Certified Randomized Model Order Reduction Methods for High-Dimensional Approximation

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in honor of Yvon Maday's chair at the Collège de France

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Motivation

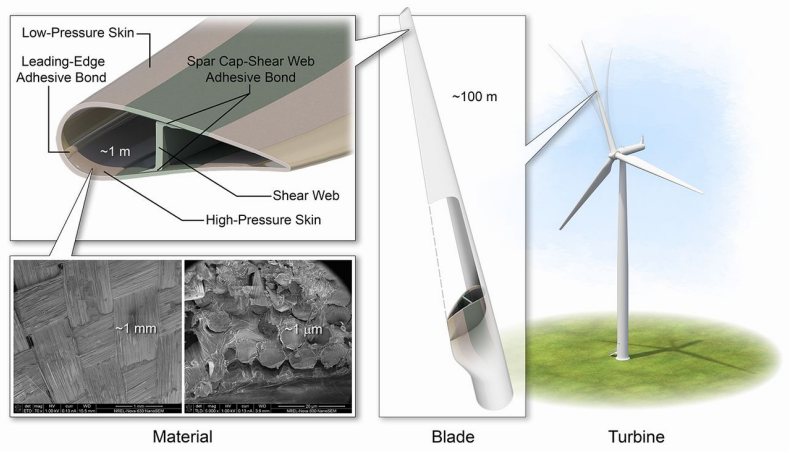

Figure: Picture by NREL, illustrations: Besiki Kazaishvili (NREL), published in [Veers et al., Science, 2019]

Large-scale and multiscale problems, Digital Twins

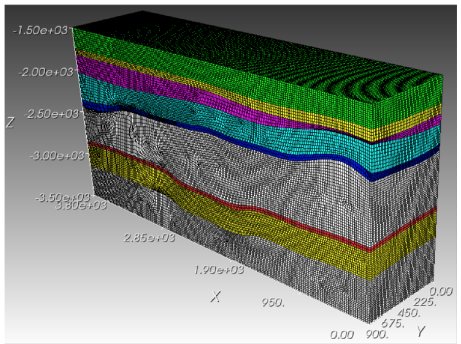

Figure: by J. Trampert

Inverse problems and Uncertainty Quantification

Goal: Discovering Low-Dimensional Structure

Goal: Approximate the set

$$\mathcal{M} := \{u(\mu) \in V : \ u(\mu) \text{ solves PDE dependent on } \mu \in \mathcal{P}\},$$

where

- $\mathcal{P}$: set of admissible parameters.
- V: Hilbert space.
- $\mathcal{P} \subset \mathbb{R}^P$: introduce probability space $(\mathcal{P}, \mathcal{F}, \rho)$.
- $\mathcal{P}$ potentially infinite dimensional: We write $u(Y)$, where $Y : \Omega \to \mathcal{P}$ random variable with values in $\mathcal{P}$ for probability space $(\Omega, \mathcal{F}, \rho)$.

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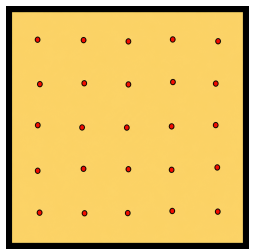
Challenge: Curse of dimensionality [Bellman 57, 61]: The cardinality of the set $\mathcal{P}_{train}$, which is used for calculations typically scales exponentially with the dimension of $\mathcal{P}$.



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Idea: Exploit that many quantities we are interested in behave like **sub-Gaussian random variables - e.g., bounded random variables are sub-Gaussian random variables.** Yields **concentration properties that remain effective in high-dimensional settings.**

Outline

- Motivation
- **Non-asymptotic bounds for Proper Orthogonal Decomposition (POD)/ Principal Component Analysis (PCA)**
- Optimal sampling
- Randomized Greedy algorithm

Marissa Whitby

(PhD Student at Stevens)

Zhiyu Yin

(former Master student at Stevens)

Tommaso Taddei

(Sapienza Università di Roma)

POD/PCA: The Optimal Approximation Space

- We seek an n-dimensional space $V_n \subset V$ minimizing

$$\min_{\dim(V_n)=n} \mathbb{E}\Big[\|u(Y) - P_{V_n} u(Y)\|_V^2\Big].$$

- Define the covariance operator

$$\Sigma = \mathbb{E}\Big(u(Y) \otimes u(Y)\Big).$$

- Then $V_n = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$, where φ_j are eigenfunctions of

$$\Sigma\varphi_j = \lambda_j \varphi_j, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots.$$

- Optimal POD error:

$$\mathbb{E}\Big[\|u(Y) - P_{V_n} u(Y)\|_V^2\Big] = \sum_{j>n} \lambda_j.$$

References: [Pearson 1901], [Jolliffe 02], [Kahlbacher, Volkwein 07], [Holmes et al 12], ...

The Monte Carlo Approximation

- Challenge: Computing the covariance operator is usually infeasible due to expectation.

$\Longrightarrow$ Build Monte Carlo approximation from snapshots $u(Y_1), \ldots, u(Y_m)$:

$$\hat{\Sigma} = \frac{1}{m} \sum_{i=1}^{m} u(Y_i) \otimes u(Y_i).$$

- Then $\hat{V}_n = \operatorname{span}\{\hat{\varphi}_1, \ldots, \hat{\varphi}_n\}$, where $\hat{\varphi}_j$ are eigenfunctions of

$$\hat{\Sigma}\hat{\varphi}_j = \hat{\lambda}_j \hat{\varphi}_j, \qquad \hat{\lambda}_1 \geqslant \hat{\lambda}_2 \geqslant \cdots .$$

- Empirical POD error:

$$\mathbb{E}\Big[\|u(Y) - P_{\hat{V}_n} u(Y)\|_V^2\Big].$$

How many snapshots do we need to get a small empirical POD/PCA error?

Theorem 1 (Non-asymptotic bounds for POD/PCA in expectation)

Assume: $u(Y)$ *is bounded random variable with values in* V*. There exists* $c_1 > 0$ *such that* $c_1(\lambda_n - \lambda_{N_V}) \leqslant (\lambda_n - \lambda_{n+1})$ *and*

$$1 \geqslant \frac{4\|u(Y)\|_{L^\infty(\Omega,V)}\sqrt{\lambda_n}}{\sqrt{m}(\lambda_n - \lambda_{n+1})} + \frac{4\|u(Y)\|^2_{L^\infty(\Omega,V)}}{m(\lambda_n - \lambda_{n+1})}.$$

Then it follows that

$$\mathbb{E}\left[\mathbb{E}\left[\|u(Y) - P_{\widehat{V}_n}u(Y)\|_V^2\right]\right] \leqslant \left[\frac{16\|u(Y)\|^2_{L^\infty(\Omega,V)}}{m(\lambda_n - \lambda_{n+1})c_1} + 1\right] \sum_{j>n}\lambda_j + \textit{Concentration term.}$$

- Neglected eigenvalues are mulitplied with inverse of number of samples. $\Longrightarrow$ Can benefit from rapid eigenvalue decay.
- No dependence on dimension of parameter space!

Bound for fixed spectral gap or polynomial eigenvalue decay

- **Fixed spectral gap:** The bound simplifies to

$$\mathbb{E}\left[\mathbb{E}\left[\|u(Y) - P_{\hat{V}_n} u(Y)\|^2\right]\right] \lesssim C_{\text{gap}} \left(\left[1 + \frac{1}{m}\right] \sum_{j>n} \lambda_j + \left[\frac{n^2}{m} + n \sum_{k>n} \frac{\lambda_k}{\lambda_{n+1}}\right] e^{-c_{\text{gap}} m}\right)$$

- **Polynomial eigenvalue decay:** For $\lambda_n = n^{-\alpha}$ and $m \sim n^{\alpha}$, the bound simplifies to

$$\mathbb{E}\left[\mathbb{E}\left[\|u(Y) - P_{\hat{V}_n} u(Y)\|^2\right]\right] \lesssim C_n \left(\left[1 + \frac{1}{m}\right] \sum_{j>n} \lambda_j + \left[n + \frac{n^{2-\frac{\alpha}{2}}}{\sqrt{m}} + \frac{n^2}{m}\right] e^{-c_n m}\right).$$

Remark: n^2 and $n^{2-\frac{\alpha}{2}}$ terms are likely pessimistic

How do the results advance the state of the art

- Closest non-asymptotic result for bounded random variables [Holodnak, Ipsen, Smith 18]: Number of samples needs to go like $1/\varepsilon^2$, where
  - ε is the target relative error in the approximation of the covariance matrix.
  - The approximation error of the covariance matrix must be smaller than the spectral gap in order to reliably identify the dominant eigenspace.

