eta)$$



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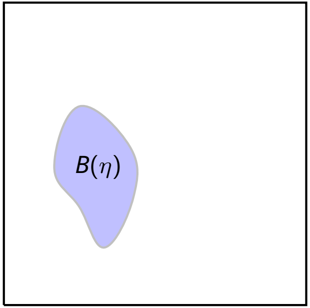


Parameter space

Theorem

(Under assumptions) For every $\varepsilon > 0$, there exist an $c_\varepsilon > 0$, such that

$$\mathbb{P}\Big(G(\eta_t) < \varepsilon\Big) \geq 1 - c_\varepsilon \theta_\beta(t), \quad \textit{for all } t \geq 0,$$

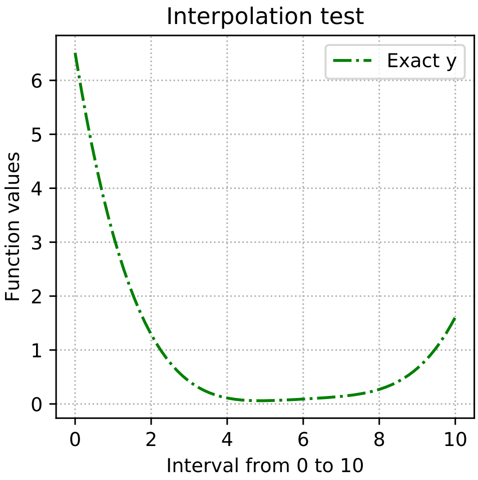
where

$$\theta_\beta = \begin{cases} e^{-\gamma\delta t} & \textit{for } \beta = 0, \\ t^{-\frac{1}{\beta}+1} & \textit{for } \beta \in (0,1), \\ \frac{1}{\log(1+t)} & \textit{for } \beta = 1. \end{cases}$$

(Nielen and Tse 2026)

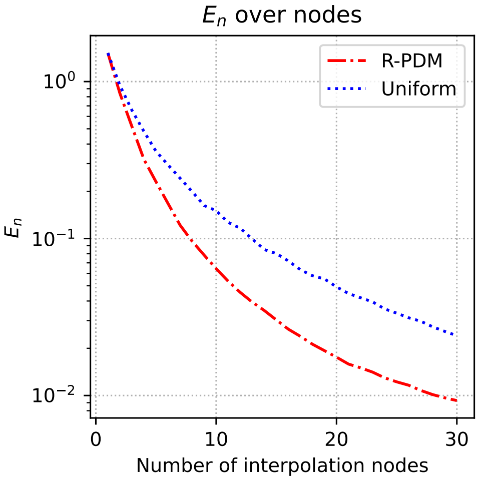
Numerical example: Interpolation

TU/e



- $f(x) = \frac{1}{200}((x-6)^4 + (x-2)^2 + 2$
- $x_0 = 0$, $x_{n+1} = 10$
- Configuration: $\eta = \{x_1, \ldots, x_n\}$
- Global error function: $G(\eta) = \|f - L_\eta\|_{L_1([0,10])}$
- Local error function: $J(q,\eta) = |f(q) - L_\eta(q)|$.

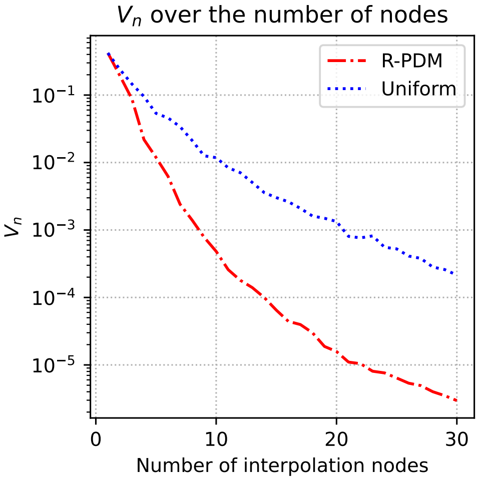
Expectation



- We run the algorithm 1000 times
- $E_n = \frac{1}{1000} \sum_{\ell=1}^{1000} \|f - L_{\eta_n}^{\ell}\|_{L_1([0,10])}$