- [Reiss, Wahl 20]: Paper requires that sub-Gaussian norm of random variable $(u(Y), v)_V$ is bounded by its variance for all $v \in V$:

$$\sup_{k \geqslant 1} \frac{1}{\sqrt{k}} \mathbb{E}\left[|(u(Y), v)_V|^k \right]^{1/k} \leqslant C_1 \mathbb{E}\left[(u(Y), v)_V^2 \right]^{1/2}.$$

This assumption is often difficult to verify and does not necessarily hold for bounded random variables (counter example!).

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- [Reiss, Wahl 20]: In numerical experiment we want to look at the POD eigenmodes (here variance goes to zero as eigenvalues go to zero):

$$\|(u(Y), v)_V\|_{L^\infty(\Omega)} \overset{?}{\leqslant} C_1 \mathbb{E}\left[(u(Y), v)_V^2\right]^{1/2}.$$

Potentially difficult for POD modes corresponding to small eigenvalues that correspond to solution features active on small subsets of $\mathcal{P}$.

Numerical results: Parametric nonlinear diffusion

$$-\operatorname{div}(d_\mu(u(x;\mu))\nabla u(x;\mu)) = 0 \quad \text{in } D,$$
$$u(x;\mu) = \mu_2 \quad \text{on } \partial D,$$
$$d_\mu(u(x;\mu)) := \frac{u(\mu)^2(1-u(\mu))^2}{(u(\mu)^3 + \frac{1}{3}(1-u(\mu))^3)^2} + 0.1 + 10^5 \mathbb{1}_{\{\mu_1 > 0.99\}} k(x),$$

where: $k(x)$ high conductivity channels, $\mu_1 \sim Unif(0,1)$, μ_2 interpolates with high probability function in H^1 on boundary

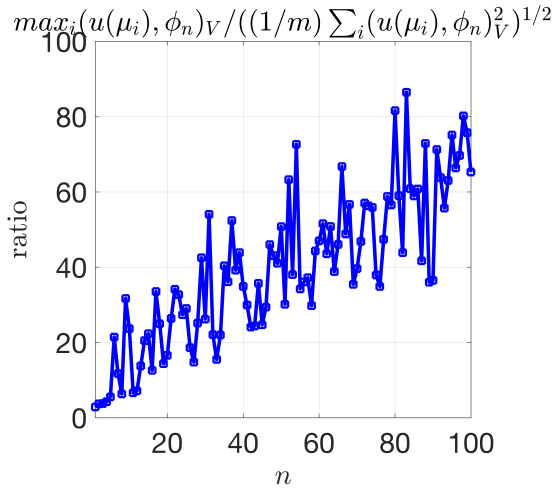
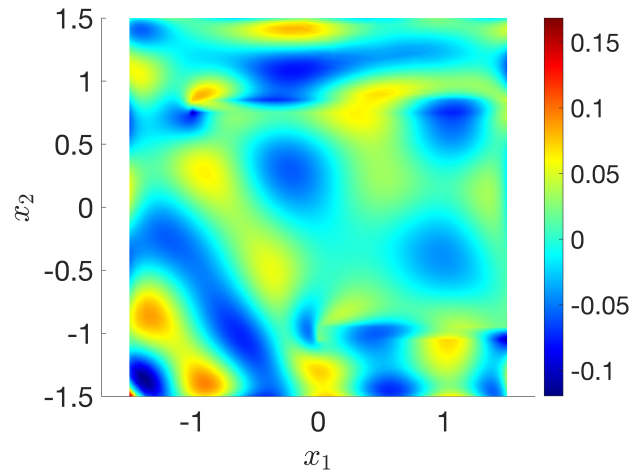


Figure: $\frac{\max_i(u(\mu_i),\widehat{\varphi}_n)_V}{\sqrt{(1/m)\sum_{i=1}^m(u(\mu_i),\widehat{\varphi}_n)_V^2}}$



POD mode associated with *max*

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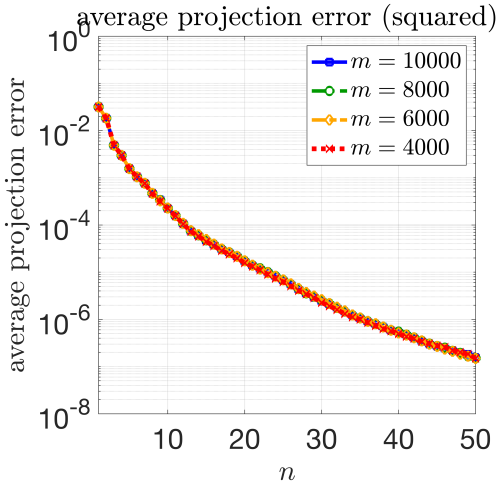

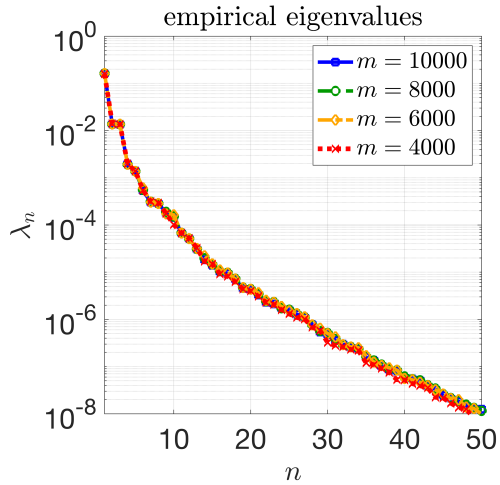

Figure: $\frac{1}{m}\sum_{i=1}^m \|u(\mu_i) - P_{\widehat{V}_n} u(\mu_i)\|_V^2$ $\widehat{\lambda}_n$

Where are we?

POD:

- For many problems a modest number of samples is sufficient to get an accurate approximation independently of the dimension of the parameter space.
- Numerical experiments suggest that assumptions in current non-asymptotic PCA/POD theory may be restrictive for parametric PDEs. Low-energy POD modes can correspond to solution features activated only on small subsets of parameter space, highlighting the relevance of our results.

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How can we improve:

- Exploit structure of problem
- Sampling strategy

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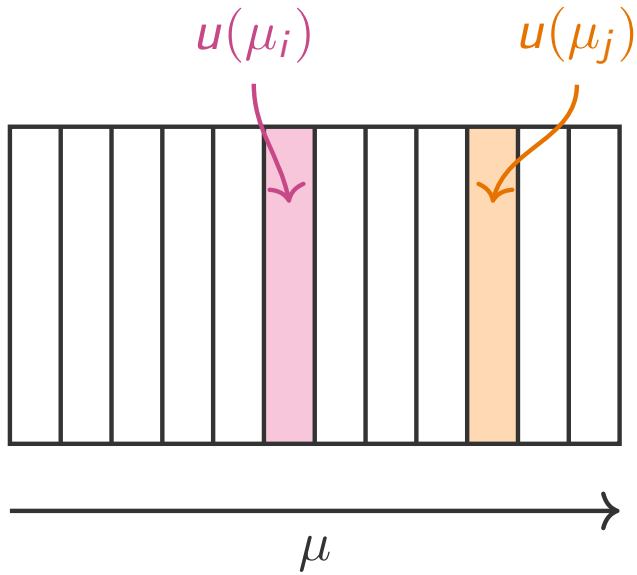
- Non-asymptotic bounds for POD/PCA
- **Optimal sampling**
- Randomized Greedy algorithm

Selecting Informative Parameters = Column Subset Selection

- $\mathcal{M} = \{u(\mu) : \mu \in \mathcal{P}\} \quad \rightsquigarrow \quad M = \left[u(\mu_1) \ \cdots \ u(\mu_N)\right]$
- Equivalent problem: Find subset of columns $C = [u(\mu_1) \ \cdots \ u(\mu_m)]$ such that $\|M - CC^\dagger M\|_F$ is small with high probability ($C^\dagger$: Moore-Penrose inverse).

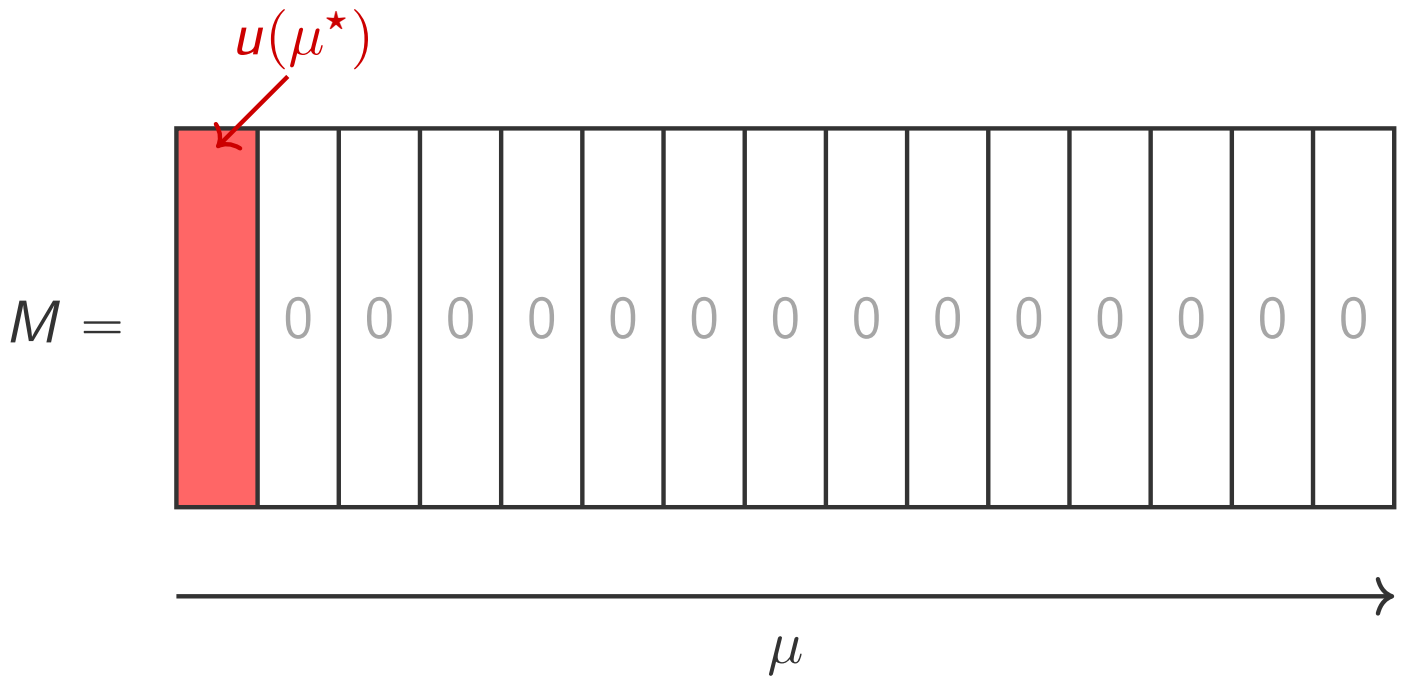
$M =$



- What are good strategies for column subset selection?

Why Uniform Sampling Can Fail

Uniform sampling: Choose parameters $\mu_1, \ldots, \mu_m$ independently and uniformly at random.



Worst-case scenario: The information relevant for approximation is highly localized: it is concentrated in a small subset of the parameter space and, equivalently, in only a few columns of the matrix.

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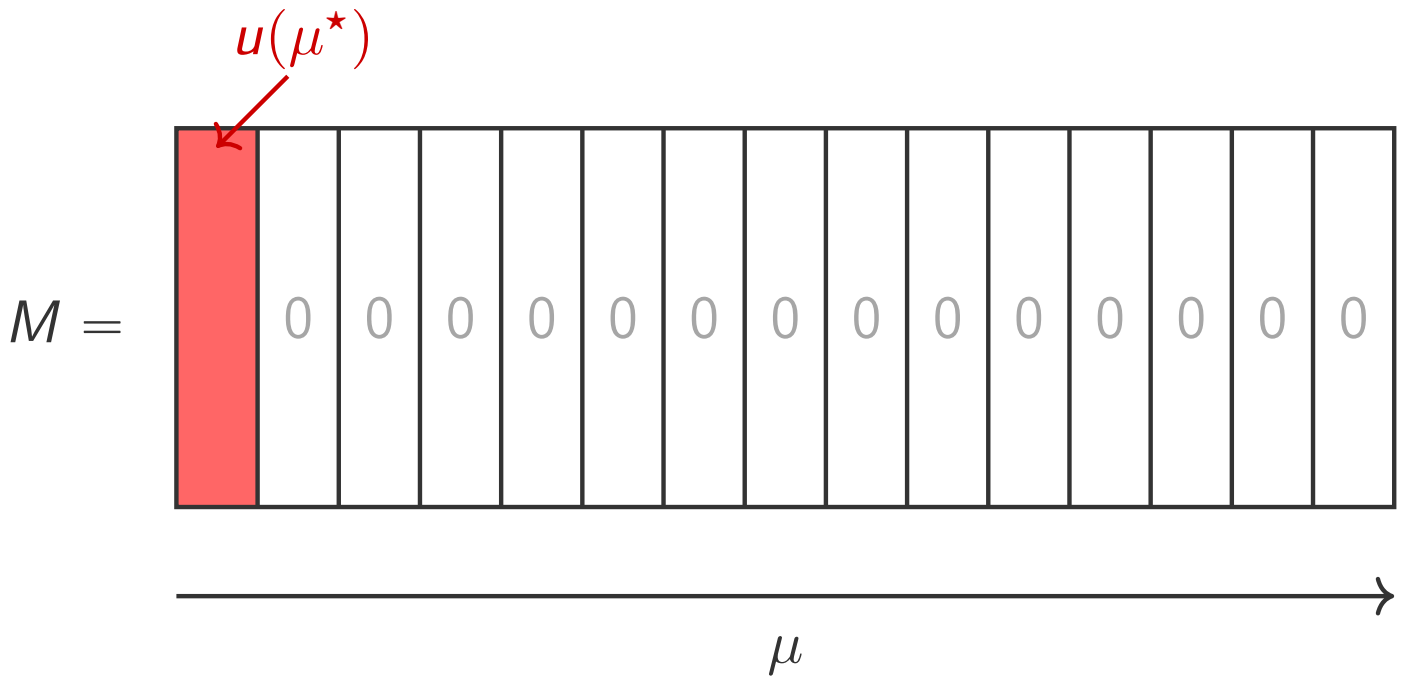

How can we identify the informative regions of the parameter space?

Squared norm sampling

Squared norm sampling: Select the parameter/column with index i with probability $p_i = \|M[:,i]\|_2^2 / (\sum_{j=1}^{N} \|M[:,i]\|_2^2)$.

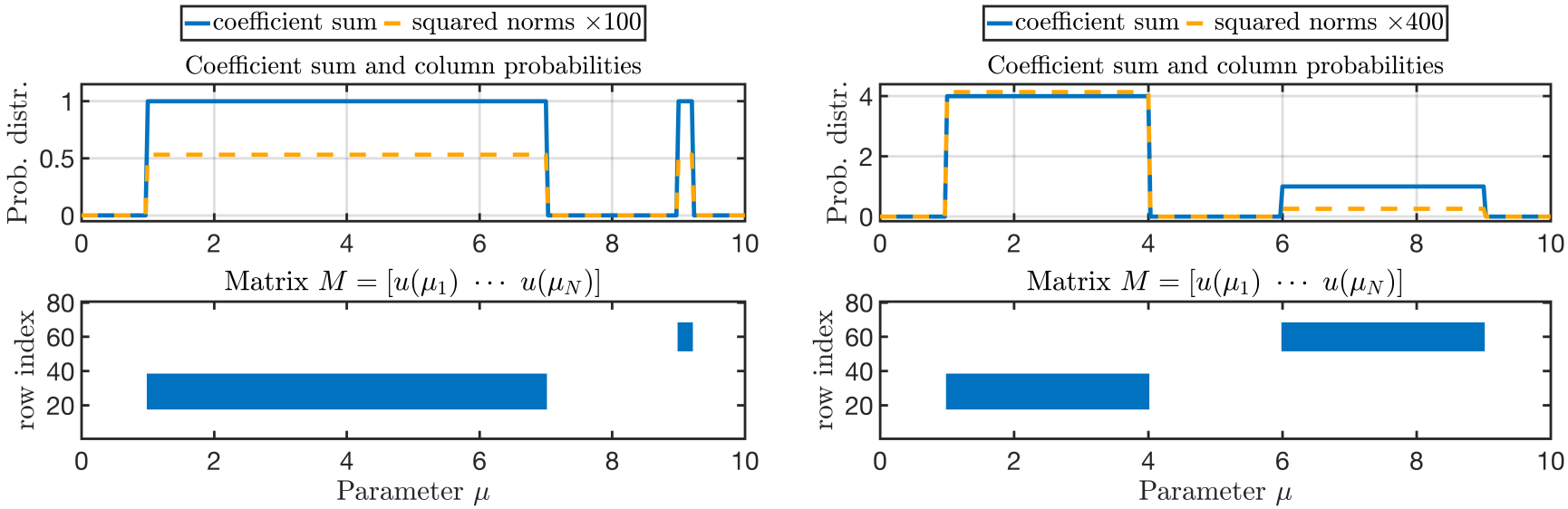

Figure: Examples adapted from [Schleuß, ter Maat, KS, SISC, 23]

Challenges: Data that has values on smaller scales (right example). Highly localized information can be still an issue.