TU/e

Variance



- We run the algorithm 1000 times
- $E_n = \frac{1}{1000} \sum_{\ell=1}^{1000} \left\| f - L_{\eta_n}^{\ell} \right\|_{L_1}$
- $V_n = \frac{1}{1000} \sum_{\ell=1}^{1000} (\| f - L_{\eta_n}^{\ell} \|_{L^1} - E_n)^2$

$$
\begin{aligned}
-\mathrm{div}(a(\omega, x)\nabla u_\omega) &= f \quad \text{in } \Gamma, \\
u_\omega &= 0 \quad \text{on } \partial\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{div}(a(\omega, x)\nabla u_\omega) &= f \quad \text{in}\, \Gamma, \\
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\end{aligned}
$$

The random field can be described via its Karhunen-Loeve (KL) expansion:

$$
a(\omega, x) = \bar{a}(x) + \sum_{k=1}^{\infty} Y_k(\omega)\Psi_k(x),
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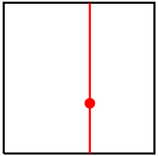
The random field can be described via its Karhunen-Loeve (KL) expansion:

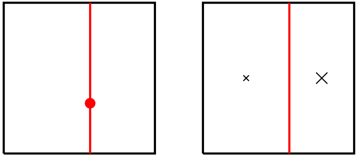
$$a(\omega,x) \approx \bar{a}(x) + \sum_{k=1}^{\mathcal{K}} Y_k(\omega)\Psi_k(x),$$

(Bachmayr and Cohen 2024; Cohen and DeVore 2015; Ghanem and Spanos 1991)

HALF splitting



HALF splitting



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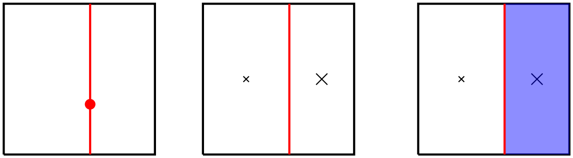

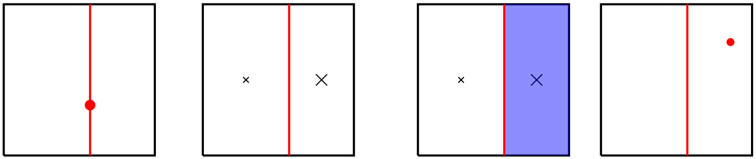

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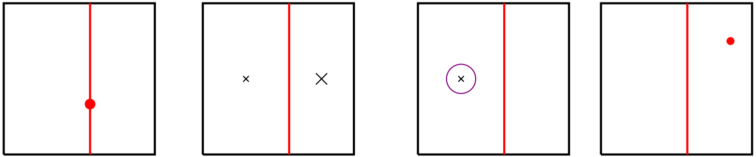

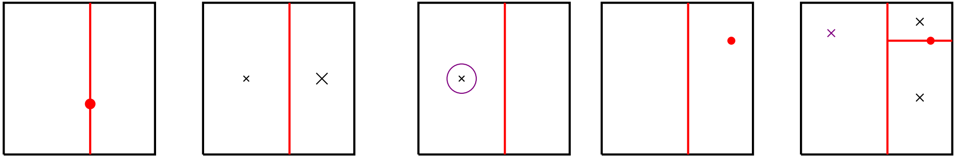

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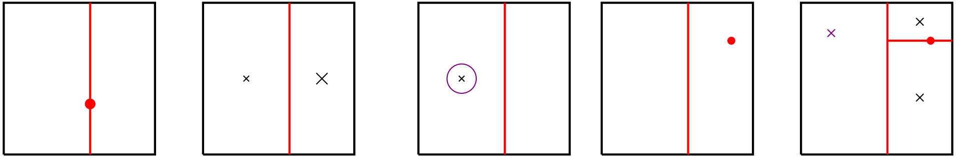