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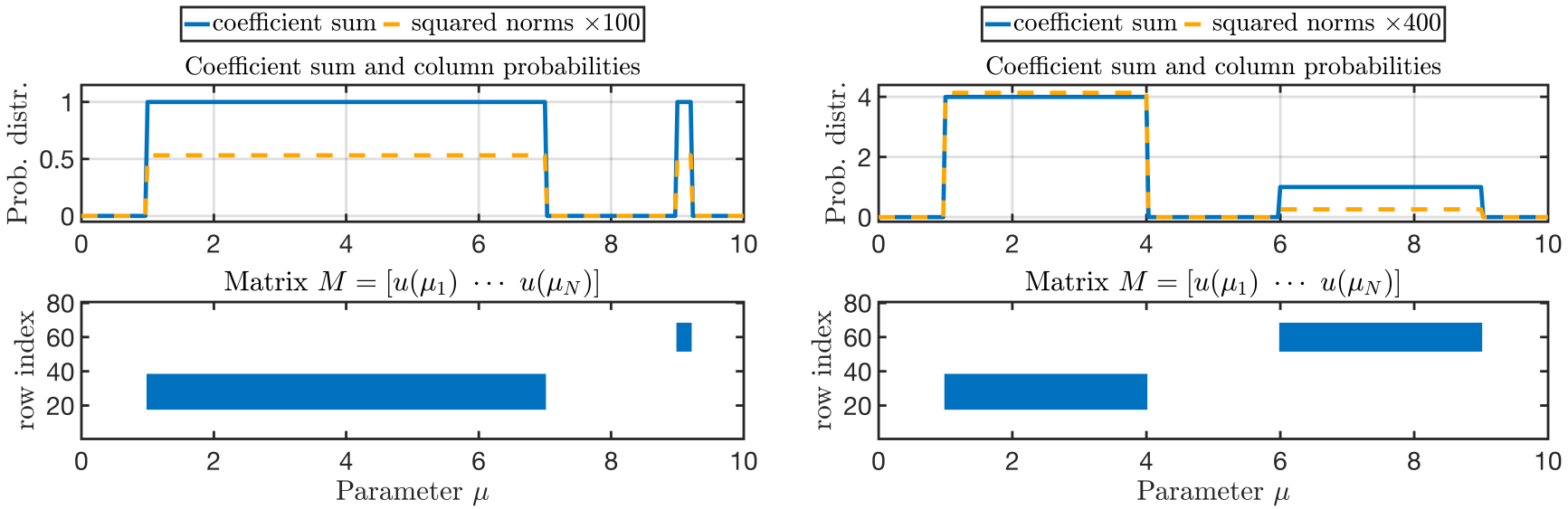

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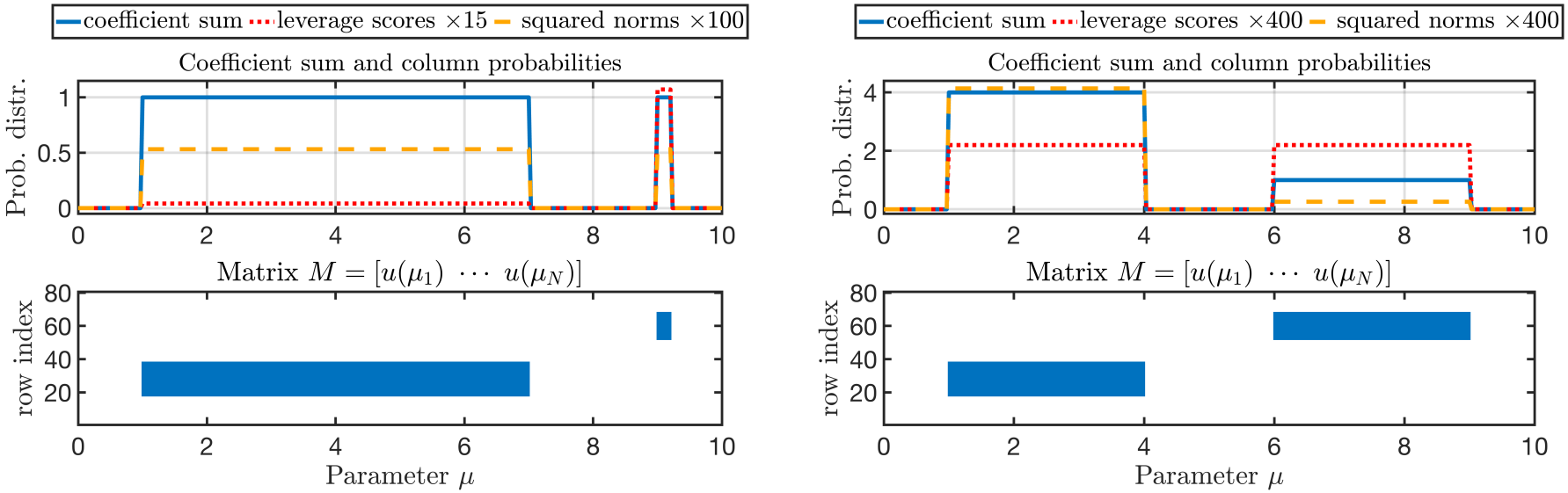
How can we sample properly from areas that are important for approximation?

Leverage score sampling

Leverage score sampling: Select the parameter/column with index i with probability $p_i = (1/n)\sum_{j=1}^{n} v_j[i]$, where $v_1, \ldots, v_n$ are the leading n right singular vectors of M.



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Leverage score sampling chooses columns which have a large influence on the best rank-n fit of M.

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Leverage score sampling yields an approximation which is, with high probability, within a factor $1+\varepsilon$ of the best rank-n approximation:

Theorem 2 ((Drineas, Mahoney, Muthukrishnan, SIMAX, 2018))

Let $\varepsilon \in (0,1]$ and $m \in \mathcal{O}(n^2/\varepsilon^2)$. Then with high probability there holds

$$\|M - CC^{\dagger}M\|_F \leqslant (1+\varepsilon)\|M - M_n\|_F,$$

where M_n: best rank-n approximation to M.

From Informative Columns to Informative Parameters

Discrete setting

- $M = \begin{bmatrix} u(\mu_1) & \cdots & u(\mu_M) \end{bmatrix}$

Continuous setting

- $\mathcal{M} = \{u(\mu) : \mu \in \mathcal{P}\}$

Approximation space: $U_n = \text{span}\{\phi_1, \ldots, \phi_n\}$ with $\{\phi_i\}_i$ orthogonal

- Leverage scores:

$$\ell_i = \frac{1}{n} \sum_{k=1}^{n} \frac{|\phi_k(\mu_i)|^2}{\|\phi_k\|_2^2}$$

- Leverage score distribution

$$p_i = \ell_i$$

- Christoffel function:

$$\mathcal{C}_n(\mu) = \sum_{k=1}^{n} \frac{|\phi_k(\mu)|^2}{\int_{\mathcal{P}} |\phi_k|^2 \, d\rho}$$

- Christoffel measure:

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}\rho}(\mu) := \mathcal{C}_n(\mu)$$

Christoffel sampling is the natural continuous analogue of leverage-score sampling and thus quantifies approximation relevance.

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Results by Cohen, Dolbeault, Migliorati, Narayan and Xu show that Christoffel sampling performs well in detecting large values → Christoffel sampling is also great for certification.

Outline

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- **Randomized Greedy algorithm**

Charles Beall (Stevens)

A randomized Greedy algorithm

- Input: Finite probability measure ρ, Error tolerance *tol*
- In each iteration
  1. Draw m_n independent samples $\mu_1, \dots, \mu_{m_n}$ from the measure
  $$\frac{\mathrm{d}\nu_n}{\mathrm{d}\rho}(\mu) := \mathcal{C}_n(\mu)$$
  2. Enrich V_n with the solution $u(\mu^*)$, where
  $$\mu^* = \arg \max_{\mu_j \in \{\mu_1, \dots, \mu_{m_n}\}} \|u(\mu_j) - u_n(\mu_j)\|_V, \quad u_n(\mu) \in V_n \text{ approximates } u(\mu)$$
- Terminate if $\|u(\mu^*) - u_n(\mu^*)\|_V \leqslant tol$.

$\mathcal{C}_n : \mathcal{P} \to \mathbb{R}_{\geqslant 0}$ denotes the **modified Christoffel function**

$$\mathcal{C}_n(\mu) := \frac{1}{2}\left(1 + \frac{\|u(\mu) - u_n(\mu)\|_V^2}{\|u - u_n\|^2_{L^2_\rho(\mathcal{P}, V)}}\right), \quad \mu \in \mathcal{P}, \tag{1}$$

Assume that the PDE is stable for ρ-almost every parameter.

A randomized Greedy algorithm

- Input: Finite probability measure ρ, Error tolerance *tol*
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- Terminate if $\|u(\mu^*) - u_n(\mu^*)\|_V \leqslant tol$.

Challenges:

- $L^2(\mathcal{P}, V)$−norm: Replace by estimator $\frac{1}{m_{L^2}} \sum_{j=1}^{m_{L^2}} \|u(\mu_j) - u_n(\mu_j)\|_V^2$ to get a practical algorithm. By Hoeffding's inequality, the practical algorithm enjoys essentially the same theoretical guarantees.
- Sampling: Slice sampling [Neal 03] worked well for us so far.

A randomized Greedy algorithm

- Input: Finite probability measure ρ, Error tolerance *tol*
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- Terminate if $\|u(\mu^*) - u_n(\mu^*)\|_V \leqslant tol$.

The error $\|u(\mu) - u_n(\mu)\|_V$ can be replaced an a posteriori error estimator (often more cost efficient).

Theorem 3 (Per-iteration certification)

$(\mathcal{P}, \mathcal{F}, \rho)$: probability space, with $\mathcal{P} \subseteq \mathbb{R}^P$, $P \geqslant 1$.
ρ and Lebesgue measure are equivalent, i.e. every set of positive ρ-measure also has positive Lebesgue measure.
Draw $m_n \in \mathbb{N}$ i.i.d. samples $\mu_1, \ldots, \mu_{m_n} \in \mathcal{P}$ according to ν_n, where

$$m_n \geqslant \frac{2}{\alpha_n^2} \ln\left(\frac{2}{\delta_n}\right), \qquad \text{for given } \alpha_n, \delta_n \in (0,1). \tag{1}$$

Then, with probability at least $1 - \delta_n$, we have

$$\|u - u_n\|_{L^\infty_\rho(\mathcal{P},V)} \leqslant \sqrt{\frac{4\,\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}}{1-\alpha_n}} \max_{1\leqslant i\leqslant m_n} \|u(\mu_i) - u_n(\mu_i)\|_V. \tag{2}$$

We can bound the error over the whole parameter set by the max over few samples!

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Union bound $\Rightarrow$ **If the algorithm reaches *tol* with output V_N, the approximation is certified for ρ-almost every parameter $\mu \in \mathcal{P}$ with high probability.**

Estimating $\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}$

- Observation: For considered numerical test cases $\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}$ did not contribute to the convergence rate.

$\Longrightarrow$ It might often not be necessary to get an accurate estimate.

- Exploit that we sample from areas with high $\mathcal{C}_n$ with high probability:
  - Consider set $G_n := \{\mu \in \mathcal{P} : S_n(\mathcal{C}_n(\mu) - \mathbb{E}_\rho[\mathcal{C}_n]) \geqslant \|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}\}$, where S_n safety factor, $\mathbb{E}_\rho[\mathcal{C}_n] = 1$.
  - Use estimator $S_n\left(\max_{1\leqslant j\leqslant m_{est}} \mathcal{C}_n(\mu_j) - 1\right)$
  - We obtain for the Christoffel measure ν_n that

$$\nu_n(\text{no draw belongs to } G_n) \leqslant \left(\frac{1}{2} + \frac{(S_n^{-1}\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})} + 1)^2}{\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}}\right)^{m_{est}}.$$

$\Longrightarrow$ Very high values will only appear on sets of small measure.

Randomized greedy yields a quasi-optimally converging approximation

- In each iteration the randomized Greedy selects a near maximizing parameter with high probability:

$$\|u - u_n\|_{L^\infty_\rho(\mathcal{P},V)} \leqslant \sqrt{\frac{4\,\|\mathcal{C}_n\|_{L^\infty_\rho(\mathcal{P})}}{1-\alpha_n}} \max_{1\leqslant i\leqslant m_n} \|u(\mu_i) - u_n(\mu_i)\|_V.$$

- The (deterministic) Greedy algorithm [Veroy, Prud'Homme, Rovas, Patera 03] selects the global maximizer in each iteration.

$\Longrightarrow$ Thanks to results in [DeVore, Petrova, Wojtaszczyk 13] for the (deterministic) Greedy algorithm, we conclude that the approximation constructed by the randomized Greedy algorithm converges at a quasi-optimal rate (slightly worse than the Kolmogorov n-width) with high probability.

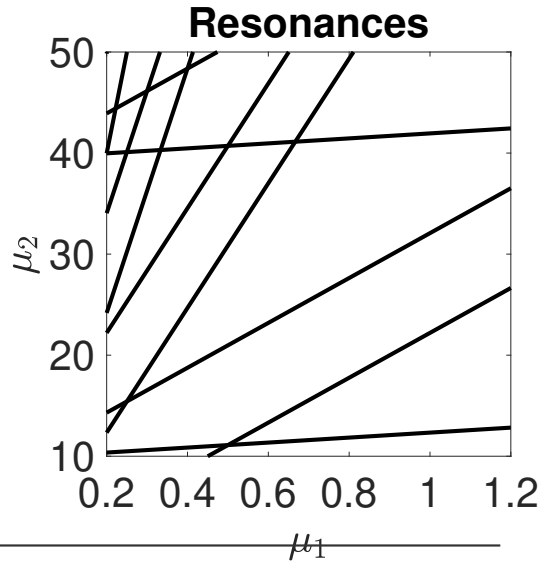
Relation to existing approaches

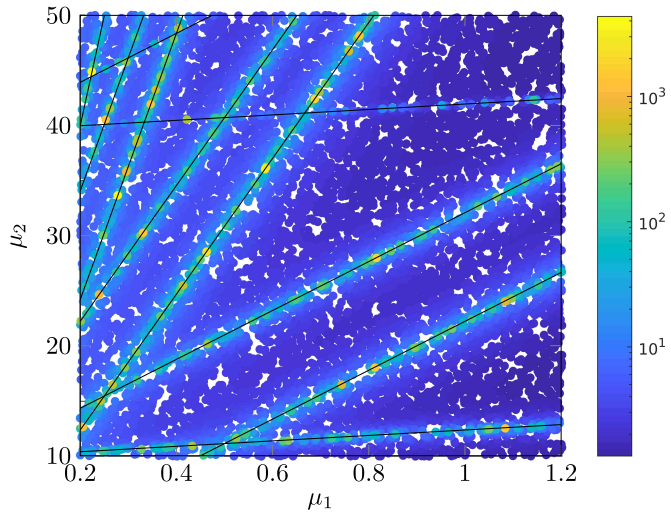
- **Randomized Greedy for model order reduction** [Cohen et al. ESAIM: M2AN '20], [Billaud-Friess et al. Adv. Comput. Math. '24], [Nielen, Tse 26]
- **Adaptive updates/partitioning of the training set:** [Sen Numer. Heat Transfr. B: Fundam. '08], [Eftang, Patera, Rønquist, SIAM J. Sci. Comput. '09], [Haasdonk, Dihlmann, Ohlberger Math. Comput. Model. Dyn. Syst. '11], [Eftang, Stamm Int. J. Numer. Methods Eng. '12], [Hesthaven, Stamm, Zhang ESAIM: M2AN '14], [Jiang, Chen, Narayan J. Sci. Comput. '17], [Jiang, Chen Int. J. Numer. Methods Eng. '20], [Nielen, Tse, Veroy-Grepl SISC 25]
- **Nonlinear optimization-based approach:** [Urban, Volkwein, Zeeb, ROM for Modeling & Comput. Reduction '14]
- **Low-rank tensor approaches:** [Khoromskij, Schwab SIAM J. Sci. Comput. '11], [Ballani, Kressner SIAM J. Sci. Comput. '16], work by R. Scheichl, A. Nouy, M. Olshanskii, ...