A closely related idea is introduced in (Guignard and Mula 2024)

Numerical example: Random fields

TU/e

Example: Uniform coefficients Source: (Chen, Quarteroni, and Rozza 2013)

$$a(\omega, x) = a_0(x) + \sum_{k=1}^{\mathcal{K}} Y_k(\omega) a_k(x),$$

where $a_0 = 1$, $\mathcal{K} = 100$, and Y_k is uniformly distribution on $[-0.5, 0.5]$.

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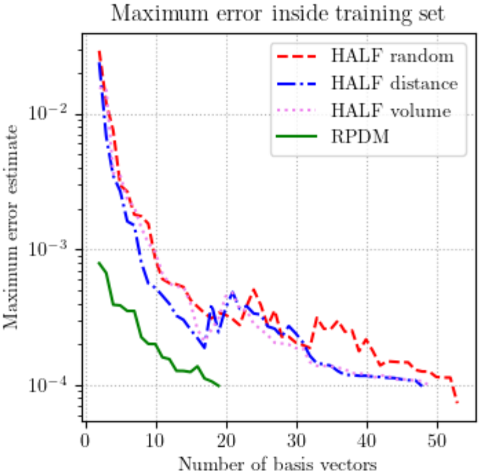
Uniform coefficients: Convergence inside training set



- We compute the reduced basis with tolerance $\varepsilon = 10^{-4}$. (Boyaval et al. 2009)

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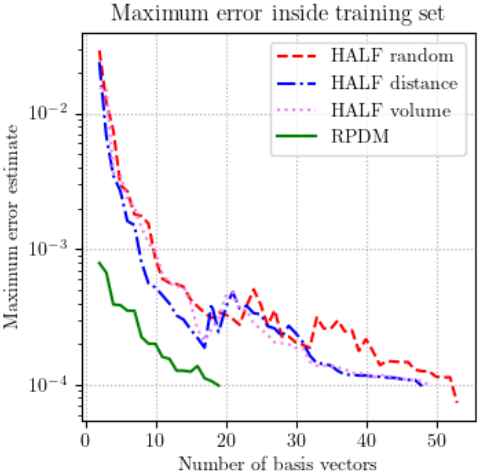

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$$\mathrm{Vol}(D) \cdot J(p_D, \eta) < \varepsilon.$$

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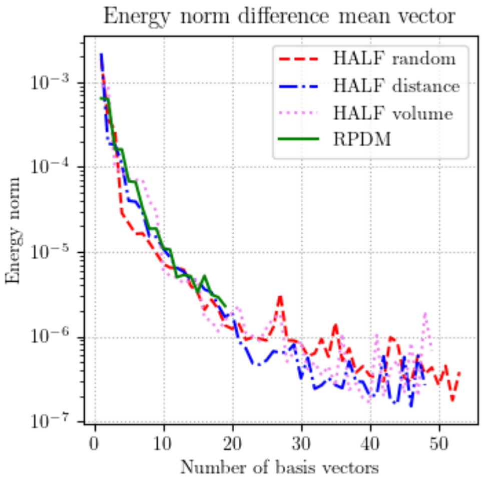
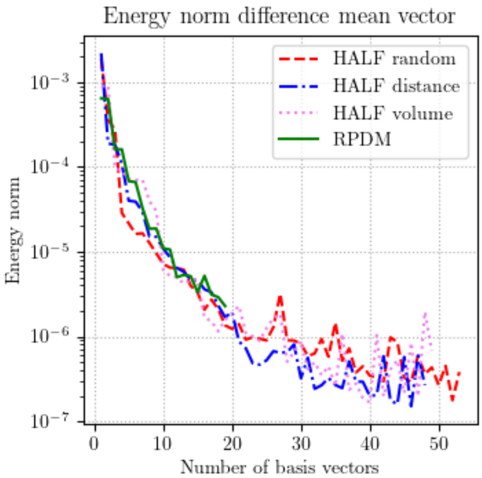
Uniform coefficients: Expectation



- $E_M = \left\| \frac{1}{M} \sum_{i=1}^{M} u_{\omega_i}(x) - \frac{1}{M} \sum_{i=1}^{M} u_{\omega_i}^{\mathrm{RB}}(x) \right\|_{A_{\bar{\omega}}}.$

Uniform coefficients: Expectation



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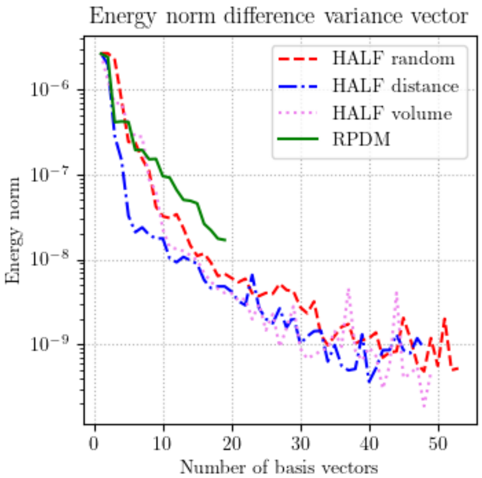
- We consider $M = 40000$ Monte Carlo samples.

Uniform coefficients: Variance



- We draw $M = 40000$ Monte Carlo samples
- $V_M = \frac{1}{M}\sum_{i=1}^{M}\left(u_{\omega_i}(x) - \mathbb{E}[u_{\omega_i}(x)]\right)^2$
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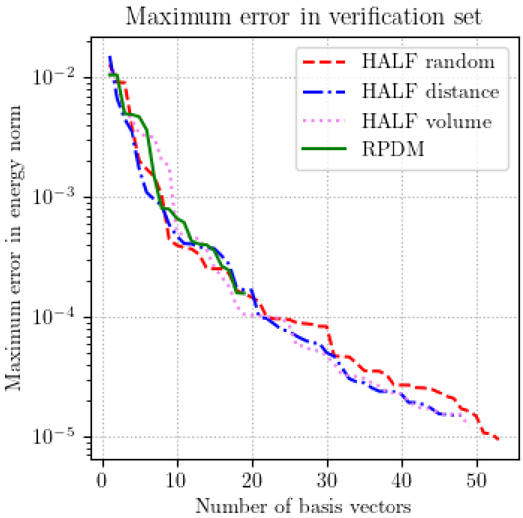
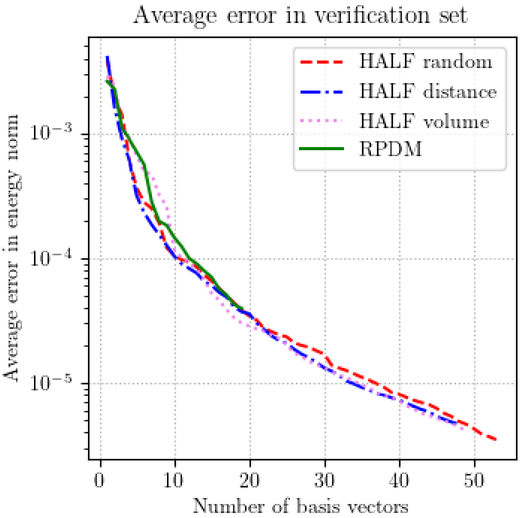
- $\|V_M - V_M^{\mathrm{RB}}\|_{A_{\bar{\omega}}}$

Numerical example: Random fields

TU/e

Uniform coefficients: Verification





Example: Lognormal coefficients (Babuška, Nobile, and Tempone 2007; Bachmayr and Cohen 2024)

$$a(\omega, x) = \exp\Big(\sum_{k=1}^{\mathcal{K}} Y_k(\omega)\Psi_k(x)\Big),$$

where $f \equiv 1$, $\mathcal{K} = 25$, and Y_k are i.i.d. standard normal.