Multiparametric Helmholtz problem[1]

Given $\mu = (\mu_1, \mu_2) \in \mathcal{P} = [0.2, 1.2] \times [10, 50]$, find $u(\mu)$ such that

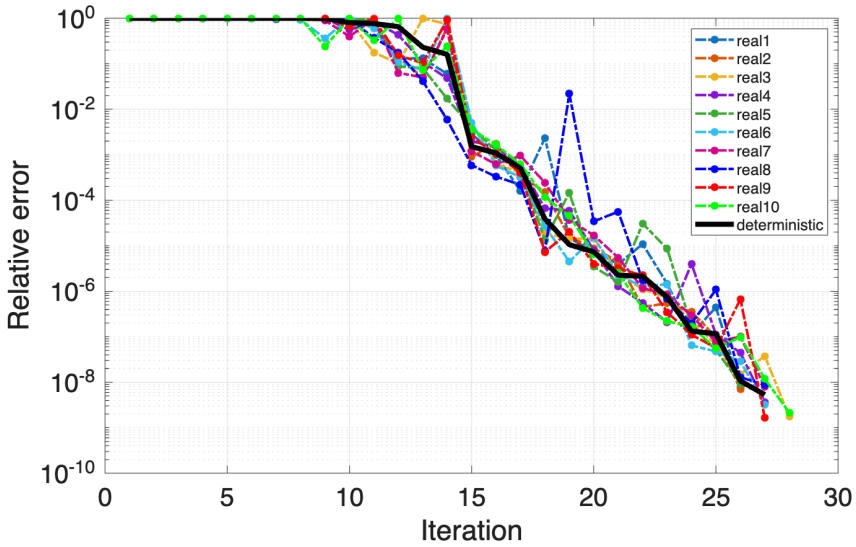
$$
\begin{aligned}
-\frac{\partial^2 u(\mu)}{\partial x^2} - \mu_1 \frac{\partial^2 u(\mu)}{\partial y^2} - \mu_2 u &= f(x,y) && \text{in } D = (0,1)^2,\\
u(\mu) &= 0 && \text{on } (0,1) \times \{0\},\\
\frac{\partial u(\mu)}{\partial y} &= \cos(\pi x) && \text{on } (0,1) \times \{1\},\\
\frac{\partial u(\mu)}{\partial x} &= 0 && \text{on } \{0,1\} \times (0,1).
\end{aligned}
$$





[1] see [KS, Zahm, Patera 19] and [Huynh et al. 10]

Convergence results



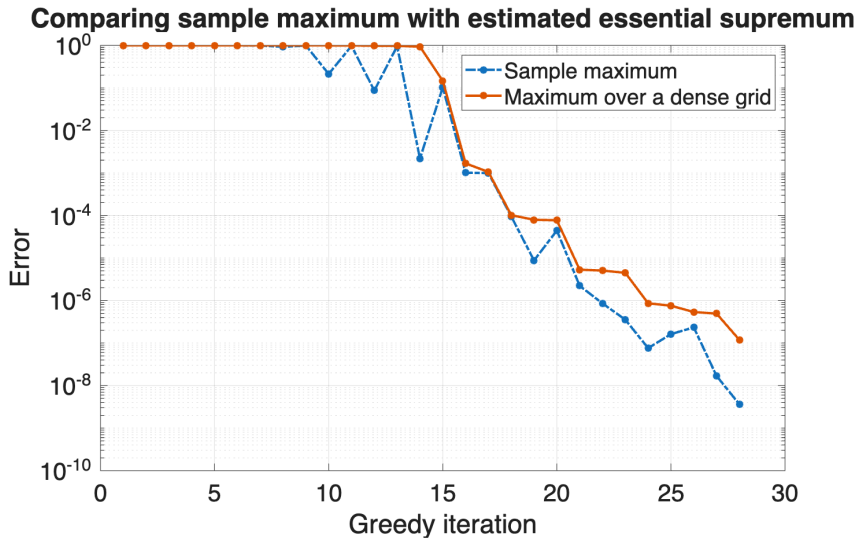


Figure: 40 samples per iteration, cardinality of training set for deterministic Greedy algorithm: 1200, maximum of error estimated on set of $22,500$ samples.

Summary

- Goal: Approximate $\mathcal{M} := \{u(\mu) : u(\mu) \text{ solves PDE dependent on } \mu \in \mathcal{P}\}$
- Idea: Exploit that under mild conditions many quantities we are interested in behave like sub-Gaussian random variables.
- POD: For many problems a modest number of samples is sufficient to get an accurate approximation independently of the dimension of the parameter space.
- Data dependent sampling strategies are well-suited to sample form areas of the parameter set that are relevant for approximation.
- Randomized Greedy algorithm provides certified and quasi-optimally converging reduced approximations for almost every $\mu \in \mathcal{P}$ by relying on a measure that ensures sampling from areas with high approximation error.

Thank you very much for your attention!

Appendix

Appendix

Theorem 4 (Non-asymptotic bounds for POD/PCA)

Assume: $u(Y)$ *is bounded random variable with values in Hilbert space* V *of dimension* N_V. *There exists* $c_1 > 0$ *such that* $c_1(\lambda_n - \lambda_{N_V}) \leqslant (\lambda_n - \lambda_{n+1})$ *and*

$$1 \geqslant \frac{4\|u(Y)\|_{L^\infty(\Omega,H)}\sqrt{\lambda_n}}{\sqrt{M}(\lambda_n - \lambda_{n+1})} + \frac{4\|u(Y)\|^2_{L^\infty(\Omega,H)}}{M(\lambda_n - \lambda_{n+1})}.$$

Then it follows that

$$\mathbb{E}\left[\mathbb{E}\left[\|u(Y) - P_{\hat{H}_n}u(Y)\|^2_H\right]\right] \leqslant c_1^{-1}\left(\frac{16\|u(Y)\|^2_{L^\infty}}{M(\lambda_n - \lambda_{n+1})}\sum_{j>n}\lambda_j\right.$$

$$+8\sum_{j\leqslant n}(\lambda_j - \lambda_{n+1})\sum_{k>n}\frac{\lambda_k}{\lambda_{n+1}}\exp\left(-\frac{M}{4}\min\left(\frac{(\lambda_j - \lambda_{n+1})^2}{4\|u(Y)\|^2_{L^\infty(\Omega,H)}\lambda_{n+1}}, \frac{(\lambda_j - \lambda_{n+1})}{2\|u(Y)\|^2_{L^\infty(\Omega,H)}}\right)\right)$$

$$+\frac{32\|u(Y)\|^2_{L^\infty}}{M}\sum_{i=1}^{n}\frac{\lambda_{n+1-i}(\lambda_n - \lambda_{n+1})}{(\lambda_{n+1-i} - \lambda_{n+1})\lambda_n}\left(\sqrt{6}\sum_{j\leqslant n}\frac{\lambda_j}{(\lambda_j - \lambda_{n+1})} + \sqrt{8}\sum_{j\leqslant n}\frac{\|u(Y)\|_{L^\infty}\sqrt{\lambda_j}}{\sqrt{M}(\lambda_j - \lambda_{n+1})}\right)$$

$$\left.\cdot\exp\left(-M\frac{(\lambda_n - \lambda_{n+1})^2}{64\lambda_n\|u(Y)\|^2_{L^\infty}}\right)\right) + \sum_{j>n}\lambda_j$$

Sketch of proof of theorem

Part I. Establish Marcinkiewicz-Zygmund (MZ) inequalities in $L^2_\nu(\mathcal{P})$:

1. We introduce the weighted estimator

$$\frac{1}{m}\sum_{j=1}^{m}\frac{1}{\mathcal{C}(\mu_j)}\phi^2(\mu_j), \quad \mu_j \overset{\text{i.i.d.}}{\sim} \nu, \quad \phi \in \text{span}\{\mathcal{E}\}, \quad \mathcal{E}(\mu) = \|u(\mu) - u_n(\mu)\|_V.$$

2. Hoeffding's inequality for bounded random variables (see e.g., [Vershynin 18]) yields:

$$\mathbb{P}_\nu\left(\left|\frac{1}{m}\sum_{j=1}^{m}\frac{1}{\mathcal{C}(\mu_j)}\phi^2(\mu_j) - \|\phi\|^2_{L^2_\rho(\mathcal{P})}\right| \geqslant \alpha\right) \leqslant 2\exp\left\{-\frac{M\alpha^2}{2}\right\}.$$

and thus MZ inequalities with probability at least $1-\delta$, $\delta = 2\exp\left\{-\frac{M\alpha^2}{2}\right\}$:

$$(1-\alpha)\|\phi\|^2_{L^2_\rho(\mathcal{P})} \leqslant \frac{1}{m}\sum_{j=1}^{m}\frac{1}{\mathcal{C}(\mu_j)}\phi^2(\mu_j) \leqslant (1+\alpha)\|\phi\|^2_{L^2_\rho(\mathcal{P})}. \tag{3}$$

Sketch of proof of theorem

Part II. Lift Marcinkiewicz-Zygmund inequalities from $L^2_\rho(\mathcal{P})$ to $L^\infty_\rho(\mathcal{P})$:

1. The definition of $\mathcal{C}$ gives the key inequality (see also e.g., [Xu, Narayan 23]):

$$\|\mathcal{E}\mathcal{C}^{-1/2}\|^2_{L^\infty_\rho(\mathcal{P})} \leqslant 2\|\mathcal{E}\mathcal{C}^{-1/2}\|^2_{L^2_\nu(\mathcal{P})}. \tag{4}$$

2. Applying Marcinkiewicz-Zygmund inequalities in $L^2_\nu(\mathcal{P})$ for $\mathcal{E}$ gives:

$$\frac{1}{\|\mathcal{C}\|_{L^\infty_\rho(\mathcal{P})}}\|\mathcal{E}\|^2_{L^\infty_\rho(\mathcal{P})} \leqslant \|\mathcal{E}\mathcal{C}^{-1/2}\|^2_{L^\infty_\rho(\mathcal{P})} \overset{(4)}{\leqslant} 2\|\mathcal{E}\mathcal{C}^{-1/2}\|^2_{L^2_\nu(\mathcal{P})} = 2\|\mathcal{E}\|^2_{L^2_\rho(\mathcal{P})}$$

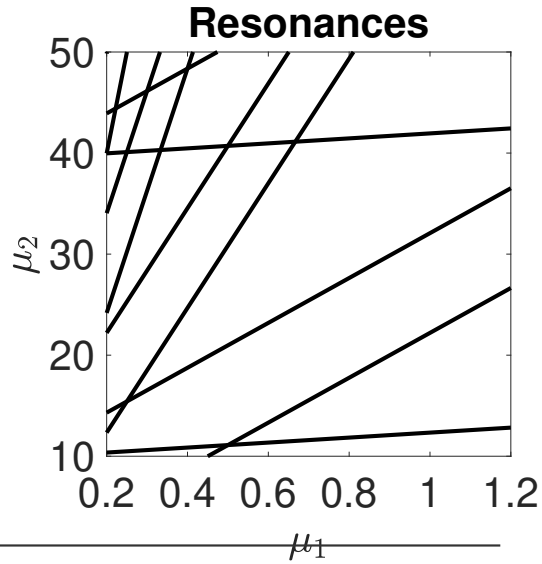
$$\overset{MZ}{\leqslant} \frac{2}{m(1-\alpha)}\sum_{j=1}^{m}|\mathcal{E}\mathcal{C}^{-1/2}(\mu_j)|^2 \leqslant \frac{4}{1-\alpha}\left\{\max_{1\leqslant j\leqslant m}\mathcal{E}(\mu_j)\right\}^2.$$

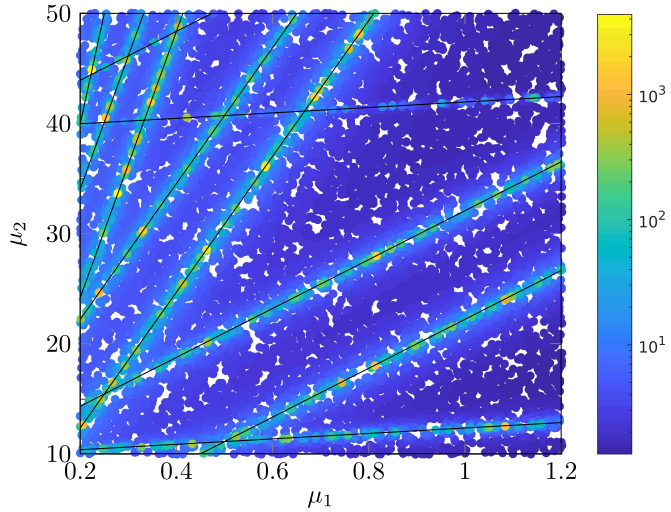
References: Part I makes use of classical arguments in sampling discretization theory (e.g., [Cohen, Migliorati 17])

Multiparametric Helmholtz problem[2]

Given $\mu = (\mu_1, \mu_2) \in \mathcal{P} = [0.2, 1.2] \times [10, 50]$, find $u(\mu)$ such that

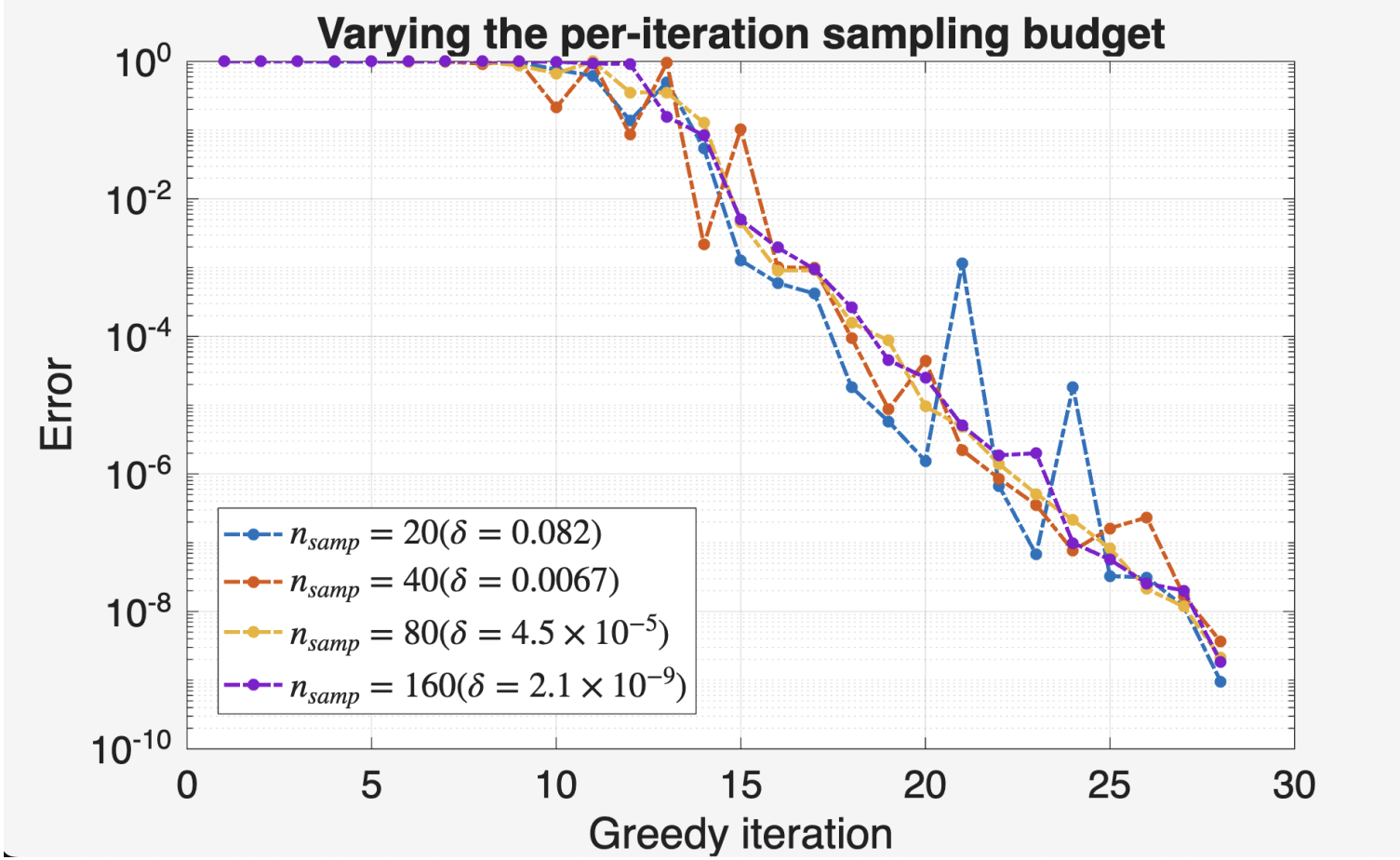
$$
\begin{aligned}
-\frac{\partial^2 u(\mu)}{\partial x^2} - \mu_1 \frac{\partial^2 u(\mu)}{\partial y^2} - \mu_2 u &= f(x,y) && \text{in } D = (0,1)^2, \\
u(\mu) &= 0 && \text{on } (0,1) \times \{0\}, \\
\frac{\partial u(\mu)}{\partial y} &= \cos(\pi x) && \text{on } (0,1) \times \{1\}, \\
\frac{\partial u(\mu)}{\partial x} &= 0 && \text{on } \{0,1\} \times (0,1).
\end{aligned}
$$