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For all $i, j \in \mathbb{N}$, there exist index $k = k(i, j)$, and we have

$$\Psi_k(x) = \frac{8}{[(2i-1)\pi][(2j-1)\pi]} \sin((i - \frac{1}{2})\pi s) \sin((j - \frac{1}{2})\pi t).$$

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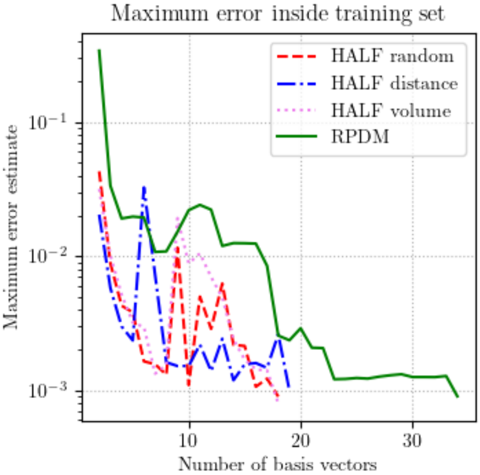
Probability integral transform: Maps uniformly distributed random variables to normally distributed Y_k. (Angus 1994)
We approximate the random field using EIM. (Barrault et al. 2004)

Lognormal coefficients: Convergence in training set



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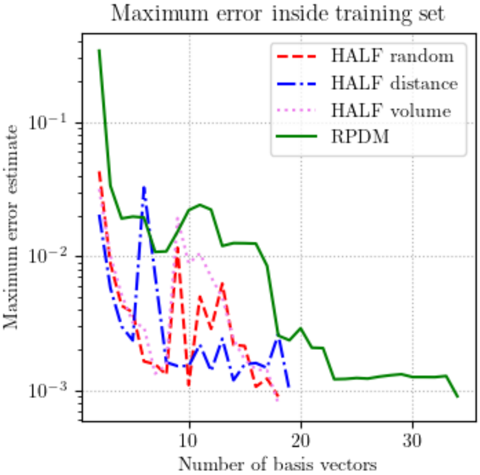

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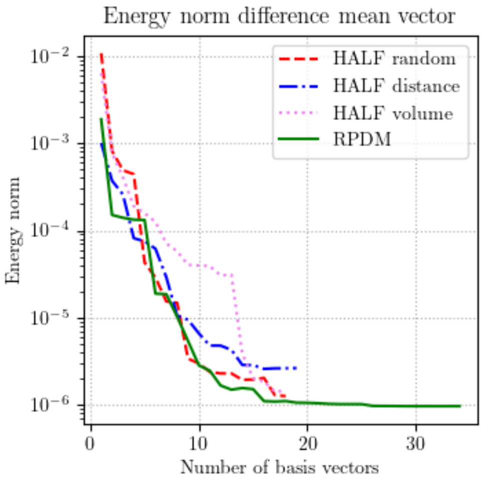

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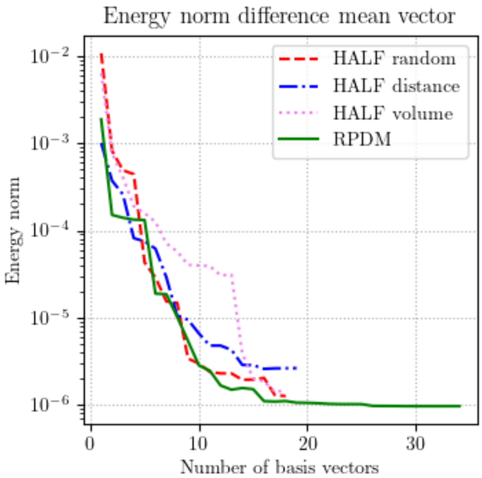