[2] see [KS, Zahm, Patera 19] and [Huynh et al. 10]

Convergence results



Parameters selected by the randomized Greedy

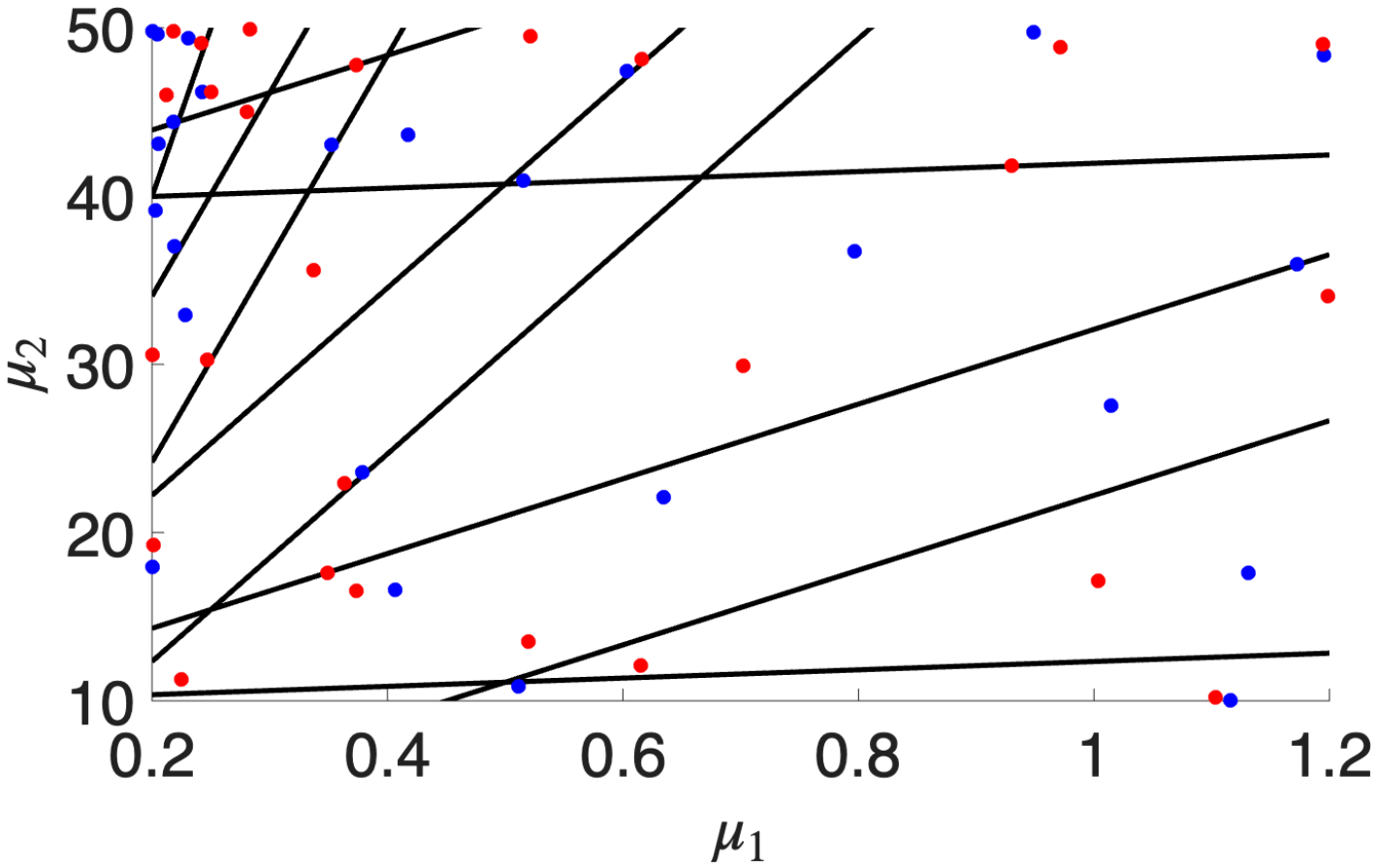


Figure: blue: randomized Greedy; red: deterministic Greedy.

Evolution of the Christoffel function

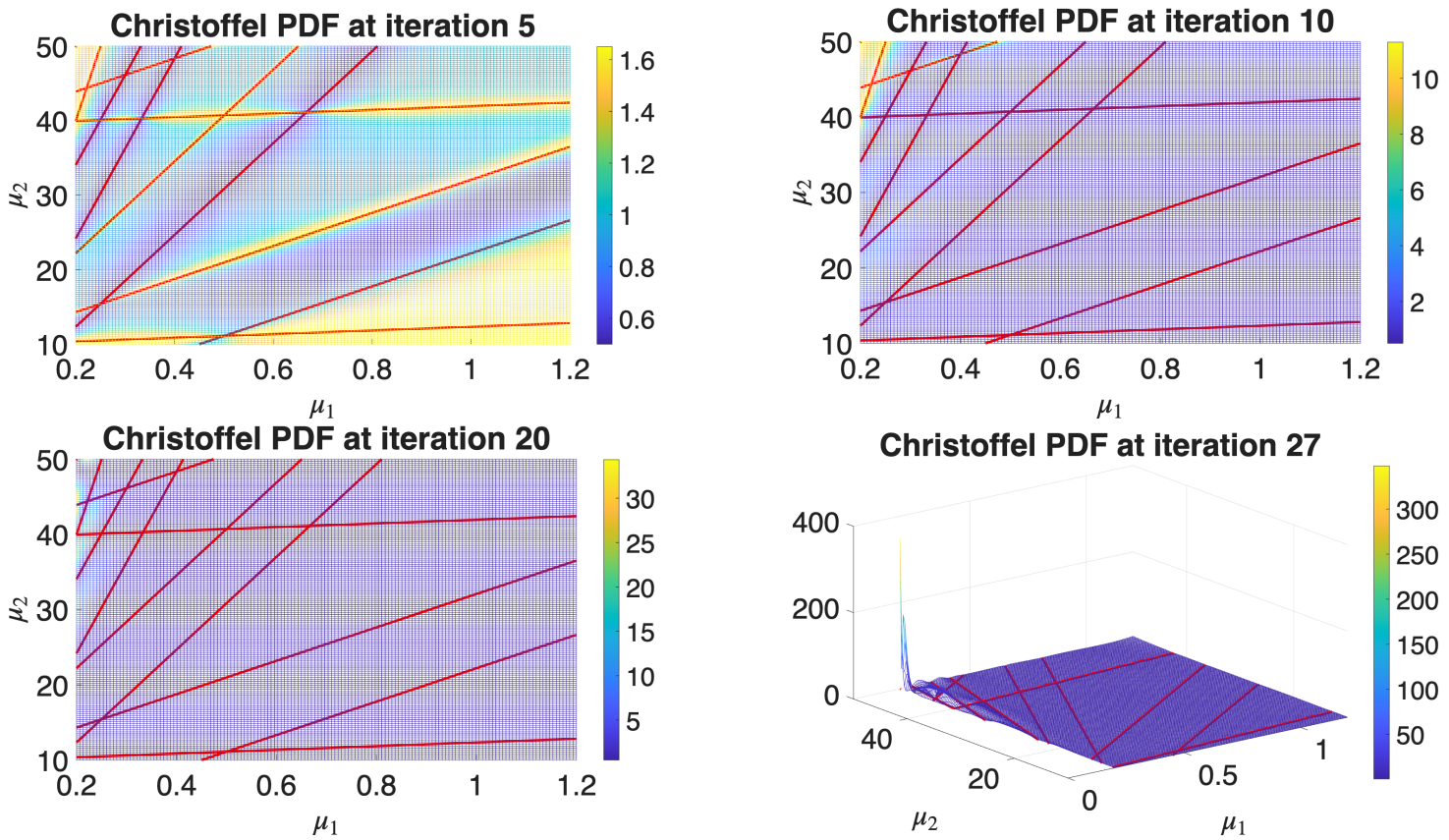


Figure: Christoffel function in iteration $5, 10, 20, 27$ (top-bottom; left-right) estimated on a uniform grid from $22,500$ samples