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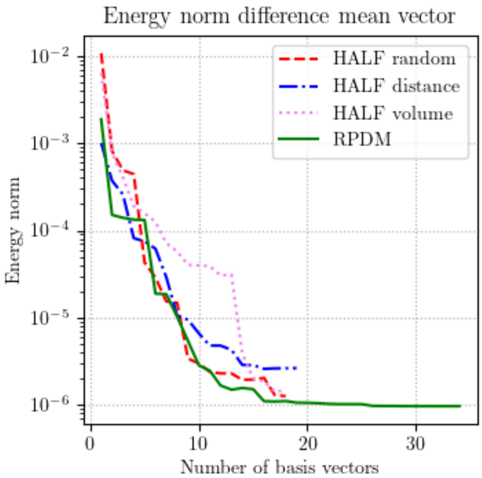
Lognormal coefficients: Expectation



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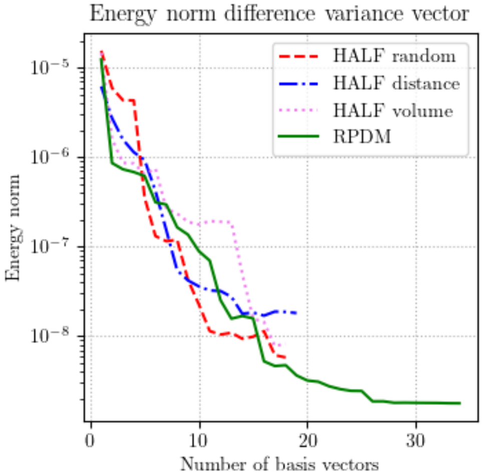
Lognormal coefficients: Variance



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- $\|V_M - V_M^{\mathrm{RB}}\|_{A_{\bar{\omega}}}$

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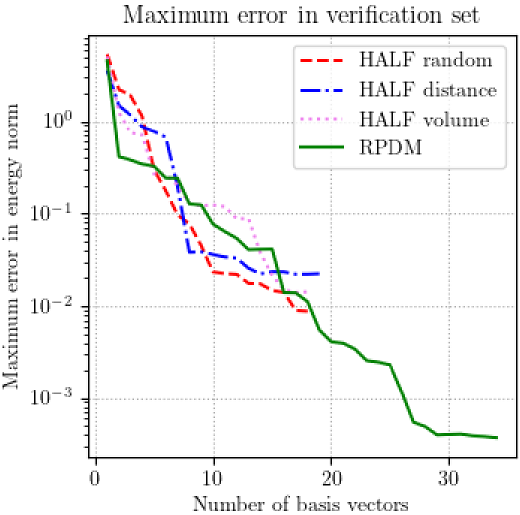
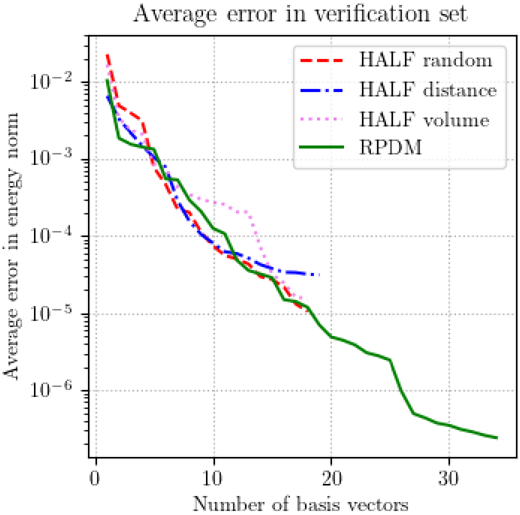
Lognormal coefficients: verification





Concluding remark

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- Scalable greedy-type method: Randomized Polytope Division Method

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- Scalable greedy-type method: Randomized Polytope Division Method
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- Applications to piecewise interpolation and random fields
- Future work: Convergence of variance

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